

# Basic Stochastic Equations

Harshad Deo

[harshad@simianquant.com](mailto:harshad@simianquant.com)

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This monograph lists the solutions of some of the basic equations of stochastic calculus.

## 1 LogNormal Asset Dynamics

This process provides a simple way to model the evolution of equities, commodities and foreign exchange assets. The change in the asset level is proportional to the current value of the level, and the drift and volatility are independent of the current asset level. Formally, for an asset  $S_t$ ,

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t) \quad (1)$$

To solve this equation, let  $X_t = \log(S_t)$ . Using Ito's lemma,

$$dX_t = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} dS_t^2 \quad (2)$$

$$= \frac{1}{S_t} \left\{ S_t(\mu_t dt + \sigma_t dW_t) \right\} - \frac{1}{2S_t^2} S_t^2 \sigma_t^2 dt \quad (3)$$

$$= \left\{ \mu_t - \frac{\sigma_t^2}{2} \right\} dt + \sigma_t dW_t \quad (4)$$

### 1.1 Constant Drift and Volatility

The simplest case is when the drift and volatility is constant. This corresponds to the classic Black-Scholes world. In this case,

$$\int_{X_1}^{X_2} dX = \left\{ \mu - \frac{\sigma^2}{2} \right\} \int_{t_1}^{t_2} dt + \sigma \int_{t_1}^{t_2} dW_t \quad (5)$$

$$\therefore X_2 = X_1 + \left\{ \mu - \frac{\sigma^2}{2} \right\} (t_2 - t_1) + \sigma \sqrt{t_2 - t_1} Z \quad (6)$$

$$\therefore S_2 = S_1 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) (t_2 - t_1) + \sigma \sqrt{t_2 - t_1} Z \right) \quad (7)$$

where  $Z$  is a Standard Normal Variable. In practice, the discount factor between two time instants is modelled using market observables, not the drift. Equation 7 can be

rewritten as:

$$S_2 = \frac{S_1}{D(t_1, t_2)} \exp\left(\frac{-\sigma(t_2 - t_1)}{2} + \sigma\sqrt{t_2 - t_1}Z\right) \quad (8)$$

Where  $D(t_1, t_2)$  is the discount factor at time  $t_2$  observed at time  $t_1$ .

## 1.2 Term Structure Models

In this case, the general solution is:

$$S_2 = S_1 \exp\left(\int_{t_1}^{t_2} \left(\mu_t - \frac{\sigma_t}{2}\right) dt + \int_{t_1}^{t_2} \sigma_t dW_t\right) \quad (9)$$

The discount factor between instants  $t_1$  and  $t_2$  is given by:

$$D(t_1, t_2) = \exp\left(-\int_{t_1}^{t_2} \mu_t dt\right) \quad (10)$$

Following market practice, equation 9 can be simplified to:

$$S_2 = \frac{S_1}{D(t_1, t_2)} \exp\left(\int_{t_1}^{t_2} \frac{-\sigma_t}{2} dt + \int_{t_1}^{t_2} \sigma_t dW_t\right) \quad (11)$$

Since all  $dW_t$  are independent and identically distributed with variance  $\sigma_t dt$ , the stochastic integral is a normal variable with a variance of  $\int \sigma_t^2 dt$ . Equation 12 can therefore be simplified to:

$$S_2 = \frac{S_1}{D(t_1, t_2)} \exp\left(\int_{t_1}^{t_2} \frac{-\sigma_t}{2} dt + \sqrt{\int_{t_1}^{t_2} \sigma_t dt} Z\right) \quad (12)$$

In practice, only the integrated volatility is observed in the market, not the term structure. If the terminal volatility is defined as:

$$\bar{\sigma} = \sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \sigma_t^2 dt} \quad (13)$$

an equation algebraically similar to 8 can be written:

$$S_2 = \frac{S_1}{D(t_1, t_2)} \exp\left(\frac{-\bar{\sigma}(t_2 - t_1)}{2} + \bar{\sigma}\sqrt{t_2 - t_1}Z\right) \quad (14)$$

If a contract depends only on the terminal value of the asset, for example a vanilla call (but not a barrier), the time dependent drift and volatility can be simplified to a single terminal drift and volatility.

$$\bar{\mu} := \frac{1}{T} \int_{t_1}^{t_2} \mu_t dt \quad (15)$$

$$dS_t = S_t(\bar{\mu} dt + \bar{\sigma} dW_t) \quad (16)$$