

Introduction to Stochastic Calculus

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Stochastic calculus models the evolution of continuous time random variables. This monograph introduces the subject and provides some useful results.

1 Wiener Process

Stochastic calculus models the evolution of random variables using *differential shocks*, which may be understood as tiny stochastic increments over time interval dt . This process is called the Wiener process and is denoted by W_t , with the increments denoted by dW_t . By definition,

$$W(t) := \int_0^t dW_t \quad (1)$$

The stochastic increments have four important properties:

1. dW_t is normally distributed with mean 0 and variance dt , i.e. $dW_t \sim \sqrt{dt}\mathcal{N}(0, 1)$.
2. The increment at time s is independent of the increment at time t . This is called the Markov property.
3. $\mathbb{E}[dW_t|\mathcal{F}_s] = 0 \quad \forall t > s$, i.e. the expectation of future increments, given the current information is 0¹. This is called the Martingale property.
4. $dW_t^2 = dt$, i.e. the square of the stochastic increments is deterministic. This is not true in the naive algebraic sense, i.e. $\frac{dW_t^2}{dt} \neq 1$, but is true inside stochastic differential equations. This property is called quadratic variance.

The formal proofs of all of these properties are very technical, out of the scope of anything less than a year long course in stochastic calculus, and unnecessary in practice. Since W_t is the sum of independent identically distributed random variables, each with mean 0 and variance dt , it is normally distributed with mean 0 and variance t .

$$W_t \sim \sqrt{t}\mathcal{N}(0, 1) \quad (2)$$

Over appropriate integration limits, a stochastic differential equation is defined as:

$$\int dX_t = \int \mu(t, X_t)dt + \int \sigma(t, X_t)dW_t \quad (3)$$

¹ \mathcal{F}_s is the set of all information available at time s , referred to as the *filtration*

In practice, the integral signs are dropped and the differential notation is used.

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \quad (4)$$

The function μ is called the drift of the process and the function σ is called the volatility of the process.

2 Ito's Lemma

Quadratic variance makes stochastic calculus different from regular calculus. In practice, one can use the shorthand: *stochastic calculus is just like regular calculus, except that the second order derivative is considered*. Ito's lemma formalizes this statement by defining the derivative of a function of a stochastic variables. Consider,

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \quad (5)$$

$$\text{Let } y = f(t, X_t) \quad (6)$$

$$\therefore dy = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\left\{\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial^2 f}{\partial X_t^2}dX_t^2 + \frac{\partial^2 f}{\partial t\partial X_t}dtdX_t\right\} + \dots \quad (7)$$

In regular calculus, the quadratic (and higher) terms are discarded because they are infinitesimally small compared with the first order terms. However, dX_t^2 cannot be discarded because it has a term of the same order as dt .

$$\therefore dy = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}\left\{\mu(t, X_t)dt + \sigma(t, X_t)dW_t\right\} + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma(t, X_t)dt \quad (8)$$

$$= \left\{\frac{\partial f}{\partial t} + \mu(t, X_t)\frac{\partial f}{\partial X_t} + \frac{\sigma(t, X_t)^2}{2}\frac{\partial^2 f}{\partial X_t^2}\right\}dt + \frac{\partial f}{\partial X_t}\sigma(t, X_t)dW_t \quad (9)$$

This can now be generalized for vector valued functions of vector valued inputs, but that is beyond the scope of this monograph.

3 Martingale

A Martingale M_t is a stochastic process with the property that future expectation, given current information, is equal to the current value. Formally,

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \quad \forall t > s \quad (10)$$

Differentiating both sides and dropping the filtration to simplify the notation,

$$\mathbb{E}[dM_t] = 0 \quad (11)$$

If M_t is an Ito process, with deterministic functions μ and σ ,

$$\mathbb{E}[\mu(t, M_t)dt + \sigma(t, M_t)dW_t] = 0 \quad (12)$$

$$\therefore \mathbb{E}[\mu(t, M_t)]dt = 0 \quad (13)$$

$$\therefore \mu(t, M_t) = 0 \quad \forall t \quad (14)$$

Therefore, a stochastic process is a martingale if it has no drift.

4 Measure

A measure may be understood as a probability distribution associated with a stochastic process. Given a set of paths taken by a stochastic process, the measure of those paths is the probability of that set being observed². Two measures are said to be equivalent if the set of paths over which they are defined are the same. In practice, this means that the two processes have the same volatility.

5 Numeraires

All financial derivatives are priced in terms of other assets, usually those whose values are fixed by the market. Any reference non-dividend paying asset is called a numeraire. Given a numeraire B_t , if there exists a measure in which all tradeable assets are martingales with respect to B_t , any contract whose value is derived from those assets is also a martingale, i.e. it is impossible to construct a portfolio of those assets and the security that is not arbitrage free.

The numeraire in which it is convenient to define an asset may not be the same one in which it is convenient to price it. Under *reasonable* constraints which are almost surely met in practice, it is possible to change the numeraire of a model. Mathematically, changing the numeraire means changing the drift of a process while keeping the volatility constant. This does not alter the set of possible paths, it only reweights them. Therefore, if a portfolio is arbitrage free in the first numeraire, it remains arbitrage free in the second numeraire. The technique is formally described by Girsanov's theorem, but can be encapsulated by a simple toolkit³

5.1 Rule 1

The price of any asset divided by the numeraire is a martingale in the measure associated with the numeraire. For example, for an arbitrary asset H and an arbitrary numeraire B ,

$$\mathbb{E}^B \left[\frac{H_T}{B_T} \right] = \frac{H_0}{B_0} \quad (15)$$

5.2 Rule 2

The time t risk neutral price is invariant to change of numeraire, i.e. for arbitrary numeraires B and Q ,

$$\text{Price}_t = \mathbb{E}^B \left[B_t \frac{\text{Payoff}_T}{B_T} \right] = \mathbb{E}^Q \left[Q_t \frac{\text{Payoff}_T}{Q_T} \right] \quad (16)$$

²When talking about continuous stochastic processes, the probability of any single path is strictly 0, therefore it is necessary to talk about a set of paths

³taken from *Interest Rate Models - Theory and Practice* by Brigo and Mercurio

5.3 Rule 3

The drift of a process changes when changing the numeraire. In the univariate case, if the numeraire of process A is being changed from B to Q ,

$$\mu_A^Q = \mu_A^B - \sigma_A \rho \left(\frac{\sigma_B}{B} - \frac{\sigma_Q}{Q} \right) \quad (17)$$

Where ρ is the correlation between the two numeraires. Alternatively

$$dW^Q = dW^B - \rho \left(\frac{\sigma_B}{B} - \frac{\sigma_Q}{Q} \right) dt \quad (18)$$

6 Useful Results

6.1 Product Rule

Ito's lemma can be used to derive the expression for the derivative of the product of two stochastic variables. For two functions of stochastic variables f and g ,

$$d(fg) = fdg + gdf + dfdg \quad (19)$$

In deterministic calculus, the last term would be dropped as it would be infinitesimal compared with the first two. In the stochastic case, it would have at least one term of the order dt .

6.2 Ito Isometry

Ito isometry relates the covariance of two stochastic processes. Formally,

$$\mathbb{E} \left[\int_x^t X_t dW_t \int_s^t Y_t dW_t \right] = \int_s^t X_t Y_t dt \quad (20)$$