

Solving the Black Scholes Equation using Finite Differences

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This monograph illustrates how the Black Scholes PDE can be solved using finite differences on a rectangular grid.

1 Equation Setup

The coefficients of derivatives of the asset value are independent of the spot level if the PDE is expressed in logspot coordinates. This improves the numerical stability of the scheme and is therefore the case considered here. The Black Scholes PDE expressed in logspot is:

$$\frac{\partial V}{\partial t} + \left(r^d - r^f - \frac{\sigma^2}{2} \right) \frac{\partial V}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} - r^d V = 0 \quad (1)$$

To simplify the notation, the dependence of the rates on time and the dependence of local volatility on logspot and time are assumed implicit. Since the payoff is specified at the terminal time T , the PDE needs to be solved backwards in time. Let $\tau = T - t$. To simplify subsequent notation, let $\mu = r^d - r^f$. Therefore, the equation in time-reversed coordinates is:

$$\frac{\partial V}{\partial \tau} = \left(\mu - \frac{\sigma^2}{2} \right) \frac{\partial V}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} - r^d V \quad (2)$$

Let m be the index along the logspot axis and n be the index along the (reversed) time axis. If the domain is discretized into \bar{m} points along the logspot axis and \bar{n} points along the time axis,

$$m \in [0, \dots, \bar{m} - 1] \quad (3)$$

$$n \in [0, \dots, \bar{n} - 1] \quad (4)$$

$$\Delta\tau_i := \tau_{i+1} - \tau_i \quad (5)$$

$$h_i = h_{i+1} - h_m \quad (6)$$

2 Logspot Derivatives on a Non Uniform Grid

The first step is to derive an expression for approximations of the partial derivatives of the asset value with respect to logspot. Ignoring the time index of the payoff,

$$V_{m+1} \approx V_m + h_m \frac{\partial V_m}{\partial x_m} + \frac{h_m^2}{2} \frac{\partial^2 V_m}{\partial x_m^2} \quad (7)$$

$$V_{m-1} \approx V_m - h_{m-1} \frac{\partial V_m}{\partial x_m} + \frac{h_{m-1}^2}{2} \frac{\partial^2 V_m}{\partial x_m^2} \quad (8)$$

Rearranging and ignoring the errors due to the approximation,

$$\begin{aligned} & \begin{bmatrix} h_m & h_m^2/2 \\ -h_{m-1} & h_{m-1}^2/2 \end{bmatrix} \begin{bmatrix} \partial V_m / \partial x_m \\ \partial^2 V_m / \partial x_m^2 \end{bmatrix} = \begin{bmatrix} V_{m+1} - V_m \\ V_{m-1} - V_m \end{bmatrix} \\ \therefore & \begin{bmatrix} \partial V_m / \partial x_m \\ \partial^2 V_m / \partial x_m^2 \end{bmatrix} = \frac{2}{h_m h_{m-1} (h_m + h_{m-1})} \begin{bmatrix} h_{m-1}^2/2 & -h_m^2/2 \\ h_{m-1} & h_m \end{bmatrix} \begin{bmatrix} V_{m+1} - V_m \\ V_{m-1} - V_m \end{bmatrix} \end{aligned}$$

$$\therefore \frac{\partial V_m}{\partial x} = -\frac{h_m}{h_{m-1}(h_m + h_{m-1})} V_{m-1} + \frac{h_m - h_{m-1}}{h_m h_{m-1}} V_m + \frac{h_{m-1}}{h_m(h_m + h_{m-1})} V_{m+1} \quad (9)$$

$$\therefore \frac{\partial^2 V_m}{\partial x^2} = \frac{2}{h_{m-1}(h_m + h_{m-1})} V_{m-1} - \frac{2}{h_m h_{m-1}} V_m + \frac{2}{h_m(h_m + h_{m-1})} V_{m+1} \quad (10)$$

To simplify subsequent notation,

$$\alpha_{1,m} := \frac{-h_m}{h_{m-1}(h_m + h_{m-1})} \quad (11)$$

$$\alpha_{2,m} := \frac{h_m - h_{m-1}}{h_m h_{m-1}} = \frac{1}{h_{m-1}} - \frac{1}{h_m} \quad (12)$$

$$\alpha_{3,m} := \frac{h_{m-1}}{h_m(h_m + h_{m-1})} \quad (13)$$

$$\alpha_{4,m} := \frac{2}{h_{m-1}(h_m + h_{m-1})} \quad (14)$$

$$\alpha_{5,m} := \frac{-2}{h_m h_{m-1}} \quad (15)$$

$$\alpha_{6,m} := \frac{2}{h_m(h_m + h_{m-1})} \quad (16)$$

$$\beta_{1,m} := \frac{\alpha_{4,m} - \alpha_{1,m}}{2} \quad (17)$$

$$\beta_{2,m} := \frac{\alpha_{5,m} - \alpha_{2,m}}{2} \quad (18)$$

$$\beta_{3,m} := \frac{\alpha_{6,m} - \alpha_{3,m}}{2} \quad (19)$$

$$(20)$$

Therefore right hand side of equation 2 can be written as:

$$\begin{aligned} & V_{m-1} \left(\mu_n \alpha_{1,m} + \frac{\alpha_{4,m} - \alpha_{1,m}}{2} \sigma_{m,n}^2 \right) + V_m \left(\mu_n \alpha_{2,m} + \frac{\alpha_{5,m} - \alpha_{2,m}}{2} \sigma_{m,n}^2 - r_n^d \right) + \\ & V_{m+1} \left(\mu_n \alpha_{3,m} + \frac{\alpha_{6,m} - \alpha_{3,m}}{2} \sigma_{m,n}^2 \right) \end{aligned} \quad (21)$$

$$= V_{m-1} \left(\mu_n \alpha_{1,m} + \sigma_{m,n}^2 \beta_{1,m} \right) + V_m \left(\mu_n \alpha_{2,m} + \sigma_{m,n}^2 \beta_{2,m} - r_n^d \right) + V_{m+1} \left(\mu_n \alpha_{3,m} + \sigma_{m,n}^2 \beta_{3,m} \right) \quad (22)$$

3 Implementing the Time Step

Equation 22 can be constructed with respect to payoff at the current timestep (called the explicit step scheme), with respect to the payoff at the next timestep (called the implicit scheme), or a linear combination of the two. In general,

$$\begin{aligned} \frac{V_m^{n+1} - V_m^n}{\Delta \tau_n} &= (\mu_n \alpha_{1,m} + \sigma_{m,n}^2 \beta_{1,m}) [(1 - \theta) V_{m-1}^n + \theta V_{m-1}^{n+1}] \\ &\quad + (\mu_n \alpha_{2,m} + \sigma_{m,n}^2 \beta_{2,m} - r_n^d) [(1 - \theta) V_m^n + \theta V_m^{n+1}] \\ &\quad + (\mu_n \alpha_{3,m} + \sigma_{m,n}^2 \beta_{3,m}) [(1 - \theta) V_{m+1}^n + \theta V_{m+1}^{n+1}] \end{aligned} \quad (23)$$

To simplify the notation,

$$\gamma_{1,m} := (\mu_n \alpha_{1,m} + \sigma_{m,n}^2 \beta_{1,m}) \Delta \tau_n \quad (24)$$

$$\gamma_{2,m} := (\mu_n \alpha_{2,m} + \sigma_{m,n}^2 \beta_{2,m} - r_n^d) \Delta \tau_n \quad (25)$$

$$\gamma_{3,m} := (\mu_n \alpha_{3,m} + \sigma_{m,n}^2 \beta_{3,m}) \Delta \tau_n \quad (26)$$

$$\begin{aligned} \therefore V_m^{n+1} - V_m^n &= \gamma_{1,m} [(1 - \theta) V_{m-1}^n + \theta V_{m-1}^{n+1}] + \gamma_{2,m} [(1 - \theta) V_m^n + \theta V_m^{n+1}] \\ &\quad + \gamma_{3,m} [(1 - \theta) V_{m+1}^n + \theta V_{m+1}^{n+1}] \end{aligned} \quad (27)$$

4 Boundary Conditions

The first boundary condition is the payoff at $\tau = 0$. This corresponds to the terminal payoff of the option. The second boundary condition is at x_0 and x_{m-1} for all τ . This could be:

1. A value, either constant or time dependent. For example, it would be zero for a double barrier
2. Zero gamma, which corresponds to a practical limit on the bounds of the grid, but not a trade specific limit

In the latter case, the value at the boundary needs to be stepped from the current value. Equation 2 with zero gamma is:

$$\frac{\partial V}{\partial \tau} = \mu \frac{\partial V}{\partial x} - r^d V \quad (28)$$

4.1 Zero Gamma at Lower Logspot Boundary

The logspot derivative is:

$$V_1 = V_0 + h_0 \frac{\partial V_0}{\partial x_0} \quad (29)$$

$$\therefore \frac{\partial V_0}{\partial x_0} = \frac{V_1 - V_0}{h_0} \quad (30)$$

Therefore, the right hand side of equation 28 is:

$$\mu_n \frac{V_1 - V_0}{h_0} - r_n^d V_0 = V_0 \left(-r_n^d - \frac{\mu_n}{h_0} \right) + \frac{\mu_n}{h_0} V_1 \quad (31)$$

Following the previous section, equation 31 can be constructed with respect to the payoff at the current timestep, the payoff at the next timestep, or a linear combination of both.

$$\frac{V_0^{n+1} - V_0^n}{\Delta \tau_n} = \left(-r_n^d - \frac{\mu_n}{h_0} \right) [(1 - \theta)V_0^n + \theta V_0^{n+1}] + \frac{\mu_n}{h_0} [(1 - \theta)V_1^n + \theta V_1^{n+1}] \quad (32)$$

To simplify the notation,

$$\gamma_4 := \left(-r_n^d - \frac{\mu_n}{h_0} \right) \Delta \tau_n \quad (33)$$

$$\gamma_5 := \frac{\mu_n \Delta \tau_n}{h_0} \quad (34)$$

$$\therefore V_0^{n+1} - V_0^n = \gamma_4 [(1 - \theta)V_0^n + \theta V_0^{n+1}] + \gamma_5 [(1 - \theta)V_1^n + \theta V_1^{n+1}] \quad (35)$$

4.2 Zero Gamma at Upper Logspot Boundary

The logspot derivative is:

$$V_{\bar{m}-2} = V_{\bar{m}-1} - h_{\bar{m}-2} \frac{\partial V_{\bar{m}-1}}{\partial x_{\bar{m}-1}} \quad (36)$$

$$\therefore \frac{\partial V_{\bar{m}-1}}{\partial x_{\bar{m}-1}} = \frac{V_{\bar{m}-1} - V_{\bar{m}-2}}{h_{\bar{m}-2}} \quad (37)$$

Therefore, the right hand side of equation 28 is:

$$\mu_n \frac{V_{\bar{m}-1} - V_{\bar{m}-2}}{h_{\bar{m}-2}} - r_n^d V_{\bar{m}-1} = \frac{-\mu_n}{h_{\bar{m}-2}} V_{\bar{m}-2} + V_{\bar{m}-1} \left(\frac{\mu_n}{h_{\bar{m}-2}} - r_n^d \right) \quad (38)$$

Following the previous section, equation 38 can be constructed with respect to the payoff at the current timestep, the payoff at the next timestep, or a linear combination of both.

$$\frac{V_{\bar{m}-1}^{n+1} - V_{\bar{m}-1}^n}{\Delta \tau_n} = \frac{-\mu_n}{h_{\bar{m}-2}} [(1 - \theta)V_{\bar{m}-2}^n + \theta V_{\bar{m}-2}^{n+1}] + \left(\frac{\mu_n}{h_{\bar{m}-2}} - r_n^d \right) [(1 - \theta)V_{\bar{m}-1}^n + \theta V_{\bar{m}-1}^{n+1}] \quad (39)$$

To simplify the notation,

$$\gamma_6 := \frac{-\mu_n \Delta \tau_n}{h_{\bar{m}-2}} \quad (40)$$

$$\gamma_7 := \left(\frac{\mu_n}{h_{\bar{m}-2}} - r_n^d \right) \Delta \tau_n \quad (41)$$

$$\therefore V_{\bar{m}-1}^{n+1} - V_{\bar{m}-1}^n = \gamma_6 [(1 - \theta)V_{\bar{m}-2}^n + \theta V_{\bar{m}-2}^{n+1}] + \gamma_7 [(1 - \theta)V_{\bar{m}-1}^n + \theta V_{\bar{m}-1}^{n+1}] \quad (42)$$

4.3 Grid Limits for Zero Gamma

Computing the exact limits on the logspot axis where the gamma of the payoff would be zero is too computationally intensive for use in a practical scheme. A useful heuristic is to use the limits where the gamma of a vanilla option would be *low*, for example 0.001. The Black-Scholes formula for the gamma of a call option is:

$$\Gamma = e^{-r_f t} \frac{\phi(d_1)}{S \sigma \sqrt{t}} \quad (43)$$

This formula, however, cannot be used without further complications:

1. The local volatility function will, in general, have a smile and strike dependence, which raises the question of which strike to use when computing the volatility.
2. A logspot solver will only have access to the local volatility function, while the formula is for the implied volatility

To sidestep these issues, there are three observations:

1. Gamma decreases with increase in volatility. The term $\phi(d_1)$ is at least an order of magnitude less than $1/\sigma$.
2. For non-pathological cases, the local volatility will be of roughly the order of magnitude as implied volatility
3. Gamma falls exponentially with distance from the forward

Therefore, a useful heuristic is to take the at-the-money-forward local volatility in formula 43. In this case,

$$\phi(d_1) = \Gamma S \sigma \sqrt{t} e^{r_f t} \quad (44)$$

$$\exp(-d_1^2/2) = \sqrt{2\pi} \Gamma S \sigma \sqrt{t} e^{r_f t} \quad (45)$$

$$-d_1^2/2 = r_f t + \log(\sqrt{2\pi} \Gamma S \sigma \sqrt{t}) \quad (46)$$

$$d_1 = \pm \sqrt{-2 \left(r_f t + \log(\sqrt{2\pi} \Gamma S \sigma \sqrt{t}) \right)} \quad (47)$$

$$\frac{\log F - \log K + \sigma^2 t/2}{\sigma \sqrt{t}} = \pm \sqrt{-2 \left(r_f t + \log \sqrt{2\pi} \Gamma + \log S + \log \sigma \sqrt{t} \right)} \quad (48)$$

$$\log K = \log F + \frac{\sigma^2 t}{2} \pm \sigma \sqrt{t} \sqrt{-2 \left(r_f t + \log \sqrt{2\pi} \Gamma + \log S + \log \sigma \sqrt{t} \right)} \quad (49)$$

$$\therefore \log K = \log F + \frac{\sigma^2 t}{2} \pm \sqrt{-\sigma^2 t \left(2r_f t + 2 \log \sqrt{2\pi} \Gamma + 2 \log S + \log \sigma^2 t \right)} \quad (50)$$

5 Solving the Time Step

Following the discussion in the previous sections, the evaluation of the payoff vector at the next timestep, V^{n+1} , can be expressed as the solution of a system of linear equations. As an illustrative example, consider the zero gamma case with $\bar{m} = 5$.

$$\begin{bmatrix} 1 - \theta\gamma_4 & -\theta\gamma_5 & 0 & 0 & 0 \\ -\theta\gamma_{1,1} & 1 - \theta\gamma_{2,1} & -\theta\gamma_{3,1} & 0 & 0 \\ 0 & -\theta\gamma_{1,2} & 1 - \theta\gamma_{2,2} & -\theta\gamma_{3,2} & 0 \\ 0 & 0 & -\theta\gamma_{1,3} & 1 - \theta\gamma_{2,3} & -\theta\gamma_{3,3} \\ 0 & 0 & 0 & -\theta\gamma_6 & 1 - \theta\gamma_7 \end{bmatrix} \begin{bmatrix} V_0^{n+1} \\ V_1^{n+1} \\ V_2^{n+1} \\ V_3^{n+1} \\ V_4^{n+1} \end{bmatrix} = \begin{bmatrix} 1 + (1 - \theta)\gamma_4 & (1 - \theta)\gamma_5 & 0 & 0 & 0 \\ (1 - \theta)\gamma_{1,1} & 1 + (1 - \theta)\gamma_{2,1} & (1 - \theta)\gamma_{3,1} & 0 & 0 \\ 0 & (1 - \theta)\gamma_{1,2} & 1 + (1 - \theta)\gamma_{2,2} & (1 - \theta)\gamma_{3,2} & 0 \\ 0 & 0 & (1 - \theta)\gamma_{1,3} & 1 + (1 - \theta)\gamma_{2,3} & (1 - \theta)\gamma_{3,3} \\ 0 & 0 & 0 & (1 - \theta)\gamma_6 & 1 + (1 - \theta)\gamma_7 \end{bmatrix} \begin{bmatrix} V_0^n \\ V_1^n \\ V_2^n \\ V_3^n \\ V_4^n \end{bmatrix} \quad (51)$$

By inspection, it is easy to observe that the system is a tridiagonal matrix and can easily be inverted using the Thomas algorithm. The first step is to diagonalize the system.

$$\delta_m := \begin{cases} 1 - \theta\gamma_4, & m = 0 \\ 1 - \theta\gamma_{2,m} + \theta\gamma_{1,m}\zeta_{m-1}, & m \in [1, \dots, \bar{m} - 2] \end{cases} \quad (52)$$

$$\zeta_m := \begin{cases} -\theta\gamma_5/\delta_0, & m = 0 \\ -\theta\gamma_{3,m}/\delta_m, & m \in [1, \dots, \bar{m} - 2] \end{cases} \quad (53)$$

$$\rho_m := \begin{cases} \{V_0^n + (1 - \theta)(\gamma_4 V_0^n + \gamma_5 V_1^n)\}/\delta_0, & m = 0 \\ \{V_m^n + (1 - \theta)(\gamma_{1,m} V_{m-1}^n + \gamma_{2,m} V_m^n + \gamma_{3,m} V_{m+1}^n) + \theta\gamma_{1,m}\rho_{m-1}\}/\delta_m, & m \in [1, \dots, \bar{m} - 2] \end{cases} \quad (54)$$

The next step is the reverse sweep using recursive substitution. Given the construction of ζ and ρ ,

$$V_{\bar{m}-1}^{n+1} = \frac{V_{\bar{m}-1}^n + (1 - \theta)(\gamma_6 V_{\bar{m}-2}^n + \gamma_7 V_{\bar{m}-1}^n) + \theta\gamma_6 \rho_{\bar{m}-2}}{1 - \theta\gamma_7 + \theta\gamma_6 \zeta_{\bar{m}-2}} \quad (55)$$

$$V_m^{n+1} = \rho_m - \zeta_m V_{m+1}^{n+1} \quad (56)$$

For the case where the value is specified at one of the terminal nodes, one fewer equation needs to be solved. In that case, the ζ and ρ values at that node are clearly zero. The rest of the system can be solved in the same way.