

Deriving the Black-Scholes Formula for a Call Option

Harshad Deo

harshad@simianquant.com

March 25, 2019

This monograph presents multiple ways of deriving the Black-Scholes formula for a call option. The specific case considered is for foreign exchange derivatives, but the mathematical and financial arguments are identical for equity and commodity derivatives. Foreign exchange was chosen because the exchange rate itself is not a tradeable asset, and this nuance is not handled properly by standard textbooks.

1 Using Martingale Pricing

Since the exchange rate is not a tradeable asset, the universe consists of the domestic money market account, B_t^d , the foreign money market account, B_t^f and the stochastic exchange rate S_t . Since the payoff is going to be dependent on the terminal value, we can simplify the notation by assuming constant risk free rates with the understanding that the values equal the terminal zero rates. Similarly, with a further assumption that **volatility is independent of strike**, we can assume a constant volatility with the understanding that its implied variance is equal to the integrated terminal variance. Let C_t denote the price of a call option at time t .

$$dB_t^d = r^d B_t^d dt \quad (1)$$

$$\therefore B_t^d = e^{r^d t} \quad (2)$$

$$dB_t^f = r^f B_t^f dt \quad (3)$$

$$\therefore B_t^f = e^{r^f t} \quad (4)$$

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad (5)$$

1.1 Using one numeraire

The most straightforward way to price the option is in the domestic money market account numeraire. In this case, the tradeable asset is the foreign money market account denominated in the domestic currency.

$$A_t := S_t e^{r^f t} \quad (6)$$

By changing the drift, we can make the process A_t a martingale with respect to B_t^d . In this measure, the call option price process will also be a martingale with respect to domestic money market numeraire.

$$d\left(\frac{A_t}{B_t^d}\right) = d\left(\exp((r^f - r^d)t)S_t\right) \quad (7)$$

$$= \exp((r^f - r^d)t)dS_t + S_t d(\exp((r^f - r^d)t)) \quad (8)$$

$$= \exp((r^f - r^d)t)S_t\{(\mu + r^f - r^d)dt + \sigma dW_t\} \quad (9)$$

By setting the drift μ to $r^d - r^f$, A_t will be a martingale with respect to B_t^d . In this case, the asset price is given by:

$$S_t = S \exp\left\{(r^d - r^f - \frac{\sigma^2}{2})t + \sigma\sqrt{t}Z\right\}, \quad Z \sim \mathcal{N}(0, 1) \quad (10)$$

The price of a call option is now the result of a tedious integration. Since the domestic interest rate is deterministic, it can be factorized out of the expectation. To simplify the notation, all expectations in this subsection are with the B^d numeraire.

$$\frac{C_0}{B_0^d} = \mathbb{E}\left[\frac{C_t}{B_t^d}\right] \quad (11)$$

$$\therefore C_0 = e^{-r^d t} \mathbb{E}[C_t] \quad (12)$$

$$= e^{-r^d t} \mathbb{E}[(S_t - K)_+] \quad (13)$$

Let \bar{Z} be the value of the standard normal variable for which $S_t = K$. Clearly $Z > \bar{Z}$ implies that $S_t > K$.

$$S_t = K \quad (14)$$

$$\implies S \exp\left\{(r^d - r^f - \frac{\sigma^2}{2})t + \sigma\sqrt{t}\bar{Z}\right\} = K \quad (15)$$

$$\implies (r^d - r^f - \frac{\sigma^2}{2})t + \sigma\sqrt{t}\bar{Z} = \ln\left(\frac{K}{S}\right) \quad (16)$$

$$\implies \bar{Z} = \frac{\ln(K/S) - (r^d - r^f - \sigma^2/2)t}{\sigma\sqrt{t}} \quad (17)$$

Considering the expectation in equation 13,

$$\mathbb{E}[(S_t - K)_+] = \mathbb{E}[(S_t - K)1_{S_t > K}] \quad (18)$$

$$= \mathbb{E}[S_t 1_{S_t > K}] - K \mathbb{E}[1_{S_t > K}] \quad (19)$$

The second term in equation 19 simpler to evaluate.

$$\mathbb{E}[1_{S_t > K}] = \mathbb{P}(S_t > K) \quad (20)$$

$$= 1 - \mathbb{P}(S_t \leq K) \quad (21)$$

$$= 1 - \Phi(\bar{Z}) \quad (22)$$

$$= \Phi(-\bar{Z}) \quad (23)$$

$$= \Phi\left(-\frac{\ln(K/S) - (r^d - r^f - \sigma^2/2)t}{\sigma\sqrt{t}}\right) \quad (24)$$

$$= \Phi\left(\frac{\ln(S/K) + (r^d - r^f - \sigma^2/2)t}{\sigma\sqrt{t}}\right) \quad (25)$$

Considering the first term in equation 19,

$$\mathbb{E}[S_t 1_{S_t > K}] = \frac{1}{\sqrt{2\pi}} \int_{\bar{z}}^{\infty} S_t \exp\left(-\frac{z^2}{2}\right) dz \quad (26)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\bar{z}}^{\infty} S \exp\left((r^d - r^f - \frac{\sigma^2}{2})t + \sigma\sqrt{t}z - \frac{z^2}{2}\right) dz \quad (27)$$

$$= \frac{S \exp((r^d - r^f - \sigma^2/2)t)}{\sqrt{2\pi}} \int_{\bar{z}}^{\infty} \exp(-z^2/2 + \sigma\sqrt{t}z) dz \quad (28)$$

$$= \frac{S \exp((r^d - r^f - \sigma^2/2)t)}{\sqrt{2\pi}} \int_{\bar{z}}^{\infty} \exp\left(-\frac{z^2 - 2\sigma\sqrt{t}z + \sigma^2 t}{2} + \sigma^2 t/2\right) dz \quad (29)$$

$$= \frac{S \exp((r^d - r^f)t)}{\sqrt{2\pi}} \int_{\bar{z}}^{\infty} \exp\left(-\frac{(z - \sigma\sqrt{t})^2}{2}\right) dz \quad (30)$$

$$= \frac{S \exp((r^d - r^f)t)}{\sqrt{2\pi}} \int_{\bar{z} - \sigma\sqrt{t}}^{\infty} e^{-z^2/2} dz \quad (31)$$

$$= \frac{S \exp((r^d - r^f)t)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 1_{\bar{z} - \sigma\sqrt{t} > 0} e^{-z^2/2} dz \quad (32)$$

$$= S e^{(r^d - r^f)t} \mathbb{P}[1_{\bar{z} - \sigma\sqrt{t} > 0}] \quad (33)$$

$$= S e^{(r^d - r^f)t} \Phi\left(-(\bar{z} - \sigma\sqrt{t})\right) \quad (34)$$

$$= S e^{(r^d - r^f)t} \Phi\left(\frac{\ln(S/K) + (r^d - r^f + \sigma^2/2)t}{\sigma\sqrt{t}}\right) \quad (35)$$

To simplify subsequent notation,

$$d_1 := \frac{\ln(S/K) + (r^d - r^f + \sigma^2/2)t}{\sigma\sqrt{t}} \quad (36)$$

$$d_2 := \frac{\ln(S/K) + (r^d - r^f - \sigma^2/2)t}{\sigma\sqrt{t}} \quad (37)$$

Substituting the results of equations 25 and 35 in equation 18,

$$\mathbb{E}[(S_t - K)_+] = S e^{(r^d - r^f)t} \Phi(d_1) - K \Phi(d_2) \quad (38)$$

Substituting equation 38 in equation 13,

$$C_0 = S e^{-r^f t} \Phi(d_1) - K e^{-r^d t} \Phi(d_2) \quad (39)$$

1.2 Using two numeraires

Following the previous section, the process $A_t = e^{r^f t} S_t$ is a martingale in the B_t^d numeraire if $\mu = r^d - r^f$. Therefore, the price of the call option in the B_d numeraire is given by:

$$C_0 = \mathbb{E}^{B^d} \left[B_0^d \frac{C_t}{B_t^d} \right] \quad (40)$$

$$\therefore C_0 = \mathbb{E}^{B^d} \left[B_0^d \frac{(S_t - K)_+}{B_t^d} \right] \quad (41)$$

$$= \mathbb{E}^{B^d} \left[B_0^d \frac{(S_t - K) 1_{S_t > K}}{B_t^d} \right] \quad (42)$$

$$= \mathbb{E}^{B^d} \left[B_0^d \frac{S_t 1_{S_t > K}}{B_t^d} \right] - K \mathbb{E}^{B^d} \left[B_0^d \frac{1_{S_t > K}}{B_t^d} \right] \quad (43)$$

$$= \mathbb{E}^A \left[A_0 \frac{S_t 1_{S_t > K}}{A_t} \right] - K \mathbb{E}^{B^d} \left[B_0^d \frac{1_{S_t > K}}{B_t^d} \right] \quad (44)$$

$$= S e^{-r^f t} \mathbb{E}^A [1_{S_t > K}] - K e^{-r^d t} \mathbb{E}^{B^d} [1_{S_t > K}] \quad (45)$$

The evaluation of the second term is identical to that in the previous section. Recalling equation 25,

$$\mathbb{E}^{B^d} [1_{S_t > K}] = \Phi \left(\frac{\ln(S/K) + (r^d - r^f - \sigma^2/2)t}{\sigma\sqrt{t}} \right) \quad (46)$$

To evaluate the expectation of the first term, we need to change the measure of process S_t so that it is a martingale with respect to numeraire A . This is straightforward to evaluate.

$$\mu^A = \mu^{B^d} - S \sigma^S \rho_{A,S} \left(\frac{\sigma^{B^d}}{B^d} - \frac{\sigma^A}{A} \right) \quad (47)$$

$$= S(r^d - r^f) - S \sigma \left(-\frac{e^{r^f t} S \sigma}{e^{r^f t} S} \right) \quad (48)$$

$$= S(r^d - r^f + \sigma^2) \quad (49)$$

Following the arguments of the previous subsection,

$$\mathbb{E}^A [1_{S_t > K}] = \Phi \left(\frac{\ln(S/K) + (r^d - r^f + \sigma^2/2)t}{\sigma\sqrt{t}} \right) \quad (50)$$

Combining equations 46 and 50 with equation 45,

$$C_0 = S e^{-r^f t} \Phi(d_1) - K e^{-r^d t} \Phi(d_2) \quad (51)$$

2 Solving the Black Scholes Equation

For a contract with value V , the general case of the Black Scholes equation is:

$$\frac{\partial V}{\partial t} + (r_t^d - r_t^f) S \frac{\partial V}{\partial S} + \frac{\sigma_{S,t}^2}{2} \frac{\partial^2 V}{\partial S^2} - r_t^d V = 0 \quad (52)$$

To solve this equation, it needs to be simplified using change of variables. Let V_0 be the value function in the original asset level.

$$V_0(S, t) := \exp\left(\int_0^t r_\tau^d d\tau\right) V_1(S, T) \quad (53)$$

$$\therefore \frac{\partial V_0}{\partial S} = \exp\left(\int_0^t r_\tau^d d\tau\right) \frac{\partial V_1}{\partial S} \quad (54)$$

$$\therefore \frac{\partial^2 V_0}{\partial S^2} = \exp\left(\int_0^t r_\tau^d d\tau\right) \frac{\partial^2 V_1}{\partial S^2} \quad (55)$$

$$\therefore \frac{\partial V_0}{\partial t} = \exp\left(\int_0^t r_\tau^d d\tau\right) \left\{ V_1 \frac{\partial}{\partial t} \int_0^t r_\tau^d d\tau + \frac{\partial V_1}{\partial t} \right\} \quad (56)$$

$$= \exp\left(\int_0^t r_\tau^d d\tau\right) \left\{ V_1 r_t^d + \frac{\partial V_1}{\partial t} \right\} \quad (57)$$

Substituting in equation 52,

$$\exp\left(\int_0^t r_\tau^d d\tau\right) \left\{ \frac{\partial V_1}{\partial t} + r_t^d V_1 + (r_t^d - r_t^f) S \frac{\partial V_1}{\partial S} + \frac{\sigma_{S,t}^2}{2} \frac{\partial^2 V_1}{\partial S^2} - r_t^d V_1 \right\} = 0 \quad (58)$$

$$\therefore \frac{\partial V_1}{\partial t} + (r_t^d - r_t^f) S \frac{\partial V_1}{\partial S} + \frac{\sigma_{S,t}^2}{2} \frac{\partial^2 V_1}{\partial S^2} = 0 \quad (59)$$

The second step is to remove the dependence of the coefficients of the Spot derivatives on the Spot.

$$x := \log(S) \quad (60)$$

$$V_1(S, t) := V_2(x, t) \quad (61)$$

$$\therefore \frac{\partial V_1}{\partial t} = \frac{\partial V_2}{\partial t} \quad (62)$$

$$\therefore \frac{\partial V_1}{\partial S} = \frac{\partial V_2}{\partial x} \frac{\partial x}{\partial S} = \frac{1}{S} \frac{\partial V_2}{\partial x} \quad (63)$$

$$\therefore \frac{\partial^2 V_1}{\partial S^2} = \frac{\partial}{\partial S} \left\{ \frac{1}{S} \frac{\partial V_2}{\partial x} \right\} = \frac{-1}{S^2} \frac{\partial V_2}{\partial x} + \frac{1}{S^2} \frac{\partial^2 V_2}{\partial x^2} \quad (64)$$

Substituting in equation 68 and collecting the terms,

$$\frac{\partial V_2}{\partial t} + \left(r_\tau^d - r_\tau^f - \frac{\sigma_{x,\tau}^2}{2} \right) \frac{\partial V_2}{\partial x} + \frac{\sigma_{x,\tau}^2}{2} \frac{\partial^2 V_2}{\partial x^2} = 0 \quad (65)$$

Since the boundary conditions are specified at $t = T$,

$$\tau := T - t \quad (66)$$

$$V_2(x, t) := V_3(x, \tau) \quad (67)$$

By inspection, equation 65 can be rewritten as:

$$\frac{\partial V_3}{\partial \tau} = \left(r_\tau^d - r_\tau^f - \frac{\sigma_{x,\tau}^2}{2} \right) \frac{\partial V_3}{\partial x} + \frac{\sigma_{x,\tau}^2}{2} \frac{\partial^2 V_3}{\partial x^2} = 0 \quad (68)$$

Under the assumption that volatility is independent of strike (and therefore of asset level),

$$y := x + \int_0^\tau (r_\theta^d - r_\theta^f - \frac{\sigma_\theta^2}{2}) d\theta \quad (69)$$

$$V_3(x, \tau) := V_4(y, \tau) \quad (70)$$

$$\therefore \frac{\partial V_3}{\partial \tau} = \frac{\partial V_4}{\partial \tau} + \frac{\partial V_4}{\partial y} \frac{\partial y}{\partial \tau} \quad (71)$$

$$= \frac{\partial V_4}{\partial \tau} + \frac{\partial V_4}{\partial y} \frac{\partial}{\partial \tau} \left\{ \int_0^\tau (r_\theta^d - r_\theta^f - \frac{\sigma_\theta^2}{2}) d\theta \right\} \quad (72)$$

$$= \frac{\partial V_4}{\partial \tau} + \frac{\partial V_4}{\partial y} (r_\tau^d - r_\tau^f - \frac{\sigma_\tau^2}{2}) \quad (73)$$

$$\therefore \frac{\partial V_3}{\partial x} = \frac{\partial V_4}{\partial y} \frac{\partial y}{\partial x} = \frac{\partial V_4}{\partial y} \quad (74)$$

$$\therefore \frac{\partial^2 V_3}{\partial x^2} = \frac{\partial^2 V_4}{\partial y^2} \frac{\partial y}{\partial x} = \frac{\partial^2 V_4}{\partial y^2} \quad (75)$$

Substituting in equation 65

$$\frac{\partial V_4}{\partial \tau} + \frac{\partial V_4}{\partial y} (r_\tau^d - r_\tau^f - \frac{\sigma_\tau^2}{2}) = (r_\tau^d - r_\tau^f - \frac{\sigma_\tau^2}{2}) \frac{\partial V_4}{\partial y} + \frac{\sigma_\tau^2}{2} \frac{\partial^2 V_4}{\partial y^2} \quad (76)$$

$$\therefore \frac{\partial V_4}{\partial \tau} = \frac{\sigma_\tau^2}{2} \frac{\partial^2 V_4}{\partial y^2} \quad (77)$$

The equation in this form is amenable to similarity reduction, which essentially means that the partial differential equation in two variables can be reduced to an ordinary differential equation in one variable. To motivate the tedious algebra that will follow, the approach is to derive a solution that satisfies equation 77. This solution should be a delta function at $\tau \rightarrow 0$, where the payoff is specified. By the linearity of equation 77, this delta function can be convolved with the payoff to compute the price in the transformed coordinates. The coordinate transformations can then be inverted to get the price in the original coordinates.

$$\gamma_\tau := \int_0^\tau \sigma_\theta^2 d\theta \quad (78)$$

$$\therefore \frac{d\gamma_\tau}{d\tau} = \sigma_\tau^2 \quad (79)$$

Let $f(\xi)$ be a function with the following properties:

1. It is defined, smooth and bounded for $-\infty < \xi < \infty$
2. It is non-negative for all values
3. The total area of the function is finite i.e. $\int_{-\infty}^\infty f(\xi) d\xi < \infty$
4. The limit as ξ approaches infinity is zero i.e. $\lim_{\xi \rightarrow \pm\infty} f(\xi) = 0$

For constants α , β and \bar{y} , let $W(y, \tau)$ be a solution of equation 77, unconstrained by the payoff boundary condition.

$$W(y, \tau; \bar{y}) := \gamma_\tau^\alpha f\left(\frac{y - \bar{y}}{\gamma_\tau^\beta}\right) \quad (80)$$

$$\eta := \frac{y - \bar{y}}{\gamma_\tau^\beta} \quad (81)$$

Substituting equation 80 in equation 77,

$$\alpha \gamma_\tau^{\alpha-1} \frac{d\gamma_\tau}{d\tau} f(\eta) - \beta \gamma_\tau^\alpha \frac{y - \bar{y}}{\gamma_\tau^{\beta+1}} \frac{d\gamma_\tau}{d\tau} \frac{df(\eta)}{d\eta} = \frac{\sigma_\tau^2}{2} \gamma_\tau^{\alpha-2\beta} \frac{d^2 f(\eta)}{d\eta} \quad (82)$$

$$\gamma_\tau^{\alpha-1} \frac{d\gamma_\tau}{d\tau} \left(\alpha f(\eta) - \beta \eta \frac{df(\eta)}{d\eta} \right) = \frac{\sigma_\tau^2}{2} \gamma_\tau^{\alpha-2\beta} \frac{d^2 f(\eta)}{d\eta} \quad (83)$$

$$\therefore \gamma_\tau^{\alpha-1} \left(\alpha f(\eta) - \beta \eta \frac{df(\eta)}{d\eta} \right) = \frac{\gamma_\tau^{\alpha-2\beta}}{2} \frac{d^2 f(\eta)}{d\eta} \quad (84)$$

Since $f(\xi)$ is an arbitrary function, the exponents of γ_τ on both sides must be the same.

$$\therefore \alpha - 1 = \alpha - 2\beta \quad (85)$$

$$\therefore \beta = \frac{1}{2} \quad (86)$$

Since we want the solution to be a delta function as $\tau \rightarrow 0$, its integral over the whole domain should be independent of τ

$$\int_{-\infty}^{\infty} \gamma_\tau^\alpha f\left(\frac{y - \bar{y}}{\gamma_\tau^\beta}\right) dy = \int_{-\infty}^{\infty} \gamma_\tau^{\alpha+\beta} f(\eta) d\eta \quad (87)$$

$$\therefore \alpha + \beta = 0 \quad (88)$$

$$\therefore \alpha = -\frac{1}{2} \quad (89)$$

Therefore, equation 84 can be written as

$$-f(\eta) - \eta \frac{df(\eta)}{d\eta} = \frac{d^2 f(\eta)}{d\eta^2} \quad (90)$$

$$\therefore \frac{d(\eta f(\eta))}{d\eta} + \frac{d^2 f(\eta)}{d\eta^2} = 0 \quad (91)$$

$$\therefore \eta f(\eta) + \frac{df(\eta)}{d\eta} = a \quad (92)$$

$$(93)$$

Since a is arbitrary, it can be set to 0 for the specific solution

$$\therefore \frac{df}{f} = -\eta d\eta \quad (94)$$

$$\therefore f(\eta) = be^{-\frac{\eta^2}{2}} \quad (95)$$

Since the area of $f(\xi)$ is finite, it can be set to 1 to fix b . By inspection and comparison with the normal density function,

$$f(\eta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\eta^2}{2}} \quad (96)$$

$$\therefore W(y, \tau; \bar{y}) = \frac{1}{\sqrt{2\pi\gamma_\tau}} \exp \left\{ -\frac{1}{2} \left(\frac{y - \bar{y}}{\sqrt{\gamma_\tau}} \right)^2 \right\} \quad (97)$$

As $\tau \rightarrow 0$, $\gamma_\tau \rightarrow 0$ and $W(y, \tau; \bar{y})$ becomes a delta function centered at \bar{y} . Therefore, for an arbitrary payoff $h(y)$ defined at $\tau = 0$,

$$\lim_{\tau \rightarrow 0} V_4(y, \tau) = \lim_{\tau \rightarrow 0} \int_{-\infty}^{\infty} h(\bar{y}) W(\bar{y}; y, \tau) d\bar{y} \quad (98)$$

Before solving the system in the original coordinates, it is useful to add some more notation.

$$\bar{r}^d := \frac{1}{T} \int_0^T r_\theta^d d\theta \quad (99)$$

$$\bar{r}^f := \frac{1}{T} \int_0^T r_\theta^f d\theta \quad (100)$$

$$\bar{\sigma} := \sqrt{\frac{1}{T} \int_0^T \sigma_\theta^2 d\theta} \quad (101)$$

With this notation, the transformations can be rewritten as:

$$V_0(S, t) = e^{\bar{r}^d T} V_1(S, t) \quad (102)$$

$$y = x + \left(\bar{r}^d - \bar{r}^f - \frac{\bar{\sigma}^2}{2} \right) \tau \quad (103)$$

$$\gamma_\tau = \bar{\sigma}^2 \tau \quad (104)$$

$$W(y, \tau, \bar{y}) = \frac{1}{\sqrt{2\pi\tau\bar{\sigma}}} \exp \left\{ -\frac{1}{2} \left(\frac{y - \bar{y}}{\bar{\sigma}\sqrt{\tau}} \right)^2 \right\} \quad (105)$$

$$\therefore \lim_{\tau \rightarrow 0} V_4(y, \tau) = \frac{1}{\sqrt{2\pi\tau\bar{\sigma}}} \lim_{\tau \rightarrow 0} \int_{-\infty}^{\infty} h(\bar{y}) \exp \left\{ -\frac{1}{2} \left(\frac{y - \bar{y}}{\bar{\sigma}\sqrt{\tau}} \right)^2 \right\} d\bar{y} \quad (106)$$

$$\therefore \lim_{\tau \rightarrow 0} V_3(x, \tau) = \frac{1}{\sqrt{2\pi\tau\bar{\sigma}}} \lim_{\tau \rightarrow 0} \int_{-\infty}^{\infty} h(\bar{y}) \exp \left\{ -\frac{1}{2} \left(\frac{x + (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)\tau - \bar{y}}{\bar{\sigma}\sqrt{\tau}} \right)^2 \right\} d\bar{y} \quad (107)$$

$$\therefore V_2(x) = \frac{1}{\sqrt{2\pi T\bar{\sigma}}} \int_{-\infty}^{\infty} h(\bar{y}) \exp \left\{ -\frac{1}{2} \left(\frac{x + (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)T - \bar{y}}{\bar{\sigma}\sqrt{T}} \right)^2 \right\} d\bar{y} \quad (108)$$

Let \bar{y} be the terminal logspot.

$$\therefore h(\bar{y}) = (e^{\bar{y}} - K)1_{(e^{\bar{y}} - K)} \quad (109)$$

$$\begin{aligned} \therefore V_2(x) &= \frac{1}{\sqrt{2\pi T}\bar{\sigma}} \int_{\log(K)}^{\infty} \exp \left\{ \bar{y} - \frac{1}{2} \left(\frac{x + (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)T - \bar{y}}{\bar{\sigma}\sqrt{T}} \right)^2 \right\} d\bar{y} \\ &\quad - \frac{K}{\sqrt{2\pi T}\bar{\sigma}} \int_{\log(K)}^{\infty} \exp \left\{ -\frac{1}{2} \left(\frac{x + (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)T - \bar{y}}{\bar{\sigma}\sqrt{T}} \right)^2 \right\} d\bar{y} \end{aligned} \quad (110)$$

To simplify the notation,

$$I_1 = \frac{1}{\sqrt{2\pi T}\bar{\sigma}} \int_{\log(K)}^{\infty} \exp \left\{ \bar{y} - \frac{1}{2} \left(\frac{x + (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)T - \bar{y}}{\bar{\sigma}\sqrt{T}} \right)^2 \right\} d\bar{y} \quad (111)$$

$$I_2 := \frac{K}{\sqrt{2\pi T}\bar{\sigma}} \int_{\log(K)}^{\infty} \exp \left\{ -\frac{1}{2} \left(\frac{x + (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)T - \bar{y}}{\bar{\sigma}\sqrt{T}} \right)^2 \right\} d\bar{y} \quad (112)$$

Considering the second integral,

$$z_1 := \frac{x + (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)T - \bar{y}}{\bar{\sigma}\sqrt{T}} \quad (113)$$

$$\therefore dz_1 = -\frac{d\bar{y}}{\bar{\sigma}\sqrt{T}} \quad (114)$$

$$\therefore I_2 = \frac{K}{\sqrt{2\pi T}\bar{\sigma}} \int_{-\infty}^{(x - \log K + (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)T)/\bar{\sigma}\sqrt{T}} \bar{\sigma}\sqrt{T} e^{-z_1^2/2} dz_1 \quad (115)$$

$$= K\Phi \left(\frac{\log(x/K) + (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)T}{\bar{\sigma}\sqrt{T}} \right) \quad (116)$$

$$d_2 := \frac{\log(S/K) + (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)T}{\bar{\sigma}\sqrt{T}} \quad (117)$$

$$\therefore I_2 = K\Phi(d_2) \quad (118)$$

Considering the first integral,

$$\bar{\mu} := (\bar{r}^d - \bar{r}^f - \bar{\sigma}^2/2)T \quad (119)$$

$$I_1 = \frac{1}{\sqrt{2\pi T}\bar{\sigma}} \int_{\log(K)}^{\infty} \exp \left\{ \bar{y} - \frac{1}{2} \left(\frac{x + \bar{\mu} - \bar{y}}{\bar{\sigma}\sqrt{T}} \right)^2 \right\} d\bar{y} \quad (120)$$

$$= \frac{1}{\sqrt{2\pi T}\bar{\sigma}} \int_{\log(K)}^{\infty} \exp \left\{ -\frac{(x + \bar{\mu})^2 + \bar{y}^2 - 2\bar{y}(x + \bar{\mu}) - 2\bar{\sigma}^2 T}{2\bar{\sigma}^2 T} \right\} d\bar{y} \quad (121)$$

$$= \frac{1}{\sqrt{2\pi T}\bar{\sigma}} \int_{\log(K)}^{\infty} \exp \left\{ -\frac{(x + \bar{\mu} + \bar{\sigma}^2 T - \bar{y})^2 + (x + \bar{\mu})^2 - (x + \bar{\mu} + \bar{\sigma}^2 T)^2}{2\bar{\sigma}^2 T} \right\} d\bar{y} \quad (122)$$

$$= \frac{1}{\sqrt{2\pi T}\bar{\sigma}} \int_{\log(K)}^{\infty} \exp \left\{ -\frac{(x + \bar{\mu} + \bar{\sigma}^2 T - \bar{y})^2 - (2x + 2\bar{\mu} + \bar{\sigma}^2 T)(\bar{\sigma}^2 T)}{2\bar{\sigma}^2 T} \right\} d\bar{y} \quad (123)$$

$$= e^{x + (\bar{r}^d - \bar{r}^f)T} \frac{1}{\sqrt{2\pi T}\bar{\sigma}} \int_{\log(K)}^{\infty} \exp \left\{ -\frac{(x + (\bar{r}^d - \bar{r}^f + \bar{\sigma}^2/2)T - \bar{y})^2}{2\bar{\sigma}^2 T} \right\} d\bar{y} \quad (124)$$

$$z_2 := \frac{x + (\bar{r}^d - \bar{r}^f + \bar{\sigma}^2/2)T - \bar{y}}{\bar{\sigma}\sqrt{T}} \quad (125)$$

$$\therefore dz_2 = -\frac{d\bar{y}}{\bar{\sigma}\sqrt{T}} \quad (126)$$

$$I_1 = e^{x+(\bar{r}^d-\bar{r}^f)T} \frac{1}{\sqrt{2\pi T}\bar{\sigma}} \int_{-\infty}^{(x-\log K+(\bar{r}^d-\bar{r}^f+\bar{\sigma}^2/2)T)/\bar{\sigma}\sqrt{T}} \bar{\sigma}\sqrt{T} e^{-z_2^2/2} dz_1 \quad (127)$$

$$= e^{x+(\bar{r}^d-\bar{r}^f)T} \Phi\left(\frac{x - \log K + (\bar{r}^d - \bar{r}^f + \bar{\sigma}^2/2)T}{\bar{\sigma}\sqrt{T}}\right) \quad (128)$$

$$d_1 := \frac{\log(S/K) + (\bar{r}^d - \bar{r}^f + \bar{\sigma}^2/2)T}{\bar{\sigma}\sqrt{T}} \quad (129)$$

$$\therefore I_1 = e^{x+(\bar{r}^d-\bar{r}^f)T} \Phi(d_1) \quad (130)$$

$$\therefore V_2(x) = e^{x+(\bar{r}^d-\bar{r}^f)T} \Phi(d_1) - K\Phi(d_2) \quad (131)$$

$$\therefore V_1(S) = S e^{(\bar{r}^d-\bar{r}^f)T} \Phi(d_1) - K\Phi(d_2) \quad (132)$$

$$\therefore V_0(S) = S e^{-\bar{r}^f T} \Phi(d_1) - K e^{-\bar{r}^d T} \Phi(d_2) \quad (133)$$