Part I
Manifolds

1 Lecture 1

Definition Let X be a set, then a *metric* on x as a function

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Definition The ϵ-neighbourhood $U(x_0, \epsilon)$ of x_0 is defined by $U(x_0, \epsilon) = \{x | d(x, x_o,) < \epsilon\}$

Definition A subset U of X is open if for each $x_0 \in U$, there exists $\epsilon > 0$ such that $U(x_0, \epsilon) \subset U$

Definition A subset B of X is closed in X if its complement is open.

Definition A function $f : X \to Y$ is continuous at $x_0 \in X$ if for each open set $V \subset Y$, containting $f(x_0)$, theres an open set $U \subset X$ containing x_0 such that $f(U) \subset V$

Definition $f : X \to Y$ is *injective* if $\forall a, b \in X f(a) = f(b) \Rightarrow a = b$

Definition $f : X \to Y$ is *differentiable* at $x_0 \in X$ if there exists an n by k matrix B such that;

$$\lim_{h \to 0} \frac{f(\mathbf{x}_0 + h) - f(\mathbf{x}_0) - Bh}{||h||} = 0 \tag{1}$$

The matrix B is unique and is the derivative of f at x_0 denoted by $Df(x_0)$

We have the partial derivative of a function f as;

$$D_i f = \lim_{t \to 0} \frac{f_i(x_0 + te_i) - f(x_0)}{t} \tag{2}$$

Definition If all partial derivatives of f_i for $i = 1, \ldots, n$ of order r or less are continuous, then f is of class C^r on X.

Definition Let $A \subset \mathbb{R}^n$ open and let $f(A) \to \mathbb{R}^n$ be of class C^r, then we define $Df(x)$ to be *non-singular* at x if $Det[f] \neq 0$.

Theorem 1.1 *If $Df(x)$ is non singular at $a \in A$, there is a neighbourhood U of A such that f carries U injectively into an open set $V \subset \mathbb{R}^n$. The inverse function f^{-1} is of class C^r on V.*

Note that the non singularity of Df implies that f is locally injective, however it may not be globally one to one on the whole domain.

2 Lecture 2

Definition Let $0 < k \leq n$, let $A \subset \mathbb{R}^k$ open, and let $\alpha : A \to \mathbb{R}^n \in C^r$, $r \geq 1$. Then the set $y = \alpha(A)$, together with the map α is called the *parametrised manifold.*

Definition Suppose M is a subspace of $\mathbb{R}^n$ such that for each $p \in M$, there is a set V containing p, open in M and set U ehcih is open in $\mathbb{R}^k$, and a continuous map $\alpha : U \to V suchthat$;

1. α is injective
2. $\alpha^{-1} : V \to U$ is continuous
3. $D\alpha(x)$ has rank k for each $x \in U$
4. α is of class $C^r, r \geq 1$

This implies that M is a k-manifold without boundries in $\mathbf{R}^n$ of class C^r. α is called the co-ordinate patch on M around p.

3 Lecture 3

4 Lecture 4

5 Lecture 5

6 Lecture 6

Definition The ϵ - neighbourhood $U(x_o, \epsilon)$ of x_0 is defined by

$$U(x_0, \epsilon) = \{x | d(x, x_0) < \epsilon\} \tag{3}$$

Definition A subset B of X is *closed* in X if its complement is open.

Definition A function $f : X \to Y$ is continuos at $x_0 \in X$ if for each open set $V \subset Y$, containing $f(x_0)$ theres an open set $U \subset containing x_0$ such that $f(U) \subset V$.

Example

$$f : \mathbb{R} \to \mathbb{R} given by f(x) = x \tag{4}$$

Definition Let M be a K-Manifold in R^n and $p \in M$.[1]

If there is a coordinate patch $\alpha : U- > V$ on M about p such that U is open in R^k then p is an *interior point*, otherwise p is a boundary point.[2]

[1] In the manifold definition we allowed two domains, open and closed.

[2] Interior point - Pre image open in R^k.

The set of boundary points is δM. δM : Boundary of M $M - \delta M$: Interior of M. Let $\alpha : U- > V$ about the point p in M.

1. If U contained in R^k open, then p is in the interior of M
2. If U contained in H^k open and $p = \alpha(x_0)$ with x_0 in H^k, then p is in the interior.
3. If U in H^k and $p = \alpha(x_0)$ with $x_0 in R^k - 1$ implies that p is in the boundary of M

Proof 1. Follows from to def

2. Let $U_0 - U intersetion H^k$ and let $V_0 = \alpha(x_0)$, then $\alpha_(u_0) : U_\rightarrow V_0$ is a patch about p with U_0 open in R^k.
3. Assume there is a 2nd patch, $\alpha_1 : U_1 \rightarrow V_1$ about p with U_1 open in R^k we will derive a contradiction. $V_0$and V_1 are both open in M which implies W, the intersection is open in M. Let $W_i = \alpha_i^- 1(w)$ for $i = 0, 1$, then W_0 is open in H^k contains $x_0$and W is open in$R^k \alpha_0^- 1 \circ \alpha_1 : W1- > W_0$ is of class C^r with non- singular derivative.

   By the Inverse function theorem $\rightarrow W_0$ must be open in R^k, however W_0 is contained in H^k and contains $X_0 \in R^k - 1 \rightarrow W_0$ not open inR^k

Theorem 6.1 *Let M be a k-manifold in $\mathbb{R}^n$ for $k \geq 2$. If δM is non empty, thenδM is a (k-1)-manifold without boundary in R^n.*

Proof Let p$\in \delta M$. Let $\alpha : U- > V$be a patch on M around P. Then U is open in $\mathbb{H}^k$and$p = \alpha(x_0)$ for some $x_0 \in \delta \mathbb{H}^k = \mathbb{R}^(k-1)xo$.

By the previous Lemma, each point in$U \cap \delta mathbb H^k$ is mapped toδM

$\alpha : U \rightarrow \delta M$

Let $U_0 \in \mathbb{R}^k - 1$, and open such that $U_0 x0 = U \cap \delta H^k$

Definition $\alpha : U_0 \rightarrow V_0$, given by $\alpha(x, 0) = \alpha(x, 0) = \alpha_0(x)$ Then α_0 is a co-od patch and$\delta M \Rightarrow \delta(\delta M) = \emptyset$

$x^2 + y^2 - 1 = 0$ is the equation for the unit circle.

Theorem 6.2 *Let O be open in $\mathbb{R}^n$; let $f : O \rightarrow \mathbb{R}$, be of class C^r. Let M be the set of points for which f(x) = 0, and N the set for which $f \geq 0$. Suppose M is not empty and Df(x) has rank 1 for all $x \in M$ Then N is a n-manifold in $\mathbb{R}^n$ and $\delta N = M$.*

7 Lecture 7

Theorem 7.1 *Let O be open in $\mathbb{R}^n$, let $f : O \rightarrow \mathbb{R}$ be of class $C^r N = \{x \in \mathbb{R}^n | f(x) \geq 0\}$, Suppose $M = \emptyset$ and $Df(x) has rank 1 \forall x \in M$. Then N is an n-manifold in $\mathbb{R}^n$ and $M = \delta N$*

Proof Let $p \in Nsuchthatf(p) > 0$ Let$U \in \mathbb{R}$, open consisting of all points x such that $f(x) > 0$. LKet $\alpha : U \to U$ the identity map. Then α is a patch on N about p, whose domain is open in $\mathbb{R}^n \Rightarrow$ p is an interior point.

Let $p \in Nsuchthatf(p) = 0$ Since $Df(p) \neq, atleast1D_if(p), i = 1, \ldots, n$ is non zero. Let $D_nf(p) \neq 0$ Define $F : O \to \mathbb{R}^n$ by

1. $F(x) = (x_1, \ldots, x_(n-1), f(x))$
2. $$DF = \begin{bmatrix} I_{n-1} & 0 \\ * & D_nf \end{bmatrix}$$

$\Rightarrow Df(p)$ is non singular $\Rightarrow \exists A \supset p$ which is mapped by F 1 - to - 1 to an open set B of $\mathbb{R}$ and $F^{-1} \in C^r$

F maps $A \cap N$ onto an open set $B \cap \mathbb{H}^n$, since $x \in N \Leftrightarrow f(x) \geq 0. \Rightarrow F^-1B \cap \mathbb{H}^n \to A \cap N$ is the required coordinate patch.

$f_i : 0 \to \mathbb{R}^n fori = 1, \ldots, n-k\ M = \{x \in \mathbb{R}^n | f_i(x) = 0 \forall i\}$ Let$D(f_1, \ldots f_(n-k))$ has rank n-k on M $\Rightarrow$ M is a k - manifold

Example

$$f_1(x_1, y_1) = x_1^2 + y_1^2 - f_1^2 = 0 f_2(x_1, y_1, x_2, y_2) = (x_2 - x_1)^2 + (y_2 - y_1)^2 f^2 = 0 \quad (5)$$

We have discussed Manifolds and there properties, the next part of the course we will look at integration in $\mathbb{R}^k$. We will have a third section called differential forms which we will look at a lot of linear algebra for. These three sections will at the end, come together to have integration on M using differential form. We will see Stokes theorem.

Part II
Integration

7.1 Integration in $\mathbb{R}^n$

Rectangle:
$Q = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$
Volume:
$V(Q) = (b_1 - a_1) \ldots (b_n - a_n)$

Definition Given $[a, b] \in \mathbb{R}$, a partition P of $[a, b]$is t_i such that;
$a = t_0 < t_1 < \ldots < t_k = b$

Definition A partition P of Q is an n-tuple $(P_1, \ldots, P_n)$ such that P is a partition of $[a_i, b_i] \forall i$.

Definition $f : Q \to \mathbb{R}$ and bounded. Let P be a partition of Q. For each rectangle $R \subset P$ we define;

1. $m_R = inf\{f(x)|x \in R\}$
   $L(f,P) = \sum_(R \subset P) m_R(f)V(R)$

2. $M_R = sup\{f(x)|x \in \mathbb{R}\}$
   $U(f,P) = \sum_(R \subset P) M_R(f)V(R)$

8 Lecture 8

We will discuss integration on $\mathbb{R}^n$ over a rectangle Q by taking a partition of the rectangle. Lecture begins with a recap from lecture 8, defining upper and lower sums.

The lower integral is given by $\int_- f = sup_P\{L(f,P)\}$ *ldots* the maximum value of the lower sum.

The upper integral: $\int^- f = inf\{U(f,P)\}$.

Definition f is integrable over Q if$\int_{-Q} f = \int^{-Q} = \int_Q f$

Example of non-integrable function:

$$g(x) = \{1, if x \in Q \tag{6}$$
$$0 if x \notin Q. \tag{7}$$

Let $Q = I_1 \times \ldots \times I_n$ with $I_j \in \mathbb{R}$ and closed.
If $f : Q \to \mathbb{R}$ is continuous, then

$$\int_Q f = \int +x_1 \in I_1 \ldots \int_{x_n \in I_n} f(x_1, \ldots x_n) \tag{8}$$

Definition Let A be a subset of $\mathbb{R}^n$ we say A has a *measure zero* in $\mathbb{R}^n$ if for every $\epsilon > 0$, there is a covering of A by countably many rectangles Q_i such that

$$\sum_i V(Q_i) < \epsilon \tag{9}$$

Theorem 8.1 *Let $Q \subset \mathbb{R}^n$bearectangle, and $f : Q \to \mathbb{R}$. Assume f is integrable over Q. If f vanishes except on a set of measure 0, then $\int_Q f = 0$.*

Proof Let P be a partition of Q, and $R \subset P$. Assume $f = 0$, except on the set E of measure 0.
$\Rightarrow R \nsubseteq E \Rightarrow f = 0$ at somepoint of R. Then $m_R(f) \leq 0, M_R(f) \geq 0$.
$\Rightarrow L(f,P) \leq 0, U(f,P) \geq 0$.
Thus $\int_- f \leq 0, \int^- \Rightarrow \int_Q f = 0$

Definition Let S be a bounded set in $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ be a bounded function. Define $f_s : \mathbb{R}^n \to \mathbb{R}$ by;

$$f_s(x) = \{f(x) for x \in S \tag{10}$$

$$0 otherwise. \tag{11}$$

Choose a rectangle Q such that $S \subset Q$. Then the integral of f over S is defined by $\int_S f = \int_Q f_s$ if the latter exists.

This is independent of the choice of the rectangle Q. Refer to diagram in notesbook.

Theorem 8.2 *Let S be a bounded set in $\mathbb{R}^n$;*
let $f : S \to \mathbb{R}$ be a bounded & continuous function. Let $A = IntS$. If f is integrable over S, then f is integrable over A and:

$$\int_S f = \int_A f \tag{12}$$

Proof Not proved in class

Definition Let S be a bounded set in $\mathbb{R}^n$. If the constant 1 is integrable over S, we say that S is rectifiable ad then the n-dim volume is defined as $V(S) = \int_S 1$

Let $A \subset \mathbb{R}$ open. Then there exists a sequence $C_1, C_2, \ldots$ of compact rectifiable subsets of A whose union is A, such that $C_N \subsetneq C_{N+1}$ for each N.

9 Lecture 9

Rectifiable Set A bounded set S such that;

$$\int_S 1 \text{ exists} \tag{13}$$

Recall: $f(x) = 1$ if, $x \in Q, 0$ otherwise.

Theorem 9.1 *Let A be open in $\mathbb{R}^n$; let $f : A \to \mathbb{R} \in C^0$. Choose a sequence C_N of compact rectifiable subsets of A, whose union is A such that;*

$$C_N \nsubseteq C_{N+1} \forall N \tag{14}$$

Then f is integrable over $A \Leftrightarrow$ the sequence $\int_{C_N} f$ is bounded.
In that case;

$$\int_A f = \lim_{N \to \infty} \int_{C_N} f \tag{15}$$

Example

$$\int_A f(x,y) = \int_A e^{-x^2-y^2} \tag{16}$$

$$= lim_{N\to\infty} \int f(x,y) \tag{17}$$

Recall from 1 dimensional change of variables;

$$\int_{g(b)}^{g(a)} f = \int_a^b (f \circ g)g' \tag{18}$$

Change of Variable: $x = g(y)$

$$f \in C^0, g \in C^{-1} \tag{19}$$

Definition Let $A \subset \mathbb{R}^n$ open. Let $g : A \to \mathbb{R}^n$ be a 1 to 1 function of class C^r such that $Det(Dg) \neq 0$ for $x \in A$. Then g is a change of variables in $\mathbb{R}^n$

Definition Let A and B be open in $\mathbb{R}^n$. If $g : A \to B$ is 1 to 1, such that both $g \in C^r(A)$ and $g^{-1} \in C^r(B)$, then g is a *diffeomorphism.*

Theorem 9.2 (Change of Variables Theorem) *Let $g : A \to B$ be a diffeomorphism. Let $f : B \to \mathbb{R}$ be continuous. Then f is integrable over B iff, $(f \circ g)|Det(Dg)|$ is integrable over A, and in this case;*

$$\int_B f = \int_A (f \circ g)|Det(Dg)| \tag{20}$$

Proof Not proved in class

Reffering to the previous example (In copy)

Example $\int_B f(x,y)$ with $f(x,y) = e^{-x^2-y^2}$ and $g(r,\phi) = (r\cos\phi, r\sin\phi)$. We have the set $A = \{r \in [0,\infty], \phi \in [0,\pi 2]\}$

Example finished in book

Theorem 9.3 *Let A be an n by n matrix. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ defined by $h(x) = A\dot{x}$. Let S be a rectifiable set in$\mathbb{R}^n$ and $T = h(s)$ Then $V(T) = |Det(A)|V(S)$*

Proof We first assume that $Det(A) \neq 0, \to$ h is diffeomoprhism of $\mathbb{R}^n$. $h : S \to T$ and T is rectifiable.

$$V(T) = \int_T 1 = \int |Det(Dh)| \text{ by change of Var theorem} \tag{21}$$

$$= \int |Det(A)| = |Det(A)|V(S). \tag{22}$$

Let $Det(A) = 0$. Then h carris s to a linear subspace V of $\mathbb{R}^n$ with $Dim(V) \to$ The n-dim volme of T is 0.

10 Lecture 10

$$\int_B f = \int_A (f \circ g)|Det[D_g]| \tag{23}$$

Let $h : \mathbb{R}^n \to \mathbb{R}^n$, $h(x) = Ax$ A is an $(nx \times n)$-matrix. Let S be a rectifiable set in $\mathbb{R}^n$, and $T = h(s)$;

$$V(T) = \int_{\_} 1 = |Det[A]|V(S) \tag{24}$$

Definition A matrix A is orthogonal if;

$$A^T \dot{A} = I_n \tag{25}$$

Note that $Det[A] = \pm 1, \text{ since } Det[A^T] = Det[A]$

Let A be an orthogonal matrices represent rotations and reflections (refer to notes for diagram). Recall the Euclidean metric in $\mathbb{R}^n$;

$$||x - y||^2 = \sum_{i=1}^{n} (x_i - y_i)^2 =< x - y, x - y > \tag{26}$$

Where $< x, y >= \sum_{i=1}^{n} x_i y_i$

Theorem 10.1 *Let $h(x) = A\dot{x}$ with A orthogonal, then $||x-y|| = ||h(x)-h(y)||$*

Proof $< h(x), h(x) >= (Ax)^T Ax = x^T A^T Ax = x^T x =< x, x > \Rightarrow h(x)$ preserves distance such maps are *isometies*. Most general isometry in $\mathbb{R}^n$ is $h(x) = Ax + p$

A change of basis of $\mathbb{R}^n$ is a map $g : \mathbb{R}^n \to \mathbb{R}^n$ which is injective, linear, and both g and its inverse are continuous. Let $\vec{x}$ be a directed line segment. Change of basis leaves $\vec{x}$ invariant.

Let a_i be the old basis and b_i the new basis, then $(b_1, \ldots, b_n) = (a_1, \ldots, a_n)C$ Where C is an n by n matrix.

$$\vec{x} = \sum_{i=1}^{n} \tag{27}$$

$$= \sum_{i=1}^{n} y_i b_i \text{ for some } y_i \tag{28}$$

$$\tag{29}$$

Proposition 10.2 $y = C^{-1}x$

Proof $b\dot{y} = zCC^{-1}x = a\dot{x} = \vec{x}$

- a transforms *covariantly* under change of basis.

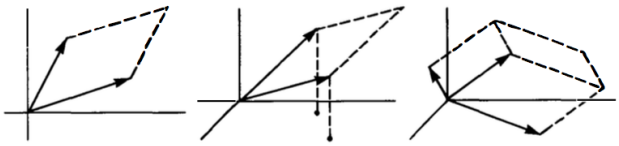
- x transforms *contravariantly* under a change of basis.

Definition Let $a_1, \ldots, a_k$ be independent vectors of $\mathbb{R}^n$. The *k-dimensional parallelopiped*

$$\rho = \rho(a_1, \ldots a_k) \text{ defined to be;} \tag{30}$$

$$\rho = \{x \in \mathbb{R}^n | x = \sum_{i=1}^{n} c_i a_i, 0 \leq c_i \leq 1\} \tag{31}$$

$$\tag{32}$$



Theorem 10.3 *Let $a_1, \ldots, a_n$ be n independent vectors in $\mathbb{R}^n$. Let $A = [a_1, \ldots a_n]$, the n by n matrix with columns a_i, Then;*

$$V(\rho(\{a_i\})) = |Det[A]| \tag{33}$$

Proof Consider $h : \mathbb{R}^n \to \mathbb{R}^n$ defined by $h(x) = Ax$, then h maps the unit basis vector e_i to a_i. Therefore the unit cube $[0,1]^n$ is mapped to $\rho(\{a_i\})$

$$V(\rho(\{a_i\})) = |Det[A]| V([0,1]^n) \tag{34}$$

$$= |Det[A]| \tag{35}$$

$$\tag{36}$$

B-dim parallelopiped with edges a, b, c has volume;

$$a(b \times C) = \begin{pmatrix} a_1 & a_1 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

See diagram for illustration of right and left handed co ordinate system.

11 Lecture 11

We begin this lecture by recapping the left and right handed triple, there determinants and respective diagrams.

Multiplying a times by gives us c. We will now generalise this notion of left and right handedness.

Definition An n-tuple $(a_1, \ldots, a_n)$ of independent vectors is called an n-frame in $\mathbb{R}^n$. We call the frame right-handed if;

$$Det[a_1, \ldots, a_n] > 0 \tag{37}$$

left-handed otherwise.

The collection of all right handed frames is called an orientation of $\mathbb{R}^n$ and similarly the collection of all left handed frames.

Theorem 11.1 *Let C be a non singular $n \times n$ matrix. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ defined by $h(x) = C\dot{x}$ if $Det[C] > 0$, the frames $\{a_i\}$ and $\{h(a_i)\}$ belong to the same orientations. If $Det[C] < 0$, they belong to opposite orientations.*

Proof Let $b_i = h(a_i) \forall i$, Then;

$$C\dot{(}a_1, \ldots a_n) = (b_1, \ldots, b_n) \tag{38}$$

$$\rightarrow Det[a_1, \ldots, a_n] = Det[b_1, \ldots, b_n] \tag{39}$$

Let $\rho(\{a_i\})$ be an n dim parallelopiped in $\mathbb{R}^n$, then;

$$V(\rho(\{a_i\})) = |Det[a_1, \ldots, a_n]| \tag{40}$$

Definition Consider the k dimensional paralleopiped in $\mathbb{R}^n$, given by $\rho(a_1, \ldots, a_k)$ its k dimensional volume is

$$V(\rho) = \sqrt{Det(A^T A)}, \tag{41}$$

Where A is the n by k matrix $[a_1, \ldots, a_k]$

Proposition 11.2 *If $a_i \in \mathbb{R}^k \times 0 \in \mathbb{R}^n$, are of the form $(z_i, 0)^T$, with $z_i \in \mathbb{R}^k$. Then $V(a_1, \ldots a_k) = |Det[z_1, \ldots, z_k]|$*

Proof $A = (z, 0)^T$ with $z = (z_1, \ldots, z_k)$ $Det[A^T A] = Det[Z^T Z] = Det[Z]^2$ $V(a_1, \ldots a_k) = |Det[Z]|$

Proposition 11.3 *Let h be an orthogonal transformation, then $V(h(a_1), \ldots, h(a_k)) = V(a_1, \ldots, a_k)$*

Proof Let $h(x) = Bx$, with B orthogonal

$$\begin{aligned} V(h(a_1), \ldots, h(a_k)) =& (Det[(BA)^T BA])^{1/2} && (42)\\ =& (Det[A^T B^T BA])^{1/2} && (43)\\ =& (Det[A^T A])^{1/2} && (44)\\ =& V(a_1, \ldots, a_k) && (45)\\ & && (46) \end{aligned}$$

Definition Let $k \leq n$. Let $A \subset \mathbb{R}^k$ open, and $\alpha : A \to \mathbb{R}^n \in C^r$. Then $Y = \alpha(A)$ together with the map α is a parametrised manifold. Its volume is given by;

$$V(Y_\alpha) = \int_A V(D\alpha) \tag{47}$$

We will now consider the motivation for the volume of a parametrised manifold. Let A be the interior of a rectangle $Q \in \mathbb{R}^k$. Suppose $\alpha : A \to \mathbb{R}^n$ can be extended to a function of class C^r in a neighbourhood of Q. We partition the rectangle Q into smaller rectangles and we can take a specific rectangle R

The edges of $\alpha(R)$ are given by $\alpha(a_i + h_i e_i)$ are given by;

$$\alpha(a_1 + h_1 e_1, a2) - \alpha(a_1, a_2) \tag{48}$$

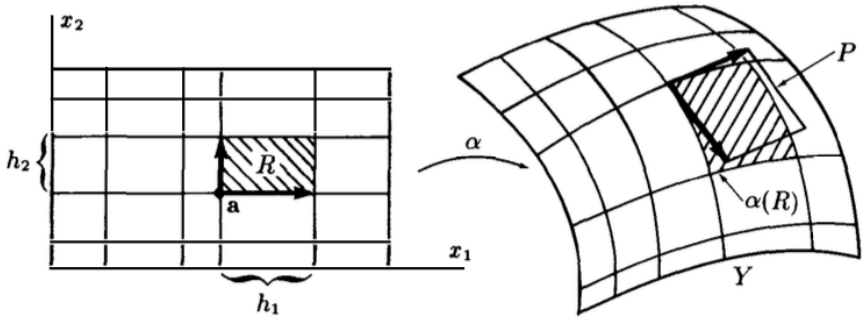
is well approximated for small h_i by;

$$\frac{\partial \alpha(a_1, a_2)}{\partial a_1} h_i = D\alpha(a) h_1 e_1 \tag{49}$$

$$V(v_1, \ldots, v_k) = V(\frac{\delta \alpha}{\delta x_1}, ldots) \times h_1, \ldots h_k \tag{50}$$

$$= V(D\alpha)V(R) \Rightarrow \int dV = \int V(D\alpha) \tag{51}$$

$$\tag{52}$$



12 Lecture 12

Fist homework is now available with deadline of the 22nd February. Recap;

$$\begin{aligned} Vol(Y_\alpha) &= \int_A V(D\alpha) \\ &= \int_{y_\alpha} dV \\ \alpha : A \to \mathbb{R}^n & \\ D\alpha \text{ is a n by k matrix} & \\ V(D\alpha) =& (Det[A^T A])^{1/2} \end{aligned}$$

example of semi circle in copybook

Definition Let $A \subset \mathbb{R}^k$ open, $\alpha : A \to \mathbb{R}^k$ and $Y = \alpha(A)$. Let $f : Y \to \mathbb{R}$. The integral of f over Y_α *with respects to the volume* is define by;

$$\int_Y f dV = \int_A (f \circ \alpha) V(D\alpha) \tag{53}$$

This can be covered by different parametrised manifolds. You should get the same length of the semi circle. The following theorem makes this statement more precise.

Theorem 12.1 *Let $g : A \to B$ be a diffeomorphism[3] of open sets in $\mathbb{R}^k$. Let $\beta : B \to \mathbb{R}^n \subset C^r(B)$, and $Y = \beta(B)$. Let $\alpha = \beta \circ g : A \to \mathbb{R}^n$, such that $Y = \alpha(A)$.*

Assume $f : Y \to \mathbb{R}$ is continuous. Then f is integrable over $Y_\beta \Leftrightarrow f$ is integrable over Y_α and $\int_{Y_\alpha} f dV = \int_{Y_\beta} f dV$

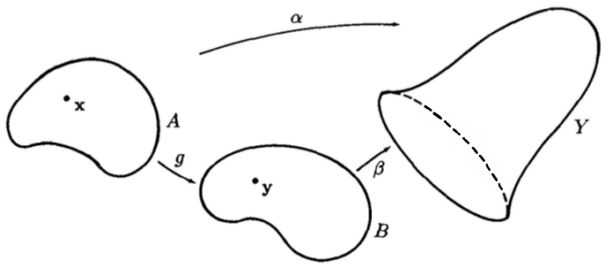


Figure 1:

[3] An invertible function mapping one differentiable manifold to another

Proof We need to show the following;

$$\int_B (f \circ \beta) V(D\beta) = \int_A (f \circ \alpha) V(D\alpha) \tag{54}$$

Using the change of variable theorem one finds;

$$\int_B (f \circ \beta) V(D\beta) = \int_A (f \circ \beta \circ g)(V(D\beta) \circ g) |Det[Dg]| \tag{55}$$

If $y = g(x)$, $(V(D\beta) \circ g)(x) = V(D\beta(y))$. Comparison shows, that we need to prove;

$$(V(D\beta) \circ g)|Det[Dg]| = V(D\alpha) \tag{56}$$

We know that $\alpha = \beta \circ g$, Chain rule gives;

$$D\alpha(x) = D\beta(y) Dg(x) \tag{57}$$

We have;

$$\begin{aligned} V(D\alpha(x))^2 =& Det[D\alpha(x)^T D\alpha(x)] & (58)\\ =& Det[(Dg(x)^T D\beta(y)^T D\beta(y) Dg(x)] & (59)\\ =& Det[Dg]^2 Det[D\beta^T D\beta] & (60) \end{aligned}$$

This shows that it does not matter which co ordinate patch you chose to compute the integrals.

Definition If $\phi : \mathbb{R}^n \to \mathbb{R}$, then the *support* of ϕ is the closure of the set;

$$\{x | \phi(x) \neq 0\} \tag{61}$$

In otherwords, if $x \notin Support(\phi)$, then there is some neighbourhood of x, U_x in which ϕ vanishes identically. Or, it is the smallest closed set containing all points x not mapped to zero.

Definition Let $M \subset \mathbb{R}^n$, be a compact k-manifold. Let $f : M \to \mathbb{R}$ be continuous. Let $S = Support(f)$. Suppose there is a coordinate patch $\alpha : U \to V$ on M such that $S \subset V$. Therefore we can assume that U is also bounded. Then *the integral of f over M* is;

$$\int_M f dV = \int_{interior(U)} (f \circ \alpha) V(D\alpha) \tag{62}$$

Recall that the interior of a set M is $M - \delta M$. Then the interior of U can be expressed as;

$$U - \delta U = \left\{ \begin{array}{cc} U & \text{if U is open } \mathbb{R}^k \\ U \cap \mathbb{H}^k & \text{if U is open in } \mathbb{H}^k \text{ but not in } \mathbb{R}^k \end{array} \right\}$$

13 Lecture 13

Illustrating what we have done this far with diagrams: see copy book.

Lemma 13.1 *If the support of f can be covered by a single co-ordinate patch, the integral* $\int_M f dV$ *is independent of the patch chosen.*

Proof We begin by using the properties of the transition functions mapping $U_1 \to U_0$. We need to show;

$$\int_{int(U_0)} (f \circ \alpha_0) V(D\alpha_0) = \int_{int(U_1)} (f \circ \alpha_1) V(D\alpha_1). \tag{63}$$

Since U_0 and V_0 contain $Supp(f)$, $W = U_0 \cap U_1$, contains $Supp(f)$. Let $W_i = \alpha_i^{-1}(W) \Rightarrow$ We can replace U_i by W_i. Since the volume is invariant under reparameterisation, the result follows immediately. Consider the transition function $\alpha_1^{-1} \circ \alpha_0^{-1} : W_0 \to W_1 \in C^r$, and non-singular derivative $Rightarrow$ $\alpha_1^{-1} \circ \alpha_0^{-1}$ is a diffeomorphism. Our previous theorem gives the equality of these integrals.

13.1 Partitions of Unity

Let $\{A_i\}$ be a collection of open sets in $\mathbb{R}^n$. Let A be there union. A sequence $\phi_1, \phi_2, \ldots,$ of continuous functions $\phi_i : \mathbb{R}^n \to \mathbb{R}$ satisfies the following;

1. $\phi_i(x) \geq 0$, $\forall x$
2. the $S_i = Support(\phi_i)|$ is contained in A
3. There exists a neighbourhood $U(x, \epsilon)$. for all $x \in A$, that intersects only finitely many S_i
4. $\sum_{i=1}^{\infty} \phi_i(x) = 1$ for each $x \in A$

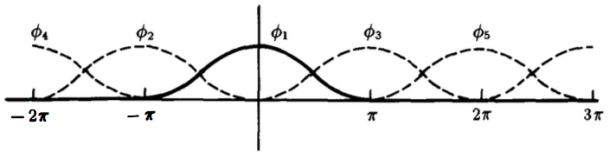


Figure 2:

Such a sequence $\{\phi_i\}$ is a *partition of unity* for A. It is dominated by $\{A_i\}$ if for all j, S_j is contained in an element of $\{A_i\}$.

We see an example of a partition of unity by translating $f(x) = (1 + Cos(x))/2$ where $x \in (-\pi, \pi)$. 0 otherwise, over the whole real line. This is of class C^1. We will now see a partition of unity of class C^∞

Theorem 13.2 *Let $A \subset \mathbb{R}^n$, open. Let $f : A \to \mathbb{R}$, be continuous. Let $\{\phi_i\}$ be a partition of unity on A having compact supports. Then $\int_A f$ exists if;*

$$\sum_{i=1}^{\infty} [\int_A \phi_i |f|] \text{ converges} \tag{64}$$

and in this case

$$\int_A f = \sum_{i=1}^{\infty} \phi_i f \tag{65}$$

Lemma 13.3 *Let M be a compact k-manifold $\in \mathbb{R}^n$ given a covering of M, by patches $\alpha_i : U_i \to V_i$, there exists a finite collection of C^∞-functions $\phi_1, \ldots, \phi_l : \mathbb{R}^n \to \mathbb{R}$. Such that:*

1. *$\phi_i(x) \geq 0$ for all x*
2. *Given i, $Supp(\phi_i)$ is compact and there is a patch $\alpha_j : U_j \to V_j$ such that $(Supp(\phi_i)) \cap M \subset V_i$*
3. *$\sum_{i=1}^{l} \phi_i(x) = 1$ for $x \in M$*

14 Lecture 14

We began to discuss the partition of unity over integrals in the last lecture. We will continue this discussion today.

14.1 Partition of Unity over Manifolds

Definition Let $f : M \to \mathbb{R}$ be continuous choose a partition of unity $\phi_1, \ldots, \phi_l$ on M dominated by a covering of M by patches. Then define

$$\int_M f dV = \sum_{i=1}^{l} \int_M \phi_i f dV \tag{66}$$

Proposition 14.1 *If the Supp(f) lies in a single patch $\alpha : U \to V$, then the definition agrees which;*

$$\int_M f dV = \int_A (f \circ \alpha) V(D\alpha), \text{ with } A = int(U) \tag{67}$$

Proof

$$\sum_{i=1}^{l} \int_M (\phi, f) dV = \sum_{i=1}^{l} \int_A (\phi \circ \alpha)(f \circ \alpha) V(D\alpha) \tag{68}$$

$$= \int_A \sum_{i=1}^{l} (\phi_i \circ \alpha)(f \circ \alpha) V(D\alpha) \tag{69}$$

$$= \int_A (f \circ \alpha) V(D\alpha) \tag{70}$$

$$\tag{71}$$

Proposition 14.2 *Definition is independent of choice of partition of unity*

Proof We begin by choosing another partition of unity and we show that the two integrals are equal to one another. Let $\psi_1, \ldots, \psi_m$, be another partition of unity. By using the previous definition we have;

$$\int_M (\psi_j, f) dV = \sum_{i=1}^{l} \int (\phi_i \psi_j f) dV \tag{72}$$

Summing over j;

$$\sum_{j=1}^{m} \sum_{i=1}^{l} \int (\phi_i \psi_j f) dV \tag{73}$$

This sum is symmetric so by taking each sum over i and j inside the integral we get;

$$\sum_{j=1}^{m} \int_M (\psi_j f) dV = \sum_{i=1}^{l} \int_M (\phi_i f) dV = \int_M f dV \tag{74}$$

It can be quite difficult to integrate certain partitions of unity so we have the following theorem;

Theorem 14.3 *Let $f : M \to \mathbb{R}$, suppose that $\alpha : A_i \to M_i$ for $i = 1, \dots N$, such that A_i is open in $\mathbb{R}^k$ and M is the disjoint union of the open sets M_i and a set K of measure zero in M. Then*

$$\int_M f dV = \sum_{i=1}^{N} \int (f \circ \alpha_i) V(D\alpha_i) \tag{75}$$

This concludes the topic of integration for the next few weeks and we now develop the technology of differential forms.

14.2 Differential Forms

We return to the topic of linear algebra and slightly generalise this before we begin to define differential forms.

Let V be a vector space. A function f is *linear* if

1. $f(v_1 + v_2) = f(v_1) + f(v_2)$, for $v_1, v_2 \in V$
2. $f(\alpha v) = alpha f(v)$, for $v \in V$ and $\alpha \in \mathbb{R}$

Definition Let V be a vector space and $V^k = V \times \dots \times V$ denote the set of all k-tuples $(v_1, \dots, v_k)$, with $v_j \in V$. Then a function;

$$f : V^k \to \mathbb{R} \text{ is } \textit{mulitilinear} \tag{76}$$

If $f(v_1, \dots v_k)$ is linear for each $i = 1, \dots, k$

It is also known as a *k-tensor*. L^k is the space of all k-tensors. $L^1(k)$ is the set of all linear transformations $f : V \to \mathbb{R}$ defined as $f(x) = \mathbf{y}^T \mathbf{x}$ $L^1(V)$ is also denoted V^* and called the *dual space* of V. $L^k(v) = V^* \otimes \dots \otimes V^*$

What is a tensor product? One example of a 2-tensor: $f(x_1, x_2) \to \mathbb{R}$ defined as $\sum_{i,j} d_{i,j} x_{1,i} x2, j$

Another example is $<,>: V \times V \to \mathbb{R} \in L^2(V)$ And another $Det(a_1, \dots a_n) \to V^n \to \mathbb{R} \in L^k(V)$

15 Lecture 15

$f \in L^k(V)$. Setting $k = 1$, $f(v) = \sum_{i=1}^{n} d_i v_i$. With $L^1(V) = V^\star$, the $dim(V) = dim(V^\star)$. Then $L^1(V)$ is just the set of all linear transformations $f : V \to \mathbb{R}$. Often times it is denoted by called the dual space of V and denoted $V^\star$.

Theorem 15.1 *Addition and multiplication of tensors:*

- $(f + g)(v_1, \dots, v_k) = f(v_1, \dots, v_k) + g(v_1, \dots, v_k)$

- $(cf)(v_1, \ldots, v_k) = c(f(v_1, \ldots, v_k))$

This implies that $L^k(V)$ is a vector space.

Multilinear transformations are entireley determined once we know its values on basis elements. This statement is proved with the following.

Lemma 15.2 *Let $a_1, \ldots, a_n$ be a basis for V. If $f, g : V^k \to$ are k-tensors on V and if*

$$f(a_{i_1}, \ldots, a_{i_k}) = g(a_{i_1}, \ldots, a_{i_k}) \tag{77}$$

for ever k-tuple $I = (i_1, \ldots, i_k)$ of integers from the set $\{1, \ldots, n\}$, then $f = g$. Note that $i_1, \ldots, i_k$ be distinct or arranged in any particular order.

Proof Given an arbitrary k-tuple of vectors $(v_1, \ldots, v_k)$ of V. We express each v_i in terms of the given basis;

$$v_i = \sum_j = 1^k a_j c_{ij}. \tag{78}$$

Computing f

$$\begin{aligned} f(v_1, \ldots, v_k) &= \sum_{j_1=1}^{n} c_{1j_1} f(a_{j_1}, v_2, \ldots, v_k) \\ &= \sum_{j_2=1}^{n} \sum_{j_1=1}^{n} c_{1j_1} c_{2j_2} f(a_{j_1}, a_{j_2}, v_2, \ldots, v_k) \\ &\vdots \\ f(v_1, \ldots, v_k) &= \sum_{1 \le j_1, \ldots, j_k \le n} c_{1j_1} c_{2j_2} \ldots c_{kj_k} f(a_{j_1}, \ldots, a_{j_k}) \end{aligned}$$

The same computation for g holds. and it follows that f and g agree on all k-tuples of vectors if they hold on all k-tuples of basis elements.

Just as we can define linear transformations V and W by arbitrarily defining their value for each basis element, a k-tensor on V can be defined by assigning its value for each k-tuple of basis elements.

This is proved in the following theorem where we claim that every k-tensor is made up of elementary k-tensors and these elementary k-tensors form a basis of $L^k(V)$.

Theorem 15.3 *Let V be a vector space with basis $a_1, \ldots, a_k$. Let $I = (i_1, \ldots, i_k)$ a k-tuple of integers from the set $\{1, \ldots, n\}$. There is a unique k-tensor ϕ_I on V. such that, for every k-tuple $J = \{j_1, \ldots, j_k\}$*

$$\phi_I(a_{j_1}, \ldots, a_{j_k}) = \left\{ \begin{array}{ll} 0 & \text{if } I \neq J \\ 1 & \text{if } J = I \end{array} \right\} \tag{79}$$

The tensors ϕ_I form a basis for $L^k(V)$ and are called Elementary tensors corresponding to the given basis for V.

We will show that these k-tensors form a basis of $L^k(V)$

Proof We will show that $f \in L^k(V)$ can be written uniquely as a linear combination of ϕ_I. Let $d_I = f(a_{i1}, \ldots, a_{i_k})$ with $I = (i_1, \ldots, i_k)$. Introduce $g = \sum_I f_I \phi_I$. By the definition of ϕ_I, $g(a_{i1}, \ldots, a_{ik}) = d_I$. By the previous lemma we can show that $f = g$.

There exist n^k different k-tuples and hence the dimension of $L^k(V) = n^k$.

15.1 K-Tensor product

Let V and W be vector spaces with basis $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$ respectively. Then $V \otimes W$ is a vector space with basis (a_i, b_j) for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. We can denots (a_1, b_j) by $a_i \otimes b_j$. Note the distinction here with cross product;

$$dim(V \otimes W) = dim(V) dim(W) \tag{80}$$
$$dim(V \times W) = dim(V) + dim(W) \tag{81}$$

Recall that $L^k(V) = V^\star \otimes V^\star$ has a dimension n^k.

16 Lecture 16

$f \in L^k(V)$ $f : V \times \ldots \times V \to \mathbb{R}$ k-tuple of Vs. $f(V_1, \ldots V_k)$ Basis elements of$L^k(v)$

Then $f = \sum d_I \phi_I$ example: $< x, y >= \sum_{i=1}^n x_i y_i = \sum_{i,j=1}^n \delta_{i,j} x_i y_i = \sum_{i=1}^n \phi_{i,i}(x, y)$

$$\begin{aligned} Det[v_1, v_2] =& v_{1,1} v_{2,2} - v_{1,2} v_{2,1} && (82)\\ =& \phi_{1,2}(v_1, v_2) - \phi_{2,1}(v_1, v_2) && (83)\\ & && (84) \end{aligned}$$

16.1 Tensor Product

Definition Let $f \in L^k(V)$ and $g \in L^l(V)$. The $(k + l)$ tensor $(f \otimes g)$ on V is defined by $(f \otimes g)(V_1, \ldots, V_{k+l}) = f(V_1, \ldots, V_k) g(V_{k+1}, \ldots, V_{k+l}) \in h^{k+l}(Vv)$ is the *tensor product* of f and g.

Definition Let $T : V \to W$ be a linear transformation. The *dual transformation* $T^\star : h^k(W) \to h^k(v)$. Let $f \in L^k(W)$ and $v_1, \ldots, v_k \in V$ then

$$(T^\star f)(v_1, \ldots, v_k) = f(T(V_1), T(V_2), \ldots, T(Vvk)) \tag{85}$$

$T^\star$ is the composite T^k and f

Theorem 16.1 *1. $T^\star$ is linear*

2. $T^\star(f \otimes g) = T^\star f \otimes T^\star g$

3. If $S : W \to X$ is a linear transformation, then $(S \circ T)^\star f = T^\star(S^\star f)$

Proof of 2

$$T^\star(f \otimes g) = f \otimes g(T(v_1), \ldots, T(v_{k+l})) \tag{86}$$
$$= f(T(V_1), \ldots, T(v_k))\dot{g}(T(V_{k+1}, \ldots, T(V_{k+l})) \tag{87}$$
$$= T^\star f \otimes T^\star g \tag{88}$$
$$\tag{89}$$

16.2 Change of basis of V

Let $\hat{e}_1, \ldots, \hat{e}_n$ be a basis for V and let $\hat{d}_1, \ldots, \hat{d}_n$ be another basis related to $\hat{e}_i$ by $(\hat{d}_1, \ldots, \hat{d}_n) = (\hat{e}_1, \ldots, \hat{e}_n)C$ where C is a $n \times n$ non-singular matrix. $\hat{d}_i = \sum_{j=1}^n C_{i,j}\hat{e}_i$ Covariant

The directed line segment $\bar{V} = \sum_{i=1}^n v_i\hat{e}_i$ is invariant under change of basis. Let $\tilde{V}$ be the co ordinates of $\vec{V}$ with respects to the basis $\hat{d}_i$, then $\tilde{V} = (V_1, \ldots, V_n) = C^{-1}v = C^{-1}(v_1, \ldots, v_n) \to \sum_{i=1}^n \tilde{v_i}\hat{d}_i = (\hat{e}_1, \ldots, \hat{e}_n)CC^{-1}v = \vec{v}$

Similarly change of basis does not change a physical tensor

$$(E_{rot} = 1/2\vec{\omega}\dot{(}\vec{r} \times \vec{p})) \tag{90}$$

$\tilde{v} = C^{-1}v \to \tilde{f}(\tilde{v_1}, \ldots, \tilde{v_k}) = f(C\tilde{v_1}, \ldots, C\tilde{v_k}) \to f$ is a covariant tensor.

$f \in h^k(v) = h* \otimes \ldots \otimes h*$ in terms of indices $\tilde{f}(\tilde{V_1}, \ldots, \tilde{v_k}) = \sum_{\{i_r\}\{j_s\}} d_{i1} \ldots$

17 Lecture 17

17.1 Change of Basis

- $\hat{d} = \hat{e}C$
- $\vec{V} = \sum v_i\hat{e}_i = \sum \tilde{v}_i\hat{d}_i$
- $\tilde{V} = C^{-1}v$

A *physical* tensor f is a tensor f such that;

$$f\hat{e}(v_1, \ldots, v_k) = f_{\hat{d}}(\tilde{v}_1, \ldots, \tilde{v_k}) \tag{91}$$
$$\Rightarrow f\hat{d}(\tilde{v}_1, \ldots, \tilde{v_k}) = f_{\hat{e}}(C\tilde{v}_1, \ldots, C\tilde{v_k}) \tag{92}$$

Example

$$< v_1, v_2 >_{\hat{e}} = v_1^T I v_2 \tag{93}$$
$$= \sum_{i=1}^n v_{1i}v_{2i} \tag{94}$$
$$< \tilde{v}_1, \tilde{v}_2 >_{\hat{d}} = \tilde{v}_1^T I \tilde{v} \tag{95}$$

We now verify the second property.

$$< C\tilde{v}_1, \tilde{v}_2 >_{\hat{e}} = \tilde{V}_1 C^T C \tilde{V}_2 = < \tilde{V}_1, \tilde{V}_2 >_{\hat{d}} \tag{96}$$

For orthogonal C. Therefore the inner product is a tensor in the physical sense.

Example

$$D(V_1, V_2)_{\hat{e}} = Det(V_1, V_2) \forall i \text{ with } dim(V) = 2 \tag{97}$$

$$D(\tilde{V}_1, \tilde{V}_2)_{\hat{d}} = Det(\tilde{V}_1, \tilde{V}_2) \tag{98}$$

$$Det(\tilde{V}_1, \tilde{V}_2)_{\hat{e}} = Det(C) Det(\tilde{V}_1, \tilde{V}_2) \tag{99}$$

$$\Rightarrow \text{D is a } \textit{pseudo-tensor} \tag{100}$$

Definition Let $k \geq 2$ A permutations of the set of integers $\{1, \ldots, k\}$ is a one to one function mapping this set to itself. Recall permutations from linear algebra 1st year.

If σ and τ are permutations, then $\sigma \circ \tau$ is also a permutation. Also, σ^{-1} is a permutation. The set of all permutations forms the group S_k, with $k!$ elements.

Definition Given $1 \leq i \leq k$, defined as

$$e_i(j) = \left\{ \begin{array}{cc} j, & for j \neq i, i+1 \\ i+1, & for j = 1 \\ i, & for j = i+1 \end{array} \right\} \tag{101}$$

e_i is an *elementary permutation.* $e_i\{1, \ldots, k\} = 1, \ldots, i-1, i+1, i, i+2, \ldots, k$
$e_i \circ e_i = 1$

Lemma 17.1 *Each $\sigma \in S_k$ equals a composite of elementary permutations.*

Proof We will prove this by induction. For $0 \leq i \leq k$, we say σ fixes the first i integers if $\sigma(j) = j$ for $1 \leq j \leq i$.

If $i = k$, σ fixes all integers. $\Rightarrow \sigma = 1$ and lemma holds for $i = k$. Assume that σ, which fixes the first i integers, can be written as composite of e_i's.

We need to first show that any σ which fixes the first $i-1$ integers can be written as a composite. Let

$\sigma(l) = i$ for $l \geq i$. We can write

$$\sigma' = \left\{ \begin{array}{cc} \sigma & if l = i \\ e_1 \circ e_{i+1} \circ \ldots \circ e_{l-1}, if & l > i \end{array} \right\} \tag{102}$$

$\Rightarrow \sigma = \sigma_{l-1} \circ \ldots \circ e_i \sigma' \rightarrow$ Lemma holds by the induction hypotheses.

Definition Let $\sigma \in S_k$ consider the set of pairs i, j, from $\{1, \ldots, k\}$ for which $\sigma(i) > \sigma(j)$. Each pair is an inversion.

If the total number of inversions is off, the *sign* of σ is negative, otherwise is is positive.

18 Lecture 18

Second homework is up next week with a deadline in week 8. Notes on change of basis are available on Blackboard and may be required for homework 2

Permutation -

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & k \\ \sigma(1) & \sigma(2) & \dots & \sigma(k) \end{pmatrix} \tag{103}$$

Definition Elementary permutation is where there is only one swap of numbers.

$$e_i = \begin{pmatrix} 1 & \dots & i & i+1 & \dots & k \\ 1 & \dots & i+1 & i & \dots & k \end{pmatrix} \tag{104}$$

Inversion pair (i, j) such that $i < j$ but $\sigma(i) > \sigma(j)$.

Lemma 18.1

- *If σ equals a composite of m elementary transformations, then $sgn(\sigma) = (-1)^m$*
- $sgn(\sigma \circ \tau) = sgn(\sigma)sgn(\tau)$
- $sgn(\sigma^{-1} = sgn(\sigma)$
- *If $p \neq q$ and if τ is the permutation that exchanges p and q leaving the others unchanged, then $sgn(\tau) = -1$*

Definition Let $f \in L^k(v)$ arbitrary. If σ is permutation of $\{1, \dots, k\}$, we define $f^\sigma(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) f^\sigma \in L^k(V)$.

F is *symmetric* if $f^e = f$ for *every elementary permutation* $e \in S_k$. f is *alternating* if $f^e = -f$ for an elementary permutation $e \in S^k$

Set of all alternating tensors is $A^k(V) \subset L^k(V)$. Consider the set of $L^1(V) = A^1(V)$

Elementary tensor for $k \geq 2$ are *not* alternating. But linear combinations of ϕ_1 are.

$$f = \phi_{i,j} - \phi_{j,i} \tag{105}$$

$$f(x, y) = x_i y_j - y_i x_j = -f(x, y) \tag{106}$$

Lemma 18.2 *Let $f \in L^k(v)$, let $\sigma, \tau \in S^k$*

- *The transformation $f \to f^\sigma$ is a linear transformation on $L^k(V)$. Moreover, $(f^\sigma)^\tau = f^{\sigma \circ \tau}$*
- $f \in A^k(V) \Leftrightarrow f^\sigma = sgn(\sigma) f$
- *If $f \in A^k$ and $v_p = v_q$ for $p \neq q$, then $f(v_1, \dots, v_k) = 0$*

Proof

- $$\begin{aligned}(af+bg)^\sigma =& af^\sigma + bg^\sigma \Rightarrow \sigma \text{ is Linear}\\ (f^\sigma)^\tau(V_1,\dots,V_k) =& f^\sigma(V_{\tau(1)},\dots,V_{\tau(k)})\\ =& f^\sigma(W_1,\dots,W_k) \text{ With } W_i = V_{\tau(i)}\\ =& f(W_{\sigma(1)},\dots,W_{\sigma(k)})\\ =& f(V_{\tau(\sigma(1))},\dots,W_{\tau(\sigma(k))})\\ =& f^{\sigma\circ\tau}(V_1,\dots,V_k)\end{aligned}$$

- Write σ as composite of elementary permutations σ_i, $\sigma = \sigma_1 \circ \dots \circ \sigma_m$ Then $f^\sigma = f^{\sigma_1\circ\dots\circ\sigma_m} = (-1)^m f = sgn(\sigma)f$
- For $V_p = V_q p \neq q$, then $f(v_1,\dots,v_p,\dots,v_q,\dots,v_k) =$ $f(v_1,\dots,v_q,\dots,v_p,\dots,v_k)$ On the other hand, let τ exchange p and q.

$$\begin{aligned}f^\tau(v_1,\dots,v_p,\dots,v_q,\dots,v_k) =& f(v_1,\dots,v_q,\dots,v_p,\dots,v_k) &(107)\\ =& -f(v_1,\dots,v_p,\dots,v_q,\dots,v_k) &(108)\\ \Rightarrow f(v_1,\dots,v_p,\dots,v_q,\dots,v_k) =& 0 &(109)\end{aligned}$$

$L^k(V)$ with $k > 0$, $A^k(V) o < k \leq dim(V)$ Let $a_1,\dots a_n$ be a basis of V. If $f, g \in A^k(V)$ and if $f(a_{i,1},\dots,a_{i,k}) = g(a_{i,1},\dots,a_{i,k})$ for every *ascending* k-tuple of integers from the set $\{1,\dots,n\}$, then $f = g$.

$$\begin{aligned}L^k(V) : f(a_1,a_1)f(a_1,a_2)f(a_2,a_1)f(a_2,a_2) &(110)\\ A^k(V) : f(a_1,a_2) &(111)\\ &(112)\end{aligned}$$

19 Lecture 19

Alternating tensors

Lemma 19.1 *let $a_1,\dots,a_n$ be an basis of V. If f&g are alternating k-tensors on V, and if;*

$$f(a_{i1},\dots,a_{ik}) = g(a_{i1},\dots,a_{ik}) \tag{113}$$

for every ascending k-tuple of integers$I = (i_1,\dots i_k)$ from the set $1,\dots n$

Proof Given Lemma 26.2 in Munkres, we need to show that f and g agree on all k-tuples of basis elements (unordered) Let $f = (j_1,\dots,j_k)$ be an unordered k-tuple if 2 of the indices of f are equal then $f(a_{j1},\dots,a_{jk}) = g(a_{i1},\dots,a_{jk}) = 0$.If all indices are different, let σ be a the permutation such that $I = (j_{\sigma(1)},\dots,j_{\sigma(k)})$ is ascending. Then $f(a_{j1},\dots a_{jk}) = sgn(\sigma)f(a_{j\sigma(1)},\dots,a_{j\sigma(k)})$ similar calculations holds for g. Since f and g agree they agree on f.

Proposition 19.2 *A basis for $A^k(V)$ is given by;*

$$\psi_I = \sum_{\sigma \in S_k} sgn(\sigma)(\phi_I)^\sigma, where I = (i_1, \ldots, i_k) \tag{114}$$

Is an ascending k-tuple from the set $\{1, \ldots, n\}$. ϕ_I is a basis element of $L^k(V)$.

It's properties are as follows;

- *For every ascending k-tuple $J = (j_1, \ldots, j_k)$*

$$\psi_I(a_{j1}, \ldots, a_{jk}) = \left\{ \begin{array}{cc} 0 & if I \neq J \\ 1 & otherwise \end{array} \right\} \tag{115}$$

- $\psi_I^\tau = sgn(\tau)\psi_I$

Proof From the definition, $\psi_I(a_{j1}, \ldots, a_{jk}) = \sum_{\sigma \in S_k} sgn(\sigma)\phi_I(a_{j\sigma(a)}, \ldots, a_{j\sigma(k)})$ At most 1 term in the sum is non-vanishing, namely when $I = (j_{\sigma(1)}, \ldots, j_{\sigma(k)})$ Since I and F are ascending, this only occurs for I=J. then $\psi_1 = 1$ else 0.

$$\psi_I^\tau = \sum_{\sigma \in S_k} sgn(\sigma)\phi_I^{\tau \circ \sigma} \tag{116}$$

$$= sgn(\tau) \sum_{\sigma \in S_k} sgn(\tau \circ \sigma)\phi_I^{\tau \circ \sigma} \tag{117}$$

$\tau \circ \sigma$ runs over all elements in S_k if f^σ does.

$$dim(A^k(V)) = nCk = \frac{n!}{(n-k)!k!} \tag{118}$$

Recall $dim(L^k(V)) = n^k$ Let $X = [x_1, \ldots, x_n]$ be a n by n matrix Then Det $X = \psi_{1,\ldots,n}(x_1, \ldots, x_n)$. Levi Ceuta symbol.

For k-tensors $f, g \in L^*(V)$, we defined the tensor product $\otimes : L^k(V) \times L^l(V) \to L^{k+l}(V)$

$$(f \otimes g)(V_1, \ldots, v_{k+l}) = f(V_1, \ldots, V_k)g(V_{k+1}, \ldots, V_{k+l}) \tag{119}$$

$$(f \otimes g)(V_1, V_1) = f(V_1)g(V_1) \neq 0 \tag{120}$$

$$\Rightarrow \otimes \text{ does not map alternating tenosrs to alt tensors.} \tag{121}$$

We want to construct a product $\wedge : A^k(V) \times A^l(V) \to A^{k+l}(V)$

Definition $A : L^k(V) \to A^k(V)$ by $AF = \sum_{sgn(\sigma \in S_k} sgn(\sigma)F^\sigma$

- A is linear
- AF is alternating

Then

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g) \tag{122}$$

$$= \frac{1}{k!l!} \sum sgn(\sigma)(f \otimes g)^{\sigma} \tag{123}$$

$$\tag{124}$$

Let $\sigma = \sigma_f \circ \sigma_g, \sigma_f$

20 Lecture 20

Theorem 20.1 *The wedge product has the following properties;*

- *Asscociativity:* $f \wedge (g \wedge h) = (f \wedge g) \wedge h$
- *Homogeneity:* $(cf) \wedge g = c(f \wedge g) = g \wedge (cg)$
- *Distributivity: If* f *and* g *have the same order,*

$$(f + g) \wedge h = f \wedge h + g \wedge h \tag{125}$$

$$h \wedge (f + g) = h \wedge f + h \wedge h \tag{126}$$

- *Anti commutativity: if* f *and* g *have the same orders* k *and* l *respectively then,*

$$g \wedge f = (-1)^{kl} f \wedge g \tag{127}$$

- *Given a basis* $a_q, \ldots, a_n$ *for* V*, let* ϕ_i *denote the dual basis for* $V^\star$*, and let* ψ_I *denote the corresponding elementary alternating tensors. If* $I = (i_1, \ldots, i_k)$ *is an ascending k-tuple of integers from* $\{1, \ldots, n\}$*, then;*

$$\psi_I = \phi_{i1} \wedge \phi_{i2} \wedge \ldots \wedge \phi_{ik} \tag{128}$$

- *If* $T : W \to W$ *is a linear transformation, and if* f *and* g *are alternating tensors on* W*, then*

$$T^\star(f \wedge g) = T^\star f \wedge T^\star g \tag{129}$$

Definition Given $\mathbf{x} \in \mathbb{R}^n$, we define a *tangent vector* to $\mathbb{R}$ at $\curvearrowleft$ to be a pair $(\mathbf{x}; \mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^n$. The set of all tangent vectors to $\mathbb{R}^n$ at $\mathbf{x}$ forms a vector space if we define;

$$(x; v) + (x; w) = (x; v + w) \tag{130}$$

$$c(x; v) = (x; cv) \tag{131}$$

To interpret this definition, we must note that x and v are both members of $\mathbb{R}^n$ but play different roles. We think of x as a point of the metric space $\mathbb{R}^n$ and picture it as a dot. We think of v as a member of the vector space $\mathbb{R}^n$ and picture it as an arrow. Thus $(x; v)$ is an arrow with starting point x.

Definition Let A be open in $\mathbb{R}^k$ or $\mathbb{H}^k$ Let $\alpha : A \to \mathbb{R}^n$ be of class C^r. Let $x \in A$, and $p = \alpha(x)$. We define a linear transformation

$$\alpha_\star : T_x(\mathbb{R}^k) \to T_p(\mathbb{R}^k) \tag{132}$$

by the equation;

$$\alpha_\star(x; v) = (p; D\alpha(x).v) \tag{133}$$

It is said to be the transformation induced by the differentiable map α.

21 Lecture 21

21.1 Tensor fields

Let $A \in \mathbb{R}^n$, open. A k-tensor field in A is a function w, such that:

$$w(x) \in L^k(T_x(\mathbb{R}^n)) \tag{134}$$

For each $x \in A$.

If $w(x) \in A^k(T_x(\mathbb{R}^n))\forall x \in \mathbb{R}^n$, then w is a k-form:

$$w \in \Omega^k(A) \tag{135}$$

If f is a function on A $(f : A \to \mathbb{R}), then f \in \Omega^0(A)$ example: 1-forms

$$\tilde{\phi}_i(x)(x; e_j) = \left\{ \begin{array}{lll} 1 & if & i = j \\ 0 & if & i \neq j \end{array} \right\} \tag{136}$$

Elementary k-forms For an ascending k-tuple $I = (i_1, \ldots i_k)$, then $\tilde{\psi}_I(x) = \tilde{\phi}_{i_k}(x)(x; e_j) \wedge \ldots \wedge \tilde{\phi}_{i_k}(x)(x; e_j)$ and $w(x) = \sum_I b_I(x)\psi_I(x)$, with $b_I(x) : \mathbb{R}^n \to \mathbb{R}$.

Let $f : \mathbb{R}^n \to \mathbb{R} \in \Omega^0(A)$, Recall $Df = (\frac{\partial f}{\partial x_1}, \ldots \frac{\partial f}{\partial x_n})$ is a linear transformation $Df \in L^1(\mathbb{R}^n)$ Taking the derivative, a zero form becomes a 1 form.

We introduce "d" the exterior derivative:

$$d : \Omega^k(A) \to \Omega^{k+1}(A) \tag{137}$$

Definition $df(x)(x; v) = Df(x)\dot{v}$ df is called the differential of f. Note that there is no infintesimal component of this. d acts linearly $d(af + bg) = adf + bdg$

Definition The ith projection function

$$\pi_i : \mathbb{R}^n \to \mathbb{R}, \pi_i(x) = x_i \tag{138}$$

Lemma 21.1

$$d\pi_i(x) = \tilde{\phi}_i \tag{139}$$

Proof

$$\pi_i(x)(x;v) = D\pi_i(x)\dot{v} = (0,\ldots,1,\ldots,0)\dot{v} = V_i \tag{140}$$

$$\tilde{\phi}_i(x)(x;v) = V_i \tag{141}$$

It is common to write x_i for π_i such that $\tilde{\phi}_i = dx_i$. dx_i is not an infintesimal small quantity:

$$dx \neq \lim_{\wedge x \to 0} \wedge x \tag{142}$$

If I is a k-tuple, then

$$dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_k} \tag{143}$$

$$w(x) = \sum_I b_I(x) dx_I \tag{144}$$

For $k > 0$, d acts on w such that:

$$dw = \sum_I db_I(x) \wedge dx_i \tag{145}$$

Determine dw for $w \in \Omega^k$ for $k = 0, 1, 2, 3$ in $\mathbb{R}^3$, $w = \sum_I f_I dx_I$ For $k = 0$ we have:

$$w = f \tag{146}$$

$$dw = D_1 f dx_1 + D_2 f dx_2 + D_3 dx_3 \tag{147}$$

$$= \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx_i \tag{148}$$

For $k = 1$ we have:

$$dw = d(\sum_{i=1}^{3} f_i dx_i) \tag{149}$$

$$= \sum_{i=1}^{3} df_i \wedge dx_i \tag{150}$$

$$= \sum_{i=1}^{3} D_j f_i dx_j \wedge dx_i \tag{151}$$

$$= D_1 f_1 dx_1 \wedge dx_1 + \ldots \tag{152}$$

Since $dx_i \wedge dx_i = 0$ (153)

$$dw = D_2 f_1 dx_2 \wedge dx_1 + D_3 f_1 dx_3 \wedge dx_1 + D_1 f_2 dx_1 \wedge \tag{154}$$

$$dx_2 + D_3 f_2 dx_3 \wedge dx_2 + D_1 f_3 dx_1 \wedge dx_3 + D_2 f_3 dx_2 \wedge dx_3 \tag{155}$$

$$= (D_1 f_2 - D_2 f_1) dx_1 \wedge dx_2 + (D_1 f_3 - D_3 f_1) dx_1 \wedge \tag{156}$$

$$dx_3 + (D_2 f_3 - D_3 f_2) dx_2 \wedge dx_3 \tag{157}$$

For k=2

$$dq = d(f_{12}dx_1 \wedge dx_2 + f_{23}dx_2 \wedge dx_3 + f_{13} \wedge dx_3) \tag{158}$$
$$= D_3 f_{12} dx_3 \wedge dx_1 \wedge dx_2 + D_1 f_{23} dx_1 \wedge dx_2 \wedge \tag{159}$$
$$dx_3 + D_2 f_{13} dx_2 \wedge dx_1 \wedge dx_3 \tag{160}$$
$$= D_3 f_{12} + D_1 f_{23} - D_2 f_{13} \tag{161}$$

k=3 dw=0

22 Lecture 22

22.1 Exterior Derivative - "d"

$$df(x)(x;v) = Df(x).v \tag{162}$$
$$f \in \Omega^0(A) \tag{163}$$
$$df \in \Omega^1(A) \tag{164}$$
$$\pi_i(x) = x_i = x_i(x) \tag{165}$$
$$d\pi_i = \tilde{\phi}_i = dx_i \tag{166}$$
$$dx_{i1} \wedge \ldots \wedge dx_{ik} = dx_I \tag{167}$$
$$w = \sum_{[I]} b_I dx_I = \sum_{[I]} b_I \tilde{\psi}_i \tag{168}$$
$$dw = \sum_{[I]} db_I \wedge dx_I \tag{169}$$

We will now list a few useful properties;

Theorem 22.1 *Let $w \in \Omega^k(A)$ and $\eta \in \Omega^l(A)$*

1. *$d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$*
2. *$d(dw) = 0$*
3. *d is linear*

Proof •

$$w = \sum_{[I]} f_I dx_I \tag{170}$$

$$\eta = \sum_{[J]} g_J dx_J \tag{171}$$

$$d(w \wedge \eta) = \sum_{[I][J]} d(f_I g_J) dx_I \wedge dx_J \tag{172}$$

Using the product rule (173)

$$d(f_I g_J)(x)(x; v) \tag{174}$$

$$= Df_I(x).v g_J(x) + f_I Dg_J(x).v \tag{175}$$

$$= (df_I g_J + f_I dg_J)(x)(x; v) \tag{176}$$

subbing this into previous reuslt (177)

$$dw \wedge \eta + (-1)^k w \wedge d\eta \tag{178}$$

$$d(df) = d(\sum_{i=1}^{n} D_i f dx_i) \tag{179}$$

$$= \sum_{i,j=1}^{n} D_j D_i f dx_j \wedge dx_i \tag{180}$$

$$= 0 \tag{181}$$

$$w \in \Omega^k(A) \tag{182}$$

$$d(dw) = d \sum_{[I]} df_I \wedge dx_I = \qquad 0 \tag{183}$$

- Exact form w is a form such that $w = d\alpha$ for some α
- Closed form w os a form such that dw=0

22.2 Hodge *-Operator

There is nothing in the book about this so notes will be made available online.

$$* : \Omega^k(A) \to \Omega^{n-k}(A) \tag{184}$$

$$\text{Let } w = \sum_{[I]} f_I dx_I \tag{185}$$

$$*w = \sum_{[I]} \epsilon_{IJ} f_I dx_J \tag{186}$$

Where

1. I is the ascending k-tuple given $(1,\ldots,n)/I$

2. $\epsilon_{IJ} = sgn(i_1,\ldots,i_k,j_1,\ldots,j_{n-k}) = \psi_{IJ}$

Example

$$w = dx_2 \Lambda dx_3 + dx_1 dx_2 \in \mathbb{R}^3 \tag{187}$$

$$*w = \epsilon_{231} dx_1 + \epsilon_{123} dx_3 = dx_1 + dx_3 \tag{188}$$

$1) *f = f dx_I with I = (1,\ldots,n) * *w = (-1)^{k(n-k)} w for w \in \Omega^k(A)$

22.3 Differential Forms and $\wedge$

Let $A \in \mathbf{R}^n$ open and let $e_1,\ldots,e_n$ be a basis of $\mathbb{R}^n$. Let $f : a \to \mathbb{R}$, and $G(x) = (x; g(x))$, a vector field.

22.3.1 Gradient

$\nabla f(x) = (x; \sum_{i=1}^n) f(x) e_i) \vec{F} = -\nabla U$

22.3.2 Divergence

$(\nabla G)(x) = \sum_{i=1}^n D_i g_i(x)$

22.3.3 Introduce the (musical) isomorphism ξ

$$\xi : \text{ Vector fields on } A \to \Omega^{-1}(A) \tag{189}$$

$$\xi G = \sum_{i=1}^n g_i dx_i, \text{where, } G(x) = (x; \sum_{i=1}^n g_i e_i) \tag{190}$$

$$df = \sum_{i=1}^n D_i f dx_i = \xi(\nabla f) \tag{191}$$

$$*d * w = * \, d \sum_{i=1}^n (-1)^{i-1} g_i dx_{(1,\ldots,n)/i} \tag{192}$$

$$= * \sum_{i=1}^n (-1)^{i-1} D_i g_i dx_i \wedge dx_{(1,\ldots,n)/i} \tag{193}$$

$$= \sum_{i=1}^n D_i g_i = \nabla G \tag{194}$$

In $\mathbb{R}^3$:

$$(\nabla \times G)(x) w = \xi G \tag{195}$$

$$*dw = \xi(\nabla \times G) \tag{196}$$

23 Lecture 23

23.1 Exterior Derivatives

$$d : \Omega^k(A) \to \Omega^{k+1}(A) \tag{197}$$

$$w = \sum_{[I]} f_I dx_i \tag{198}$$

$$dw = \sum_{[I]} df_I \wedge dx_i \tag{199}$$

$$df = \sum_i D_i f dx_i \tag{200}$$

$$d^2 w = 0 \tag{201}$$

23.2 Hodge $\star$ operator

$$\star = \Omega^k(A) \to \Omega^{n-k}(A) \tag{202}$$

$$\star w = \sum_{[I]} \epsilon_{IJ} f_I dx_j, \text{ with,} \tag{203}$$

$$\epsilon_{IJ} = sgn(i_1, \ldots, i_k, j_1, \ldots, j_{n-k}) \tag{204}$$

$$J = (j_1, \ldots, j_{n-k}) = (1, \ldots, n)/I \tag{205}$$

23.3 Maxwell equations

- Electric Field: $E(t, x) \in T_x(\mathbb{R}^3)$
- Magnetic Field: $B(t, x) \in T_x(\mathbb{R}^3)$ Where x is a vector in $\mathbb{R}^3$ and t is a time unit in $\mathbb{R}$.

23.3.1 Homogenous equations

$$\nabla . \vec{B} = 0 \tag{206}$$

$$\nabla x E + \frac{\partial B}{\partial t} = 0 \tag{207}$$

23.3.2 Non-homogneous equations

Sourceless

$$\nabla . E = o \tag{208}$$

$$\nabla x B + \frac{\partial E}{\partial t} = 0 \tag{209}$$

$$F \in \Omega^2(\mathbb{R}^4) \tag{210}$$

$$F = E_1 dt \wedge dx_1 + dt \wedge dx_2 + dt \wedge dx_3 - B_3 dx_1 \wedge dx_2 + B_2 dx_1 \wedge dx_3 - B_1 dx_2 \wedge dx_3 \tag{211}$$

$$\tag{212}$$

Freeform of the equation: Which terms to contribute:

$$dF = (D_2E_1 - D_1E_2 - D_tB_3)dt \wedge dx_1 \wedge dx_2 + (D_3E_2 - D_2E_3 - D_tB_1)dt \wedge dx_2 \wedge dx_3 + (D_3E_1 - D_1E_3 + D_tE$$

$$dF = 0 \Leftrightarrow \{\nabla.B = 0, text\nabla xE + \frac{\partial B}{\partial t} = 0\} \tag{214}$$

$$\star F = \star(E_1 dt \wedge dx_1 + \ldots) \tag{215}$$

$$= \epsilon_{t123} E_1 dx_2 \wedge dx_3 + \ldots \tag{216}$$

$$= E_1 dx_2 \wedge dx_3 \Rightarrow \text{Vector potential } A \in \Omega^{-1}(\mathbb{R}^4) \text{ such that } F = dA = F \begin{pmatrix} E \to & -B \\ B \to & -E \end{pmatrix} \tag{217}$$

$$\Rightarrow d \star F = 0 \Leftrightarrow \text{non homogenous, maxwell equation} \tag{218}$$

23.4 The action of a differentiable map

$$\alpha : A \to \mathbb{R}^n \in C^\infty \tag{219}$$

$$\text{Recall pushforward } \alpha_\star : T_x(\mathbb{R}^k) \to T_{\alpha(x)}(\mathbb{R}^n) \tag{220}$$

$$\alpha_\star(x; v) = (\alpha(x), D\alpha(x).v) \tag{221}$$

Recall a coordinate map of a manifold:

$$T : V \to W \text{ induces a map (pull-back)} \tag{222}$$

$$T^\star : L^k(W) \to L^k(V) \tag{223}$$

$$(T^\star f)(v_1, \ldots, v_k) = f(T(v_1), \ldots, T(v_k), \text{ with } f \in L^k(W) \tag{224}$$

$\alpha_\star$ is a map between vector spaces which implies $\alpha_\star$ or α induces a pull back map on $\alpha^\star$.

Definition Let $A \in \mathbb{R}^k$ open, and $B \in \mathbb{R}^n$ open. $\alpha : A \to \mathbb{R}^n$ such that $\alpha(A) \subset B$, Then $\alpha^\star \Omega^l(B) \to \Omega^l(A)$

$$(\alpha^\star w)(x)((x; v_1), \ldots (x; v_l)) = w(\alpha(x))(\alpha_\star(x; v_1), \ldots, (\alpha_\star(x; v_l)) \tag{225}$$

Theorem 23.1 *Let $A \in \mathbb{R}^k$, $B \in \mathbb{R}^m$, $C \in \mathbb{R}^n$ all open.*

- $\alpha : A \to \mathbb{R}^m$ *such that* $\alpha(A) \subset B$

- $B \to \mathbb{R}^n$ *such that* $\beta(B) \subset C$

$w, \eta \in \Omega^p(C), \theta \in \Omega^l(C)$.

1. $\beta^\star$ *is linear.* $\beta^\star(aw + b\eta) = a\beta^\star w + b\beta^\star \eta$
2. $\beta^\star(w \wedge \theta) = \beta^\star w \wedge \beta^\star \theta$
3. $(\beta \circ \alpha)^\star = \alpha^\star(\beta^\star w)$

Theorem 23.2 *Let* $A \subset \mathbb{R}^k$*, open and* $\alpha : A \to \mathbb{R}^n$*, let* $x \in \mathbb{R}^k$ *and* $\alpha(x) = y$*. Let* dx_i *and* dy_j *be the elemetnary 1-forms in* $\mathbb{R}^k$ *and* $\mathbb{R}^n$ *respectively. Then:*

$$\alpha^\star(dy_i) = d\alpha_i \tag{226}$$

nb:

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{227}$$

24 Lecture 24

24.1 The action of a differentiable map on differential forms

Proof Followinf the definintion of $\alpha^\star$:

$$\begin{aligned}
(\alpha^\star(dy_i))(x)(x;v) =& dy_i(y)(\alpha_\star(x;v)) && (228)\\
=& dy_i(y)(y, D\alpha.v) && (229)\\
=& \text{ith component of } D\alpha.v && (230)\\
=& \sum_{j=1}^{k} D_j\alpha_i.v_j && (231)\\
=& \sum_{j=1}^{k} \frac{\partial \alpha_i}{\partial x_j} dx_j(x)(x;v) && (232)\\
\Rightarrow& \text{ removing the arguments} && (233)\\
& \alpha^\star(dy_i) = d\alpha_i && (234)
\end{aligned}$$

If $I = (i_1, \ldots, i_k)$ is an ascending k-tuple from $1, \ldots, n$, then:

$$\alpha^\star(dy_I) = (det\frac{\partial \alpha_I}{\partial x})dx_1 \wedge \ldots \wedge x_k \tag{235}$$

with;

Theorem 24.1 *Let* $w \in \Omega^l(B)$*,* B *open in* $\mathbb{R}^n$ *and* $\alpha(A) \subset B$ *Then;*

$$\alpha^\star(dw) = d(\alpha^\star w) \tag{236}$$

Proof 1. $l = 0, w = f, x \in A, y = \alpha(x) \in B$ LHS:

$$\alpha^\star(df) = \alpha^\star(\sum_{i=1}^{n} D_i f dy_i) \tag{237}$$

$$= \sum_{i=1}^{n} ((D_i f) \circ \alpha) d\alpha_i \tag{238}$$

RHS:

$$d(\alpha^\star f) = d(f \circ \alpha) \tag{239}$$

$$= \sum_{j=1}^{k} D_j (f \circ \alpha) dx_j \tag{240}$$

Using the chain rule:

$$D(f \circ \alpha)(x) = Df(y).D\alpha(x) \tag{241}$$

$$D_j(f \circ \alpha)(x) = \sum_{i=1}^{n} D_i f(y) D_j \alpha_i(x) \tag{242}$$

$$d(\alpha^\star f) = \sum_{j=1}^{k} \sum_{i=1}^{n} (D_i f(y)) D_j \alpha_i(x) dx_j \tag{243}$$

$$= \sum_{i=1}^{n} (D_i f(y)) d\alpha_i = \alpha^\star(df) \tag{244}$$

2. $l > 0$. Linearity of d and $\alpha^\star$ imply that a proof for $w = f dy_I$ is sufficient. We begin with the RHS:

$$\alpha^\star(dw) = \alpha^\star(df \wedge dy_I) \tag{245}$$

$$= \alpha^\star(df) \wedge \alpha^\star(dy_I) \tag{246}$$

With the LHS:

$$d(\alpha^\star w) = d(\alpha^\star(f dy_I)) \tag{247}$$

$$= d(\alpha^\star f \wedge \alpha^\star(dy_I)) \tag{248}$$

$$= d(\alpha^\star f) \wedge \alpha^\star(dy_I) + \alpha^\star f \wedge d(\alpha^\star(dy_I)) \tag{249}$$

Since $d(\alpha^\star(dy_I)) = d(d\alpha_{i1} \wedge \ldots \wedge d\alpha_{ik}) = 0$

25 Lecture 25

Definition Let $w \in \Omega^k(A)$, then it is uniquely written as $w = f dx_1 \wedge \ldots \wedge dx_k$. Then we define

$$\int_A W = \int_A f. \tag{250}$$

Integration over Y_{α_0}. Let $w \in \Omega^k(B)$ with B open in $\mathbb{R}^n$ and such that $Y_\alpha \subset B$;

$$\int_{Y_\alpha} w = \int_A \alpha^\star w \tag{251}$$

$\beta(B) = \alpha(A) = Y$

$$\int_{Y_\alpha} f = \int_B (f \circ \beta) V(D\beta) = same integral parameterised over A \tag{252}$$

Theorem 25.1 *Let $g : A \to B$ be a diffeomorphism of open sets. Assume that $Det[Dg]$ does not change sign on A. Let $\beta : B \to \mathbb{R}^n \in C^\infty$, $Y = \beta(B)$. Let $\alpha = \beta \circ g$ such that $\alpha(A) = Y$. Let $w \in \Omega^k(C)$ with $C \in \mathbb{R}n$ such that $Y \subset C$, then;*

$$\int_{Y_\alpha} w = \pm \int_{Y_\beta}, \text{if it exists}, \pm = sgn(det(Dg)) \tag{253}$$

Proof Set $\eta = \beta^\star w$, then:

$$\int_{Y_\alpha} w = \int_A \alpha^\star w = \int_A g^\star(\beta^\star w) = \int_A g^\star \eta \tag{254}$$

Then

$$\begin{aligned} g^\star \eta =& (f \circ g) g^\star (dy_1 \wedge \ldots \wedge dy_k) && (255)\\ =& (f \circ g) Det(Dg) dx_1 \wedge \ldots \wedge dx_k && (256)\\ \int_A g^\star \eta =& \int_A (f \circ g) Det(Dg) && (257)\\ =& \int_B f Det(Dg) | Det(Dg^{-1}) && (258)\\ =& sgn(Det(Dg)) \int_B f && (259) \end{aligned}$$

$$\int_{Y_\alpha} w = sgn(Det(Dg)) \int_B \eta = sgn(Det(Dg)) \int_{Y_\beta} w \tag{260}$$

$\int_M w = \int_{\int U} \alpha^\star w$, $\alpha : W \to V \subset M$.

25.1 Orientable Manifolds

See book for diagram on orientation reversing and preserving. In general, let $h : \mathbb{R}^n \to \mathbb{R}^n$, given by $h(x) = C.x$.

- If $detC > 0 \Rightarrow h$ is orientation preserving
- If $detC < 0 \Rightarrow h$ is orientation reversing

Recall pushforward maps as linear maps on the tangent spaces. A diffeomorphism $g : A \to B$ induces a pushforward map on tangent spaces. $g_\star : T_x(A) \to T_{g(x)}(B)$. Recall that $g_\star(x)(x; v) = (g(x); Dg.v)$.

- If $detDg > 0 \Rightarrow g$ is orientation preserving
- If $detDg < 0 \Rightarrow g$ is orientation reversing

We want to say whether or not our transition function is orientation reversing or preserving. Let M be a k-manifold. Given $\alpha_i : U_i \to V_i \subset M$ for $i = 0, 1.$ with $V_0 \cap V_1 \neq 0$. We say α_0 & α_1 overlap positively if $\alpha_1^{-1} \circ \alpha_0^{-1}$ is orientation preserving.

26 Lecture 26

Consider diffeomorphism, $g : A \to B$. This induces pushforward map $g_\star(x) : T_x(\mathbb{R}^k) \to T_{g(x)}(\mathbb{R}^k), x \in A$. $g_\star(x; v) = (g(x), Dg(x).v)$. gi is orientation preserving if $Det(Dg) > 0$ and orientation reversing if $Det(Dg) < 0$.

If $g = \alpha_1^{-1} \circ \alpha_0$ is orientation preserving, α_0 and α_1 *overlap* positively, and otherwise negatively.

Definition M is *orientable* if it can be covered by coordinate patches, each pair of which overlap positively. Otherwise it is non orientable.

Definition An orientation of M is a set of coordinate patches which cover M and which overlap positively pairwise, together with all patches which overlap these positively.

Definition An oriented manifold is a manifold together with a choice of orientation.

Orientation of a 1-manifold defines a unit tangent vector field. Let $\alpha(t_0) = p$. The tangent vector $T(p)$ is,

$$(p; \frac{D\alpha(t_0)}{||D\alpha(t_0)||}) \tag{261}$$

$\beta(s_0) = p$ then;

$$= (p; \frac{D\beta(s_0)}{||D\beta(s_0)}) \tag{262}$$

Oriented 1-manifold=directed curve - refer to book

Allow $\alpha : U \to V$ with $U \subset L^1$. $L^1 = x \in \mathbb{R}, x \leq 0$.

26.1 $(n-1)$ manifold in $\mathbb{R}^n$

Here we can define a unit normal vector $\alpha : U \to V$ and let $\alpha(x) = p \in M$ column vecotrs of $D\alpha(p)$ gives us a basis for $T_p(M)$.

$$(p, \frac{\partial \alpha}{\partial x_i}) \text{ for } i = 1, \ldots, n \tag{263}$$

$Det(\vec{n}, D_\alpha(p)) \Rightarrow$ This fixes $\vec{n}(p)$

27 Lecture 27

27.1 Recap

- **Orientable manifolds** $g = \alpha_1^{-1} \circ \alpha_0 : W_0 \to W_1$ α_0 and α_1 overlap positively if $detDg > 0$.
- **(n-1) manifold M in** $\mathbb{R}^n$ orientation $\Rightarrow$ unit normal vector $\vec{n}$ field to M.

$$\vec{n}(p) \in T_p(\mathbb{R}^n) \text{ such that } ||\vec{n}|| = 1 \text{ and,} \tag{264}$$

$$det(\vec{n}, \frac{\partial \alpha}{\partial x_1}, \ldots, \frac{\partial \alpha}{\partial x_{n-1}}) > 0 \tag{265}$$

and n is perpendicular/orthognal to all partial derivatives of α for all x_i

Mobius strip is an example of a non orientable manifold. In general dimensions, how can we reverse the orientation of M? Taking the negative of the basis vectors. This is orientation reversing map.

Definition The reflection: $r : \mathbb{R}^k \to \mathbb{R}^k$ defined as;

$$r(x_1, \ldots, x_k) = (-x_1, \ldots, x_k) \tag{266}$$

for $k > 1$, $r : \mathbb{H}^k \to \mathbb{H}^k$. for k=1, $r : \mathbb{H}^1 \to \mathbb{L}^1$.

Definition Given an oriented manifold M, with coordinate patches α_i belonging to the orientation, then the reverse orientation is given by:

$$\beta_i = \alpha_i \circ r : r(U_i) \to V_i \tag{267}$$

$$DetD(\alpha_i^{-1} \circ \beta_i) = DetDr = -1 \tag{268}$$

thus they overlap negatively.

We are developing this technology of orientation in order to introduce stokes theorem.

Theorem 27.1 *Let $k \geq 1$, if M is orientable with non-empty boundary (which is itself a manifold), then the δM (the boundary) is an orientable manifold. (proof omitted). By restricting orientation for the boundary of M, by using the similar proof in the beginning of the course we can prove this.*

Definition With M as above and given an orientation of M, then we say that the *induced orientation* is defined by;

- k even: restricting coordinate patches on M.
- k odd: the orientation opposite to restricting the orientation on M.

Example In copy book

27.2 Integration on an oriented manifold

M is an oriented k manifold. $\omega \in \Omega^k(B)$, $B \subset \mathbb{R}^n$ open. $M \subset B$.

- Cover M by coordinate patches which belong to its orientation.
- Choose a partition of unity $\phi_1, \ldots, \phi_l$ on M which is dominated by the covering.

$$\int_M \omega = \sum_{i=1}^{l} [\int \phi_i \omega] \tag{269}$$

We can do this to our coordinate patches;

$$\int_M \phi_i \omega = \int_{U_j} \alpha_j^\star (\phi_i \omega) \tag{270}$$

for some $\alpha_j : U_j \to V_j$.

27.2.1 Generalised Stokes Theorem

Theorem 27.2 *M is a k-manifold, and give δM the induced orientation. Let $\omega \in \Omega^{k-1}(B)$, then*

$$\int_M dw = \int_{\delta M} wifboundarynonempty, 0otherwise \tag{271}$$

M is a compact oriented k-manifold in $\mathbb{R}^n$. δM has the induced orientation $\omega \in \Omega^{k-1}$ such that $M \subset B$. We will not prove the general case of stokes theorem but we will use it to derive the divergence theorem.

Example Refer to copy book

$$\omega = xdy - ydx \in \Omega^1(\mathbb{R}^2) \tag{272}$$

We have the coordinate patch $\alpha : A \to \mathbb{R}^2$ where $A = (\theta, w) \in \mathbb{H}^2 | -\pi < \theta < \pi, \omega \in [0, 1]$. and we define α as:

$$\alpha(\theta, \omega) - ((1-\omega)cos\theta, (1-\omega)sin\theta) = (x, y) \tag{273}$$

Determine the pull back equation $\alpha^\star\omega$

$$\begin{aligned}\alpha^\star\omega =&(1-w)cos\theta d((1-w)sin\theta) &(274)\\ &-(1-w)sin\theta d((1-w)cos\theta) &(275)\\ =&(1-w)^2 d\theta &(276)\\ \alpha^\star(d\omega) =&d(\alpha^\star) = z(1-w)d\theta\wedge dw &(277)\\ \int_M d\omega = &\int_A \alpha^\star d\omega = z\int_A (1-w)d\theta\wedge dw = 2\pi &(278)\\ \int_{\delta M} w = &\int_{(-\pi,\pi)} \alpha^\star\omega &(279)\\ =&\int_{(-\pi,\pi)} d\theta = 2\pi &(280)\\ & &(281)\end{aligned}$$

Theorem 27.3 (Gauss' Theorem)

$$\int_M \nabla G dV = \int_{\delta M} G.N dA \tag{282}$$

Where M is a n-manifold in $\mathbb{R}^n$

In order to prove this theorem, we must prove some preliminary results.

Orientation on M $\leftrightarrow$ unit normal vector field

$$N(p) = (p, \vec{n}(p)) \in T_p(\mathbb{R}^n) \tag{283}$$

where $\vec{n}(p) \perp T_p(M)$ and $\vec{n}(p) \perp \frac{\partial\beta}{\partial x_i}$ for $i \in [1, n-1]$ and $det(\vec{n}, D\beta) > 0$. The solution for n=3:

$$seecopybook. \tag{284}$$

For general n:

$$c_i(p) = (-1)^{i-1} det(D\beta_{\hat{i}} \tag{285}$$

$$\beta_{\hat{i}} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{i-1} \\ \beta_{i+1} \\ \vdots \\ \beta_n \end{pmatrix} \tag{286}$$

Lemma 27.4 *Let G be a vector field in* $\mathbb{R}^n$ *with* $B \in \mathbb{R}^n$*, open and* $M \subset B$*. Let*

$$G(y) = (y; g(y)) = (y; \sum_{i=1}^{n} g_i(y)e_i) \tag{287}$$

$$\omega = \star \xi G = \star \sum_{i=1}^{n} g_i dy_i \tag{288}$$

$$= \sum_{i=1}^{n} (-1)^{i=1} g_i dy_1 \wedge \ldots \wedge dy_{i-1} \wedge dy_{i+1} \wedge \ldots \wedge dy_n \tag{289}$$

$$\tag{290}$$

Lemma 27.5

$$\int_M \omega = \int_M G.NdV \tag{291}$$

Proof Assume ω is supported on a sinlge coordinate patch $\beta : U \to B$:

$$\int_M \omega = \int_U \beta^\star \omega = \int_U \sum_i (-1)^{i-1} (g_i \circ \beta) detD\beta_{\hat{i}} \tag{292}$$

Now turning our attention to the RHS:

$$\int_M GNdV = \int_U (G \circ \beta)(N.\beta)V(D\beta) \tag{293}$$

$$= \int_U (g \circ \beta) C(\beta(x)) \tag{294}$$

$$= \int_U \sum_{i=1}^{n} (g_i \circ \beta) C_i(\beta(x)) \tag{295}$$

one notices, that the left hand side is infact equal to the rhs.

Theorem 27.6 (Gauss Theorem) *G is a tangent vector field in $\mathbb{R}^n$. M is an orientable n-dim manifold with boundary δM. Let N be the outward pointing unit vector field on $M\delta$. Then;*

$$\int_M \nabla G dV = \int_{\delta M} G.NdA \tag{296}$$

Refer to copy book for diagram:

Proof Using the lhs off gauss theorem which is also the RHS of stokes theorem;

$$d\omega = d \star \sum_{i=1}^{n} g_i dy_i \tag{297}$$

$$= \nabla G dy_1 \wedge \ldots \wedge dy_n \tag{298}$$

$$\int_M d\omega = \int \nabla G dy_1 \wedge \ldots \wedge dy_n \tag{299}$$

$$\int_M dy_1 \wedge \ldots \wedge dy_n = \int \beta^\star \eta = \int (h \circ \beta) detD\beta \tag{300}$$

natural orientation $0 < detD\beta = V(D\beta)$

28 Lecture 29

missed

29 Lecture 30

Recap from last lecture:

- $$\alpha^\star\omega = \omega(u+v, u-v, u^2+v^2)(u+v, u-v, u^2+v^2; \begin{matrix} v_1+v_2 \\ v_1-v_2 \\ 2uv_1+cvv_2 \end{matrix}) \quad (301)$$