

Deriving the Black-Scholes Equation

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This monograph presents multiple ways of deriving the Black-Scholes equation with one stochastic variable. The specific case considered is for foreign exchange derivatives, but the mathematical and financial arguments are identical for equity and commodity derivatives. Foreign exchange was chosen because the exchange rate itself is not a tradeable asset, and this nuance is not handled properly by standard textbooks. The assumptions are:

1. The underlying asset follows a lognormal process. The only constraint on the drift and volatility is that they are deterministic
2. A replicating portfolio can be constructed consisting of the asset and the derivative
3. Transaction costs are zero in both directions for all notionals
4. The interest rate to borrow and lend are the same
5. The participants have sufficient credit worthiness that the interest rate is independent of the notional

To make the notation less verbose, the subscript of time is eliminated for all variables, however there is no constraint on the parameters being constant across time. This derivation therefore subsumes that of *Local Volatility* pricing.

1 Black-Scholes Equation in Spot

Since the exchange rate is not a tradeable asset, the universe consists of a domestic money market account B^d , a foreign money market account, B^f and a stochastic exchange rate rate S , i.e. one unit in the foreign money market account is worth S units in the domestic money market account. S follows a lognormal stochastic differential equation. Therefore,

$$dB^d = r^d B^d dt \quad (1)$$

$$dB^f = r^f B^f dt \quad (2)$$

$$dS = S(\mu dt + \sigma dW_t) \quad (3)$$

In this universe, a claim $V(S, t)$ is to be priced in the domestic currency.

1.1 Using Portfolio Replication

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 \quad (4)$$

$$= \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right] dt + \sigma S \frac{\partial V}{\partial S} dW_t \quad (5)$$

To eliminate the risk, a portfolio consisting of 1 unit of the contingent claim and Δ units of the foreign money market account will be built.

$$\Pi = V - \Delta S B^f \quad (6)$$

$$d\Pi = dV - \Delta (B^f dS + S dB^f) \quad (7)$$

$$= dV - \Delta [B^f S (\mu dt + \sigma dW_t) + S B^f r^f dt] \quad (8)$$

$$= dV - \Delta B^f S [(r^f + \mu) dt + \sigma dW_t] \quad (9)$$

$$= \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - \Delta S B^f (r^f + \mu) \right] dt + \left[\sigma \frac{\partial V}{\partial S} - \Delta B^f \sigma \right] S dW_t \quad (10)$$

The condition to make this portfolio risk free is

$$\Delta = \frac{1}{B^f} \frac{\partial V}{\partial S} \quad (11)$$

Therefore, the growth rate of the portfolio is

$$\frac{d\Pi}{dt} = \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - \frac{1}{B^f} \frac{\partial V}{\partial S} S B^f (r^f + \mu) \quad (12)$$

$$= \frac{\partial V}{\partial t} - S r^f \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \quad (13)$$

Since the portfolio is risk-neutral, it grows at rate of the domestic money market account

$$\frac{d\Pi}{dt} = r^d \Pi dt \quad (14)$$

$$= r^d \left[V - \frac{1}{B^f} \frac{\partial V}{\partial S} B^f S \right] \quad (15)$$

Equating the two sides,

$$\frac{\partial V}{\partial t} - S r^f \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} = r^d \left[V - \frac{\partial V}{\partial S} S \right] \quad (16)$$

$$\therefore \frac{\partial V}{\partial t} + (r^d - r^f) S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - r^d V = 0 \quad (17)$$

1.2 Using Martingale Pricing

The claim $V(S, t)$ will be priced in the domestic money market account numeraire. In this case, the tradeable asset is the foreign money market account denominated in the domestic currency. The first step is to find a measure in which the tradeable asset is a martingale in the numeraire.

$$A_t := S_t e^{r^f t} \quad (18)$$

$$d\left(\frac{A_t}{B_t}\right) = d\left(\exp((r^f - r^d)t)S_t\right) \quad (19)$$

$$= \exp((r^f - r^d)t)dS_t + S_t d(\exp((r^f - r^d)t)) \quad (20)$$

$$= \exp((r^f - r^d)t)S_t\{(\mu + r^f - r^d)dt + \sigma dW_t\} \quad (21)$$

Therefore A_t is a martingale if $\mu = r^d - r^f$. Since we want $V(S, t)$ to be a martingale in the domestic money market numeraire, its drift should be zero. The assumption of constant drift and volatility (equal to their terminal values), is valid if the payoff is only dependent on the terminal value of the process. While this was true for A_t , it is, in general, not true for an arbitrary claim $V(S, t)$. Therefore, to explicitly account for the term structure of the rates, we will reintroduce the time subscripts.

$$d\left(\frac{V(S, t)}{B_t^d}\right) = d\left[\frac{V(S, t)}{\exp\int_0^t r_\tau^d d\tau}\right] \quad (22)$$

$$= d\left[\exp\left(-\int_0^t r_\tau d\tau\right)V(S, t)\right] \quad (23)$$

$$= d\left(\exp - \int_0^t r_\tau d\tau\right)V(S, t) + \left(\exp - \int_0^t r_\tau d\tau\right)d(V(S, t)) \quad (24)$$

$$= B_t^d \left[-r_t^d V dt + \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 \right] \quad (25)$$

$$= B_t^d \left[\left\{ \frac{\partial V}{\partial t} + (r_t^d - r_t^f) \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - r_t^d V \right\} dt + (\dots) dW_t \right] \quad (26)$$

$$\therefore \frac{\partial V}{\partial t} + (r^d - r^f)S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - r^d V = 0 \quad (27)$$

2 Black-Scholes in LogSpot

The Black-Scholes equation is non-linear in the asset value. This makes pricing using numerical methods tricky. To simplify these cases, the equation is often expressed in logspot coordinates.

$$X := \ln(S) \quad (28)$$

$$V(S, t) := \bar{V}(X, t) \quad (29)$$

$$\frac{\partial X}{\partial S} = \frac{1}{S} \quad (30)$$

$$\frac{\partial^2 X}{\partial S^2} = -\frac{1}{S^2} \quad (31)$$

$$\frac{\partial V}{\partial t} = \frac{\partial \bar{V}}{\partial t} \quad (32)$$

$$\frac{\partial V}{\partial S} = \frac{\partial \bar{V}}{\partial X} \frac{\partial X}{\partial S} \quad (33)$$

$$= \frac{1}{S} \frac{\partial \bar{V}}{\partial X} \quad (34)$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 \bar{V}}{\partial X^2} \left(\frac{\partial X}{\partial S} \right)^2 + \frac{\partial \bar{V}}{\partial X} \frac{\partial^2 X}{\partial S^2} \quad (35)$$

$$= \frac{1}{S^2} \left[\frac{\partial^2 \bar{V}}{\partial X^2} - \frac{\partial \bar{V}}{\partial X} \right] \quad (36)$$

Substituting in the Black Scholes formula,

$$\frac{\partial \bar{V}}{\partial T} + (r^d - r^f) S \frac{1}{S} \frac{\partial \bar{V}}{\partial X} + \frac{\sigma^2 S^2}{2} \frac{1}{S^2} \left[\frac{\partial^2 \bar{V}}{\partial X^2} - \frac{\partial \bar{V}}{\partial X} \right] - r^d \bar{V} = 0 \quad (37)$$

$$\therefore \frac{\partial \bar{V}}{\partial t} + \left(r^d - r^f - \frac{\sigma^2}{2} \right) \frac{\partial \bar{V}}{\partial X} + \frac{\sigma^2}{2} \frac{\partial^2 \bar{V}}{\partial X^2} - r^d \bar{V} = 0 \quad (38)$$