

# Local Volatility In Practice

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The local volatility model can be used to price exotic derivatives consistently with vanilla options without requiring a numerical scheme like finite differences to price the vanilla options. This monograph derives several variations of the expression for local volatility in terms of the implied volatility of vanilla options for the case of lognormal asset dynamics with one stochastic variable.

Let  $r_t^a$  be the risk neutral drift of the asset and  $r_t^n$  be the risk neutral drift of the numeraire. Let  $S_t$  be the stochastic process equation that drives the price of the asset with a volatility  $\sigma_L$  that depends on time and the asset level.

$$\mu_t := r_t^n - r_t^a \quad (1)$$

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_L(S, t) dW_t \quad (2)$$

The function  $\sigma_L(S, t)$  is called the local volatility of the process.

## 1 Forward Kolmogorov Equation

The forward Kolmogorov equation is also called the **Fokker-Plank** equation. It defines the probability distribution of the asset level at a future time, called the forward transition probability. Let  $B(S)$  be a function such that:

1. It is finite and defined for all real values of  $S$
2. It is *sufficiently smooth* to ensure that all the required derivatives are defined
3. It smoothly falls off to zero outside a compact region *sufficiently fast* that the required integrals in this section evaluate to zero.

Using Ito's lemma,

$$dB = \frac{\partial B}{\partial S} dS + \frac{1}{2} \frac{\partial^2 B}{\partial S^2} dS^2 \quad (3)$$

$$= \frac{\partial B}{\partial S} S(\mu_t dt + \sigma_L(S, t) dW_t) + \frac{1}{2} \frac{\partial^2 B}{\partial S^2} S^2 \sigma_L^2(S, t) dt \quad (4)$$

$$= \left[ \frac{\partial B}{\partial S} S \mu_t + \frac{1}{2} S^2 \sigma_L^2(S, t) \frac{\partial^2 B}{\partial S^2} \right] dt + \frac{\partial B}{\partial S} S \sigma_L(S, t) dW_t \quad (5)$$

$$\therefore \mathbb{E}[dB] = \mathbb{E}\left[\frac{\partial B}{\partial S}S\mu_t + \frac{1}{2}S^2\sigma_L^2(S,t)\frac{\partial^2 B}{\partial S^2}\right]dt \quad (6)$$

$$\therefore \frac{d}{dt}\mathbb{E}[B] = \mathbb{E}\left[\frac{\partial B}{\partial S}S\mu_t + \frac{1}{2}S^2\sigma_L^2(S,t)\frac{\partial^2 B}{\partial S^2}\right] \quad (7)$$

Let  $p(s,t)$  be the (risk-neutral) forward transition probability for the asset at time  $t^1$ , i.e.

$$\int_a^b p(s,t)ds = \mathbb{P}(S_t \in (a,b)) \quad (8)$$

$$\therefore \frac{d}{dt} \int_{-\infty}^{\infty} B(s)p(s,t)ds = \int_{-\infty}^{\infty} p(s,t) \left( \frac{\partial B}{\partial S}S\mu_t + \frac{1}{2}S^2\sigma_L^2(s,t)\frac{\partial^2 B}{\partial S^2} \right) ds \quad (9)$$

Considering the left hand side of equation 9,

$$\frac{d}{dt} \int_{-\infty}^{\infty} B(s)p(s,t)ds = \int_{-\infty}^{\infty} B(s) \frac{\partial p(s,t)}{\partial t} ds \quad (10)$$

Considering the right hand side of equation 9,

$$I_1 := \int_{-\infty}^{\infty} \frac{\partial B}{\partial S}S\mu_t p(s,t)ds \quad (11)$$

$$I_2 := \int_{-\infty}^{\infty} \frac{1}{2} \frac{\partial^2 B}{\partial S^2} S^2 \sigma_L^2(s,t) p(s,t) ds \quad (12)$$

Integrating  $I_1$  by parts,

$$I_1 = \mu_t s p(s,t) B(s) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} B \mu_t \frac{\partial}{\partial S} (s p(s,t)) ds \quad (13)$$

$$= - \int_{-\infty}^{\infty} B(s) \mu_t \frac{\partial}{\partial S} (s p(s,t)) ds \quad (14)$$

Integrating  $I_2$  by parts,

$$I_2 = \frac{1}{2} s^2 \sigma_L^2(s,t) p(s,t) \frac{\partial B}{\partial S} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2} \frac{\partial B}{\partial S} \frac{\partial}{\partial S} (s^2 \sigma_L^2(s,t) p(s,t)) ds \quad (15)$$

$$= \frac{1}{2} B(s) \frac{\partial}{\partial S} (s^2 \sigma_L^2(s,t) p(s,t)) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{2} B(s) \frac{\partial}{\partial S^2} (s^2 \sigma_L^2(s,t) p(s,t)) ds \quad (16)$$

$$= \int_{-\infty}^{\infty} B(s) \frac{\partial}{\partial S^2} \left( \frac{1}{2} s^2 \sigma_L^2(s,t) p(s,t) \right) ds \quad (17)$$

Combining equations 10, 14 and 17, equation 9 can be rewritten as:

$$\int_{-\infty}^{\infty} B(s) \frac{\partial p(s,t)}{\partial t} ds = \int_{-\infty}^{\infty} B(s) \left( -\mu_t \frac{\partial}{\partial S} s p(s,t) + \frac{1}{2} \frac{\partial}{\partial S^2} s^2 \sigma_L^2(s,t) p(s,t) \right) ds \quad (18)$$

Since  $B(s)$  is arbitrary,

$$\frac{\partial}{\partial t} p(s,t) = -\mu_t \frac{\partial}{\partial S} s p(s,t) + \frac{1}{2} \frac{\partial}{\partial S^2} s^2 \sigma_L^2(s,t) p(s,t) \quad (19)$$

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<sup>1</sup>Since  $S_t$  is a continuous random variable, the probability of it being exactly any given value is strictly zero

## 2 As a Function of Call Option Value

In this section, the forward transition probability relation derived in equation 19 is expressed in the form of a more concrete quantity - the value of a vanilla call option. Let  $C(K, T)$  be the value of a vanilla call option with strike  $K$  at time  $T$ . Without loss of generality,

$$C(K, T) = \exp\left(-\int_0^T r_\tau^n d\tau\right) \mathbb{E}[(S_T - K)1_{S_T > K}] \quad (20)$$

$$= \exp\left(-\int_0^T r_\tau^n d\tau\right) \int_{-\infty}^{\infty} (s_T - K)1_{s_T > K} p(s, T) ds \quad (21)$$

$$= \exp\left(-\int_0^T r_\tau^n d\tau\right) \int_K^{\infty} (s_T - K) p(s, T) ds \quad (22)$$

$$\therefore \frac{\partial}{\partial K} C(K, T) = \exp\left(-\int_0^T r_\tau^n d\tau\right) \left\{ (s - K)p(s, T) \Big|_{s=K}^{s=\infty} - \int_K^{\infty} p(s, T) ds \right\} \quad (23)$$

$$= -\exp\left(-\int_0^T r_\tau^n d\tau\right) \int_K^{\infty} p(s, T) ds \quad (24)$$

$$\therefore \frac{\partial^2}{\partial K^2} C(K, T) = \exp\left(-\int_0^T r_\tau^n d\tau\right) \left\{ -p(s, T) \Big|_{s=K}^{s=\infty} - \int_K^{\infty} \frac{\partial}{\partial K} p(s, T) ds \right\} \quad (25)$$

$$= \exp\left(-\int_0^T r_\tau^n d\tau\right) p(K, T) \quad (26)$$

Therefore, the forward transition probability can be expressed as a function of the call option value

$$p(K, T) = \exp\left(\int_0^T r_\tau^n d\tau\right) \frac{\partial^2}{\partial K^2} C(K, T) \quad (27)$$

Substituting 27 into the left hand side of equation 19,

$$\frac{\partial}{\partial T} \exp\left(\int_0^T r_\tau^n d\tau\right) \frac{\partial^2}{\partial K^2} C(K, T) \quad (28)$$

$$= \exp\left(\int_0^T r_\tau^n d\tau\right) \left( r_T^n \frac{\partial^2}{\partial K^2} C(K, T) + \frac{\partial^2}{\partial K^2} \frac{\partial}{\partial T} C(K, T) \right) \quad (29)$$

$$= \exp\left(\int_0^T r_\tau^n d\tau\right) \frac{\partial^2}{\partial K^2} \left\{ r_T^n C(K, T) + \frac{\partial}{\partial T} C(K, T) \right\} \quad (30)$$

Substituting 27 into the right hand side of equation 19,

$$- \mu_T \frac{\partial}{\partial K} K p(K, t) + \frac{1}{2} \frac{\partial}{\partial K^2} K^2 \sigma_L^2(K, t) p(K, t) \quad (31)$$

$$= \exp\left(\int_0^T r_\tau^n d\tau\right) \left\{ -\mu_T \frac{\partial}{\partial K} \left( K \frac{\partial^2}{\partial K^2} C(K, T) \right) + \frac{1}{2} \frac{\partial^2}{\partial K^2} \left( K^2 \sigma_L^2(K, T) \frac{\partial^2}{\partial K^2} C(K, T) \right) \right\} \quad (32)$$

$$= \exp\left(\int_0^T r_\tau^n d\tau\right) \left\{ -\mu_T \frac{\partial^2}{\partial K^2} \left( K \frac{\partial}{\partial K} C(K, T) - C(K, T) \right) + \frac{1}{2} \frac{\partial^2}{\partial K^2} \left( K^2 \sigma_L^2(K, T) \frac{\partial^2}{\partial K^2} C(K, T) \right) \right\} \quad (33)$$

$$= \exp\left(\int_0^T r_\tau^n d\tau\right) \frac{\partial^2}{\partial K^2} \left\{ -\mu_T K \frac{\partial}{\partial K} C(K, T) + \mu_T C(K, T) + \frac{K^2 \sigma_L^2(K, T)}{2} \frac{\partial^2}{\partial K^2} C(K, T) \right\} \quad (34)$$

Equating equations 30 and 34 and integrating twice with respect to  $K$ ,

$$r_T^n C(K, T) + \frac{\partial}{\partial T} C(K, T) = -\mu_T K \frac{\partial}{\partial K} C(K, T) + \mu_T C(K, T) + \frac{K^2 \sigma_L^2(K, T)}{2} \frac{\partial^2}{\partial K^2} C(K, T) \quad (35)$$

Rearranging,

$$\frac{1}{2} \sigma_L^2(K, T) = \frac{\frac{\partial}{\partial T} C(K, T) + K \mu_T \frac{\partial}{\partial K} C(K, T) + r_T^n C(K, T)}{K^2 \frac{\partial^2}{\partial K^2} C(K, T)} \quad (36)$$

### 3 As a Function of Implied Volatility

While the expression for local volatility derived in equation 36 is more concrete than that derived in equation 19, it is not usable in practice because the values of call options are usually the result of a modelling exercise, rather than the input. In this section, the equation for local volatility will be derived in terms the implied volatility surface, which is often directly modelled.

#### 3.1 Implied Volatility

In the presence of a volatility smile, no general closed form solution exists for the price of a call option. However, in the spirit of defining a set of wrong numbers to enter into a wrong formula to get the right answer, a function called implied volatility,  $\sigma(K, T)$  is defined such that the price of the option is correctly computed. Concretely,

$$\bar{r}_T^a := \frac{1}{T} \int_0^T r_\tau^a d\tau \quad (37)$$

$$\bar{r}_T^d := \frac{1}{T} \int_0^T r_\tau^n d\tau \quad (38)$$

$$\bar{\mu}_T^d := \frac{1}{T} \int_0^T \mu_\tau d\tau = \bar{r}_T^d - \bar{r}_T^a \quad (39)$$

$$d = \frac{\log(S/K) + (\bar{r}_T^d - \bar{r}_T^a)T}{\sigma \sqrt{T}} \quad (40)$$

$$d_1 = d + \frac{\sigma \sqrt{T}}{2} \quad (41)$$

$$d_2 = d - \frac{\sigma \sqrt{T}}{2} \quad (42)$$

$$C = e^{-\bar{r}_T^a T} S_T \Phi(d_1) - e^{-\bar{r}_T^n T} K \Phi(d_2) \quad (43)$$

However, the algebra in the subsequent sections can be simplified if the formula is expressed in terms of the integrated variance.

$$\Sigma(K, T) := \sigma^2(K, T)T \quad (44)$$

$$d = \frac{\log(S) - \log(K) + (\bar{r}_T^n - \bar{r}_T^a)T}{\sqrt{\Sigma}} + \frac{\sqrt{\Sigma}}{2} \quad (45)$$

$$d_1 = d + \frac{\sqrt{\Sigma}}{2} \quad (46)$$

$$d_2 = d - \frac{\sqrt{\Sigma}}{2} \quad (47)$$

### 3.2 Useful Lemmas

To simplify the tedious algebra that follows, it is useful to prove a few lemmas.

**Lemma 3.1.**

$$\frac{\partial}{\partial T} \bar{r}_T^n T = r_T^n \quad (48)$$

$$\frac{\partial}{\partial T} \bar{r}_T^a T = r_T^a \quad (49)$$

*Proof.*

$$\frac{\partial}{\partial T} \bar{r}_T^n T = \frac{\partial}{\partial T} \int_0^T r_\tau^n d\tau = r_T^n \quad (50)$$

$$\frac{\partial}{\partial T} \bar{r}_T^a T = \frac{\partial}{\partial T} \int_0^T r_\tau^a d\tau = r_T^a \quad (51)$$

□

**Lemma 3.2.**

$$\frac{\phi(d_2)}{\phi(d_1)} = \frac{S}{K} e^{(\bar{r}_T^n - \bar{r}_T^a)T} \quad (52)$$

*Proof.*

$$\frac{\phi(d_2)}{\phi(d_1)} = \frac{\exp(-d_2^2/2)}{\exp(-d_1^2/2)} \quad (53)$$

$$= \exp\left(-\frac{1}{2}(d_2^2 - d_1^2)\right) \quad (54)$$

$$= \exp\left(-\frac{1}{2}(d_1 + d_2)(d_2 - d_1)\right) \quad (55)$$

$$= \exp\left(-\frac{1}{2}\left(2\frac{\log(S/K) + (\bar{r}_T^n - \bar{r}_T^a)T}{\sqrt{\Sigma}}\right)(-\sqrt{\Sigma})\right) \quad (56)$$

$$= \exp\left(\log(S/K) + (\bar{r}_T^n - \bar{r}_T^a)T\right) \quad (57)$$

$$= \frac{S}{K} e^{(\bar{r}_T^n - \bar{r}_T^a)T} \quad (58)$$

□

**Lemma 3.3.**

$$e^{-\bar{r}_T^a T} S\phi(d_1)\alpha - e^{-\bar{r}_T^n T} K\phi(d_2)\beta = e^{-\bar{r}_T^a T} S\phi(d_1)(\alpha - \beta) \quad (59)$$

*Proof.*

$$e^{-\bar{r}_T^a T} S\phi(d_1)\alpha - e^{-\bar{r}_T^n T} K\phi(d_2)\beta \quad (60)$$

$$= e^{-\bar{r}_T^a T} S\phi(d_1) \left( \alpha - \beta \frac{e^{-\bar{r}_T^n T} K\phi(d_2)}{e^{-\bar{r}_T^a T} S\phi(d_1)} \right) \quad (61)$$

$$= e^{-\bar{r}_T^a T} S\phi(d_1) \left( \alpha - \beta \frac{K}{S e^{(\bar{r}_T^n - \bar{r}_T^a) T}} \frac{\phi(d_2)}{\phi(d_1)} \right) \quad (62)$$

$$= e^{-\bar{r}_T^a T} S\phi(d_1) \left( \alpha - \beta \frac{K}{S e^{(\bar{r}_T^n - \bar{r}_T^a) T}} \frac{S}{K} e^{(\bar{r}_T^n - \bar{r}_T^a) T} \right), \quad \text{using lemma 3.2} \quad (63)$$

$$= e^{-\bar{r}_T^a T} S\phi(d_1)(\alpha - \beta) \quad (64)$$

□

**Lemma 3.4.**

$$\frac{\partial C}{\partial T} = -r_T^a e^{-\bar{r}_T^a T} S\Phi(d_1) + r_T^n e^{-\bar{r}_T^n T} K\Phi(d_2) + \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S\phi(d_1) \frac{\partial \Sigma}{\partial T} \quad (65)$$

*Proof.*

$$\frac{\partial C}{\partial T} = \frac{\partial}{\partial T} \left[ e^{-\bar{r}_T^a T} S\Phi(d_1) - e^{-\bar{r}_T^n T} K\Phi(d_2) \right] \quad (66)$$

Using lemma 3.1

$$\frac{\partial C}{\partial T} = -r_T^a e^{-\bar{r}_T^a T} S\Phi(d_1) + e^{-\bar{r}_T^a T} S\phi(d_1) \frac{\partial d_1}{\partial T} + r_T^n e^{-\bar{r}_T^n T} K\Phi(d_2) - e^{-\bar{r}_T^n T} K\phi(d_2) \frac{\partial d_2}{\partial T} \quad (67)$$

Using lemma 3.3

$$\frac{\partial C}{\partial T} = -r_T^a e^{-\bar{r}_T^a T} S\Phi(d_1) + r_T^n e^{-\bar{r}_T^n T} K\Phi(d_2) + e^{-\bar{r}_T^a T} S\phi(d_1) \left( \frac{\partial d_1}{\partial T} - \frac{\partial d_2}{\partial T} \right) \quad (68)$$

$$= -r_T^a e^{-\bar{r}_T^a T} S\Phi(d_1) + r_T^n e^{-\bar{r}_T^n T} K\Phi(d_2) + e^{-\bar{r}_T^a T} S\phi(d_1) \left( \frac{\partial(d_1 - d_2)}{\partial T} \right) \quad (69)$$

$$= -r_T^a e^{-\bar{r}_T^a T} S\Phi(d_1) + r_T^n e^{-\bar{r}_T^n T} K\Phi(d_2) + e^{-\bar{r}_T^a T} S\phi(d_1) \left( \frac{\partial \sqrt{\Sigma}}{\partial T} \right) \quad (70)$$

$$= -r_T^a e^{-\bar{r}_T^a T} S\Phi(d_1) + r_T^n e^{-\bar{r}_T^n T} K\Phi(d_2) + \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S\phi(d_1) \frac{\partial \Sigma}{\partial T} \quad (71)$$

□

**Lemma 3.5.**

$$\frac{\partial C}{\partial K} = -e^{-\bar{r}_T^n T} \Phi(d_2) + \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \frac{\partial \Sigma}{\partial K} \quad (72)$$

*Proof.*

$$\frac{\partial C}{\partial K} = \frac{\partial}{\partial K} \left[ e^{-\bar{r}_T^a T} S_T \Phi(d_1) - e^{-\bar{r}_T^n T} K\Phi(d_2) \right] \quad (73)$$

$$= e^{-\bar{r}_T^a T} S_T \phi(d_1) \frac{\partial d_1}{\partial K} - e^{-\bar{r}_T^n T} K \phi(d_2) \frac{\partial d_2}{\partial K} - e^{-\bar{r}_T^n T} \Phi(d_2) \quad (74)$$

$$(75)$$

Using lemma 3.3

$$\frac{\partial C}{\partial K} = -e^{-\bar{r}_T^a T} \Phi(d_2) + e^{-\bar{r}_T^a T} S_T \phi(d_1) \left( \frac{\partial d_1}{\partial K} - \frac{\partial d_2}{\partial K} \right) \quad (76)$$

$$= -e^{-\bar{r}_T^a T} \Phi(d_2) + e^{-\bar{r}_T^a T} S_T \phi(d_1) \left( \frac{\partial(d_1 - d_2)}{\partial K} \right) \quad (77)$$

$$= -e^{-\bar{r}_T^a T} \Phi(d_2) + e^{-\bar{r}_T^a T} S_T \phi(d_1) \left( \frac{\partial \sqrt{\Sigma}}{\partial K} \right) \quad (78)$$

$$= -e^{-\bar{r}_T^a T} \Phi(d_2) + \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \frac{\partial \Sigma}{\partial K} \quad (79)$$

□

**Lemma 3.6.**

$$\frac{\partial \phi(\alpha)}{\partial x} = -\alpha \phi(\alpha) \frac{\partial \alpha}{\partial x} \quad (80)$$

*Proof.*

$$\phi(\alpha) = \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2} \quad (81)$$

$$\frac{\partial \phi(\alpha)}{\partial x} = \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2} \frac{-2\alpha}{2} \frac{\partial \alpha}{\partial x} \quad (82)$$

$$= -\alpha \phi(\alpha) \frac{\partial \alpha}{\partial x} \quad (83)$$

□

**Lemma 3.7.**

$$\frac{\partial d}{\partial K} = \frac{-1}{K\sqrt{\Sigma}} - \frac{d}{2\Sigma} \frac{\partial \Sigma}{\partial K} \quad (84)$$

$$\frac{\partial d_1}{\partial K} = \frac{-1}{K\sqrt{\Sigma}} - \frac{d}{2\Sigma} \frac{\partial \Sigma}{\partial K} + \frac{1}{4\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \quad (85)$$

$$\frac{\partial d_2}{\partial K} = \frac{-1}{K\sqrt{\Sigma}} - \frac{d}{2\Sigma} \frac{\partial \Sigma}{\partial K} - \frac{1}{4\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \quad (86)$$

*Proof.*

$$\frac{\partial d}{\partial K} = \frac{\partial}{\partial K} \frac{\log(S) - \log(K) + (\bar{r}_T^n - \bar{r}_T^a)T}{\sqrt{\Sigma}} \quad (87)$$

$$= \frac{1}{\sqrt{\Sigma}} \frac{\partial}{\partial K} (\log(S) - \log(K) + (\bar{r}_T^n - \bar{r}_T^a)T) + (\log(S) - \log(K) + (\bar{r}_T^n - \bar{r}_T^a)T) \frac{\partial}{\partial K} \frac{1}{\sqrt{\Sigma}} \quad (88)$$

$$= \frac{-1}{K\sqrt{\Sigma}} + (\log(S) - \log(K) + (\bar{r}_T^n - \bar{r}_T^a)T) \frac{-1}{2} \frac{1}{\Sigma\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \quad (89)$$

$$= \frac{-1}{K\sqrt{\Sigma}} - \frac{d}{2\Sigma} \frac{\partial \Sigma}{\partial K} \quad (90)$$

$$\frac{\partial d_1}{\partial K} = \frac{\partial d}{\partial K} + \frac{1}{2} \frac{\partial}{\partial K} \sqrt{\Sigma} = \frac{-1}{K\sqrt{\Sigma}} - \frac{d}{2\Sigma} \frac{\partial \Sigma}{\partial K} + \frac{1}{4\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \quad (91)$$

$$\frac{\partial d_2}{\partial K} = \frac{\partial d}{\partial K} - \frac{1}{2} \frac{\partial}{\partial K} \sqrt{\Sigma} = \frac{-1}{K\sqrt{\Sigma}} - \frac{d}{2\Sigma} \frac{\partial \Sigma}{\partial K} - \frac{1}{4\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \quad (92)$$

□

**Lemma 3.8.**

$$\frac{\partial \Sigma}{\partial T} = 2T\sigma \frac{\partial \sigma}{\partial T} + \sigma^2 T \quad (93)$$

$$\frac{\partial \Sigma}{\partial K} = 2T\sigma \frac{\partial \sigma}{\partial K} \quad (94)$$

$$\frac{\partial^2 \Sigma}{\partial K^2} = 2T \left( \frac{\partial \sigma}{\partial K} \right)^2 + 2T\sigma \frac{\partial^2 \sigma}{\partial K^2} \quad (95)$$

*Proof.*

$$\frac{\partial \Sigma}{\partial T} = \frac{\partial}{\partial T} \sigma^2 T \quad (96)$$

$$= 2T\sigma \frac{\partial \sigma}{\partial T} + \sigma^2 T \quad (97)$$

$$\frac{\partial \Sigma}{\partial K} = \frac{\partial}{\partial K} \sigma^2 T \quad (98)$$

$$= 2T\sigma \frac{\partial \sigma}{\partial K} \quad (99)$$

$$\frac{\partial^2 \Sigma}{\partial K^2} = 2T \frac{\partial}{\partial K} \sigma \frac{\partial \sigma}{\partial K} \quad (100)$$

$$= 2T \left( \frac{\partial \sigma}{\partial K} \right)^2 + 2T\sigma \frac{\partial^2 \sigma}{\partial K^2} \quad (101)$$

□

### 3.3 Tedious Algebra

Using lemma 3.4 and lemma 3.5, the numerator of equation 36 can be expressed in terms of the implied volatility.

$$\begin{aligned} \frac{\partial C}{\partial T} + K\mu_T \frac{\partial}{\partial K} C(K, T) + r_T^a C(K, T) &= -r_T^a e^{-\bar{r}_T^a T} S\Phi(d_1) + r_T^n e^{-\bar{r}_T^n T} K\Phi(d_2) \\ &\quad + \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S\phi(d_1) \frac{\partial \Sigma}{\partial T} \\ &\quad + K\mu_T \left[ -e^{-\bar{r}_T^n T} \Phi(d_2) + \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \frac{\partial \Sigma}{\partial K} \right] \\ &\quad + r_T^a \left[ e^{-\bar{r}_T^a T} S_T \Phi(d_1) - e^{-\bar{r}_T^n T} K\Phi(d_2) \right] \end{aligned} \quad (102)$$



$$= \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \left( \frac{\partial \Sigma}{\partial T} + K \mu_T \frac{\partial \Sigma}{\partial K} \right) \quad (103)$$

Using lemma 3.8,

$$\frac{\partial C}{\partial T} = \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \left( 2T\sigma \frac{\partial \sigma}{\partial T} + \sigma^2 T + 2KT\mu_T\sigma \frac{\partial \sigma}{\partial K} \right) \quad (104)$$

Considering the second derivative of the call option price with respect to strike,

$$\frac{\partial^2 C}{\partial K^2} = \frac{\partial}{\partial K} \left\{ -e^{-\bar{r}_T^a T} \Phi(d_2) + \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \frac{\partial \Sigma}{\partial K} \right\} \quad (105)$$

$$\begin{aligned} &= \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \frac{\partial^2 \Sigma}{\partial K^2} + \frac{1}{2} e^{-\bar{r}_T^a T} S_T \phi(d_1) \frac{\partial \Sigma}{\partial K} \frac{\partial}{\partial K} \frac{1}{\sqrt{\Sigma}} \\ &\quad + \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \frac{\partial \Sigma}{\partial K} \frac{\partial}{\partial K} \phi(d_1) - e^{-\bar{r}_T^a T} \phi(d_2) \frac{\partial d_2}{\partial K} \end{aligned} \quad (106)$$

Using lemma 3.6,

$$\begin{aligned} \frac{\partial^2 C}{\partial K^2} &= \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \frac{\partial^2 \Sigma}{\partial K^2} - \frac{1}{4\Sigma\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \left( \frac{\partial \Sigma}{\partial K} \right)^2 \\ &\quad + \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) (-d_1) \frac{\partial \Sigma}{\partial K} \frac{\partial d_1}{\partial K} - e^{-\bar{r}_T^a T} \phi(d_2) \frac{\partial d_2}{\partial K} \end{aligned} \quad (107)$$

$$= \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \left[ \frac{\partial^2 \Sigma}{\partial K^2} - \frac{1}{2\Sigma} \left( \frac{\partial \Sigma}{\partial K} \right)^2 - d_1 \frac{\partial \Sigma}{\partial K} \frac{\partial d_1}{\partial K} - e^{(\bar{r}_T^a - \bar{r}_T^n)T} \frac{\phi(d_2)}{\phi(d_1)} \frac{2\sqrt{\Sigma}}{S_T} \frac{\partial d_2}{\partial K} \right] \quad (108)$$

Using lemma 3.2

$$\frac{\partial^2 C}{\partial K^2} = \frac{1}{2\sqrt{\Sigma}} e^{-\bar{r}_T^a T} S_T \phi(d_1) \left[ \frac{\partial^2 \Sigma}{\partial K^2} - \frac{1}{2\Sigma} \left( \frac{\partial \Sigma}{\partial K} \right)^2 - d_1 \frac{\partial \Sigma}{\partial K} \frac{\partial d_1}{\partial K} - \frac{2\sqrt{\Sigma}}{K} \frac{\partial d_2}{\partial K} \right] \quad (109)$$

Considering the first two terms of equation 109 along with lemma 3.8

$$\frac{\partial^2 \Sigma}{\partial K^2} - \frac{1}{2\Sigma} \left( \frac{\partial \Sigma}{\partial K} \right)^2 = 2T \left( \frac{\partial \sigma}{\partial K} \right)^2 + 2T\sigma \frac{\partial^2 \sigma}{\partial K^2} - \frac{1}{2\sigma^2 T} \left( 2T\sigma \frac{\partial \sigma}{\partial K} \right)^2 \quad (110)$$

$$= 2T\sigma \frac{\partial^2 \sigma}{\partial K^2} \quad (111)$$

Considering the latter two terms of equation 109 along with lemma 3.7,

$$\begin{aligned} -d_1 \frac{\partial \Sigma}{\partial K} \frac{\partial d_1}{\partial K} - \frac{2\sqrt{\Sigma}}{K} \frac{\partial d_2}{\partial K} &= - \left( d + \frac{\sqrt{\Sigma}}{2} \right) \frac{\partial \Sigma}{\partial K} \left( \frac{-1}{k\sqrt{\Sigma}} - \frac{d}{2\Sigma} \frac{\partial \Sigma}{\partial K} + \frac{1}{4\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \right) \\ &\quad - \frac{2\sqrt{\Sigma}}{K} \left( \frac{-1}{K\sqrt{\Sigma}} - \frac{d}{2\Sigma} \frac{\partial \Sigma}{\partial K} - \frac{1}{4\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \right) \end{aligned} \quad (112)$$

$$= \left( d + \frac{\sqrt{\Sigma}}{2} \right) \frac{\partial \Sigma}{\partial K} \left( \frac{1}{k\sqrt{\Sigma}} + \frac{d}{2\Sigma} \frac{\partial \Sigma}{\partial K} - \frac{1}{4\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \right) + \frac{2\sqrt{\Sigma}}{K} \left( \frac{1}{K\sqrt{\Sigma}} + \frac{d}{2\Sigma} \frac{\partial \Sigma}{\partial K} + \frac{1}{4\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \right) \quad (113)$$

$$\begin{aligned}
&= \frac{d}{K\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} + \frac{d^2}{2\Sigma} \left( \frac{\partial \Sigma}{\partial K} \right)^2 - \frac{d}{4\sqrt{\Sigma}} \left( \frac{\partial \Sigma}{\partial K} \right)^2 \\
&\quad + \frac{1}{2K} \frac{\partial \Sigma}{\partial K} + \frac{d}{4\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} - \frac{1}{8} \left( \frac{\partial \Sigma}{\partial K} \right)^2 \\
&\quad + \frac{2}{K^2} + \frac{d}{K\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} + \frac{1}{2K} \frac{\partial \Sigma}{\partial K}
\end{aligned} \tag{114}$$

$$= \frac{2}{K^2} + \frac{d^2}{2\Sigma} \left( \frac{\partial \Sigma}{\partial K} \right)^2 + \frac{2d}{K\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} + \frac{1}{K} \frac{\partial \Sigma}{\partial K} - \frac{1}{8} \left( \frac{\partial \Sigma}{\partial K} \right)^2 \tag{115}$$

$$= \frac{2}{K^2} \left( 1 + \frac{d^2 K^2}{4\Sigma} \left( \frac{\partial \Sigma}{\partial K} \right)^2 + \frac{dK}{\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \right) + \frac{1}{K} \frac{\partial \Sigma}{\partial K} - \frac{1}{8} \left( \frac{\partial \Sigma}{\partial K} \right)^2 \tag{116}$$

$$= \frac{2}{K^2} \left( 1 + \frac{dK}{2\sqrt{\Sigma}} \frac{\partial \Sigma}{\partial K} \right)^2 + \frac{1}{K} \frac{\partial \Sigma}{\partial K} - \frac{1}{8} \left( \frac{\partial \Sigma}{\partial K} \right)^2 \tag{117}$$

Using lemma 3.8

$$-d_1 \frac{\partial \Sigma}{\partial K} \frac{\partial d_1}{\partial K} - \frac{2\sqrt{\Sigma}}{K} \frac{\partial d_2}{\partial K} = \frac{2}{K^2} \left( 1 + \frac{dK}{2\sigma\sqrt{T}} 2T\sigma \frac{\partial \sigma}{\partial K} \right)^2 + \frac{2T\sigma}{K} \frac{\partial \sigma}{\partial K} - \frac{1}{8} (4\sigma^2 T^2) \left( \frac{\partial \sigma}{\partial K} \right)^2 \tag{118}$$

$$= \frac{2}{K^2} \left( 1 + dK\sqrt{T} \frac{\partial \sigma}{\partial K} \right)^2 + \frac{2T\sigma}{K} \frac{\partial \sigma}{\partial K} - \frac{\sigma^2 T^2}{2} \left( \frac{\partial \sigma}{\partial K} \right)^2 \tag{119}$$

Combining equations 111 and 119 with equation 109

$$\frac{\partial^2 C}{\partial K^2} = \frac{1}{2\sqrt{\Sigma}} e^{-r_T^a T} S_T \phi(d_1) \left[ 2T\sigma \frac{\partial^2 \sigma}{\partial K^2} + \frac{2}{K^2} \left( 1 + dK\sqrt{T} \frac{\partial \sigma}{\partial K} \right)^2 + \frac{2T\sigma}{K} \frac{\partial \sigma}{\partial K} - \frac{\sigma^2 T^2}{2} \left( \frac{\partial \sigma}{\partial K} \right)^2 \right] \tag{120}$$

Combining equations 104 and 120 with equation 36

$$\frac{\sigma_L^2(K, T)}{2} = \frac{1}{K^2} \frac{2T\sigma \frac{\partial \sigma}{\partial T} + \sigma^2 T + 2KT\mu_T \sigma \frac{\partial \sigma}{\partial K}}{2T\sigma \frac{\partial^2 \sigma}{\partial K^2} + \frac{2}{K^2} \left( 1 + dK\sqrt{T} \frac{\partial \sigma}{\partial K} \right)^2 + \frac{2T\sigma}{K} \frac{\partial \sigma}{\partial K} - \frac{\sigma^2 T^2}{2} \left( \frac{\partial \sigma}{\partial K} \right)^2} \tag{121}$$

Simplifying,

$$\sigma_L^2(K, T) = \frac{2T\sigma \frac{\partial \sigma}{\partial T} + \sigma^2 T + 2KT\mu_T \sigma \frac{\partial \sigma}{\partial K}}{\left( 1 + dK\sqrt{T} \frac{\partial \sigma}{\partial K} \right)^2 + TK^2 \sigma \frac{\partial^2 \sigma}{\partial K^2} + T\sigma K \frac{\partial \sigma}{\partial K} - \frac{\sigma^2 T^2 K^2}{4} \left( \frac{\partial \sigma}{\partial K} \right)^2} \tag{122}$$

## 4 Parameterization Using Logmoneyness

While the expression derived in equation 122 can be used in practice, it depends on the value of the instantaneous risk neutral drift (**not the integrated and averaged value**) at the terminal node. This inconvenience can be rectified if the implied volatility is parameterized in terms of logmoneyness,  $x$ , which is observable in the market.

$$x(K, T) := \log(K) - \log(S) - \int_0^T \mu_\tau d\tau \quad (123)$$

$$\therefore \frac{\partial x}{\partial T} = -\mu_T \quad (124)$$

$$\therefore \frac{\partial x}{\partial K} = \frac{1}{K} \quad (125)$$

$$\sigma(K, T) := \bar{\sigma}(x, T) \quad (126)$$

$$\therefore \frac{\partial \sigma}{\partial T} = \frac{\partial \bar{\sigma}}{\partial T} + \frac{\partial \bar{\sigma}}{\partial x} \frac{\partial x}{\partial T} \quad (127)$$

$$= \frac{\partial \bar{\sigma}}{\partial T} - \mu_T \frac{\partial \bar{\sigma}}{\partial x} \quad (128)$$

$$\therefore \frac{\partial \sigma}{\partial K} = \frac{\partial \bar{\sigma}}{\partial x} \frac{\partial x}{\partial K} \quad (129)$$

$$= \frac{1}{K} \frac{\partial \bar{\sigma}}{\partial x} \quad (130)$$

$$\therefore \frac{\partial^2 \sigma}{\partial K^2} = \frac{-1}{K^2} \frac{\partial \bar{\sigma}}{\partial x} + \frac{1}{K} \frac{\partial^2 \bar{\sigma}}{\partial x^2} \frac{\partial x}{\partial K} \quad (131)$$

$$= \frac{1}{K^2} \left( \frac{\partial^2 \bar{\sigma}}{\partial x^2} - \frac{\partial \bar{\sigma}}{\partial x} \right) \quad (132)$$

$$\therefore d = \frac{-x}{\sigma \sqrt{T}} \quad (133)$$

Substituting into the numerator of equation 122,

$$2T\sigma \frac{\partial \sigma}{\partial T} + \sigma^2 T + 2KT\mu_T \sigma \frac{\partial \sigma}{\partial K} = 2T\bar{\sigma} \left( \frac{\partial \bar{\sigma}}{\partial T} - \mu_T \frac{\partial \bar{\sigma}}{\partial x} \right) + \bar{\sigma}^2 T + 2KT\mu_T \bar{\sigma} \frac{1}{K} \frac{\partial \bar{\sigma}}{\partial x} \quad (134)$$

$$= \bar{\sigma}^2 T + 2T\bar{\sigma} \frac{\partial \bar{\sigma}}{\partial T} \quad (135)$$

Substituting into the denominator of equation 122

$$\begin{aligned} & \left( 1 + dK\sqrt{T} \frac{\partial \sigma}{\partial K} \right)^2 + TK^2 \sigma \frac{\partial^2 \sigma}{\partial K^2} + T\sigma K \frac{\partial \sigma}{\partial K} - \frac{\sigma^2 T^2 K^2}{4} \left( \frac{\partial \sigma}{\partial K} \right)^2 \\ &= \left( 1 + dK\sqrt{T} \frac{1}{K} \frac{\partial \bar{\sigma}}{\partial x} \right)^2 + TK^2 \bar{\sigma} \frac{1}{K^2} \left( \frac{\partial^2 \bar{\sigma}}{\partial x^2} - \frac{\partial \bar{\sigma}}{\partial x} \right) + T\bar{\sigma} K \frac{1}{K} \frac{\partial \bar{\sigma}}{\partial x} - \frac{\bar{\sigma}^2 T^2 K^2}{4} \left( \frac{1}{K} \frac{\partial \bar{\sigma}}{\partial x} \right)^2 \end{aligned} \quad (136)$$

$$= \left( 1 + d\sqrt{T} \frac{\partial \bar{\sigma}}{\partial x} \right)^2 + T\bar{\sigma} \left( \frac{\partial^2 \bar{\sigma}}{\partial x^2} - \frac{\partial \bar{\sigma}}{\partial x} \right) + T\bar{\sigma} K \frac{1}{K} \frac{\partial \bar{\sigma}}{\partial x} - \frac{\bar{\sigma}^2 T^2 K^2}{4} \left( \frac{1}{K} \frac{\partial \bar{\sigma}}{\partial x} \right)^2 \quad (137)$$

$$= \left(1 - \frac{x}{\bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial x}\right)^2 + T\bar{\sigma} \frac{\partial^2 \bar{\sigma}}{\partial x^2} - \frac{\bar{\sigma}^2 T^2}{4} \left(\frac{\partial \bar{\sigma}}{\partial x}\right)^2 \quad (138)$$

Combining equations 135 and 138 with equation 122,

$$\sigma_L^2(x, T) = \frac{\bar{\sigma}^2 T + 2T\bar{\sigma} \frac{\partial \bar{\sigma}}{\partial T}}{\left(1 - \frac{x}{\bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial x}\right)^2 + T\bar{\sigma} \frac{\partial^2 \bar{\sigma}}{\partial x^2} - \frac{\bar{\sigma}^2 T^2}{4} \left(\frac{\partial \bar{\sigma}}{\partial x}\right)^2} \quad (139)$$

## 5 Parameterization Using Integrated Variance

In practice, it is often more convenient to model integrated variance, using say interpolation, than it is to model implied volatility.

$$\Sigma(x, T) := \bar{\sigma}^2(x, T) \quad (140)$$

$$\therefore \frac{\partial \Sigma}{\partial T} = \bar{\sigma}^2 + 2T\bar{\sigma} \frac{\partial \bar{\sigma}}{\partial T} \quad (141)$$

$$\therefore \frac{\partial \Sigma}{\partial x} = 2T\bar{\sigma} \frac{\partial \bar{\sigma}}{\partial x} \quad (142)$$

$$\therefore \frac{\partial \bar{\sigma}}{\partial x} = \frac{1}{2T\bar{\sigma}} \frac{\partial \Sigma}{\partial x} \quad (143)$$

$$\therefore \frac{\partial^2 \Sigma}{\partial x^2} = 2T\bar{\sigma} \frac{\partial^2 \bar{\sigma}}{\partial x^2} + 2T \left(\frac{\partial \bar{\sigma}}{\partial x}\right)^2 \quad (144)$$

$$= 2T\bar{\sigma} \frac{\partial^2 \bar{\sigma}}{\partial x^2} + 2T \left(\frac{1}{2T\bar{\sigma}} \frac{\partial \Sigma}{\partial x}\right)^2 \quad (145)$$

$$= 2T\bar{\sigma} \frac{\partial^2 \bar{\sigma}}{\partial x^2} + \frac{1}{2\Sigma} \left(\frac{\partial \Sigma}{\partial x}\right)^2 \quad (146)$$

$$\therefore T\bar{\sigma} \frac{\partial^2 \bar{\sigma}}{\partial x^2} = \frac{1}{2} \frac{\partial^2 \Sigma}{\partial x^2} - \frac{1}{4\Sigma} \left(\frac{\partial \Sigma}{\partial x}\right)^2 \quad (147)$$

Clearly the numerator in equation 139 equals  $\partial \Sigma / \partial T$ . Substituting the values in the denominator,

$$\begin{aligned} \left(1 - \frac{x}{\bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial x}\right)^2 + T\bar{\sigma} \frac{\partial^2 \bar{\sigma}}{\partial x^2} - \frac{\bar{\sigma}^2 T^2}{4} \left(\frac{\partial \bar{\sigma}}{\partial x}\right)^2 &= \left(1 - \frac{x}{\bar{\sigma}} \frac{1}{2T\bar{\sigma}} \frac{\partial \Sigma}{\partial x}\right)^2 + \frac{1}{2} \frac{\partial^2 \Sigma}{\partial x^2} \\ &\quad - \frac{1}{4\Sigma} \left(\frac{\partial \Sigma}{\partial x}\right)^2 - \frac{\bar{\sigma}^2 T^2}{4} \left(\frac{1}{2T\bar{\sigma}} \frac{\partial \Sigma}{\partial x}\right)^2 \end{aligned} \quad (148)$$

$$= \left(1 - \frac{x}{2\Sigma} \frac{\partial \Sigma}{\partial x}\right)^2 + \frac{1}{2} \frac{\partial^2 \Sigma}{\partial x^2} - \left(\frac{\partial \Sigma}{\partial x}\right)^2 \left(\frac{1}{4\Sigma} + \frac{1}{16}\right) \quad (149)$$

Therefore, expression derived in equation 139 can be reparameterized using integrated variance as:

$$\sigma_L^2(x, T) = \frac{\frac{\partial \Sigma}{\partial T}}{\left(1 - \frac{x}{2\Sigma} \frac{\partial \Sigma}{\partial x}\right)^2 + \frac{1}{2} \frac{\partial^2 \Sigma}{\partial x^2} - \left(\frac{\partial \Sigma}{\partial x}\right)^2 \left(\frac{1}{4\Sigma} + \frac{1}{16}\right)} \quad (150)$$

## 6 General Reparameterization

It is often inconvenient to model volatility directly in terms of strike or logmoneyness, especially for foreign exchange. The general approach in that case is to use function composition, i.e. model the volatility using a *natural* parameter, for example the delta, and compute that parameter using the logstrike and/or logmoneyness. While correct, this approach can become tedious and lead to counter-intuitive terms when evaluating local volatility. In this section, a general scheme for evaluating the local volatility using a *natural* parameter is discussed.

### 6.1 Implied Volatility

Let  $\hat{x}$  be a natural parameter to model the implied volatility, for example, it could be the delta. Let  $\hat{\sigma}$  be the implied volatility expressed in this natural parameter.

$$\hat{x} := \hat{x}(x, T) \quad (151)$$

$$\hat{\sigma} := \hat{\sigma}(\hat{x}, T) \quad (152)$$

$$\bar{\sigma}(x, t) := \hat{\sigma}(\hat{x}, T) \quad (153)$$

$$\therefore \frac{\partial \bar{\sigma}}{\partial T} = \frac{\partial \hat{\sigma}}{\partial T} + \frac{\partial \hat{\sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial T} \quad (154)$$

$$\therefore \frac{\partial \bar{\sigma}}{\partial x} = \frac{\partial \hat{\sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} \quad (155)$$

$$\therefore \frac{\partial^2 \bar{\sigma}}{\partial x^2} = \frac{\partial^2 \hat{\sigma}}{\partial \hat{x}^2} \left( \frac{\partial \hat{x}}{\partial x} \right)^2 + \frac{\partial \hat{\sigma}}{\partial \hat{x}} \frac{\partial^2 \hat{x}}{\partial x^2} \quad (156)$$

Substituting in the numerator of equation 139,

$$\bar{\sigma}^2 T + 2T\bar{\sigma} \frac{\partial \bar{\sigma}}{\partial T} = \hat{\sigma}^2 T + 2T\hat{\sigma} \left( \frac{\partial \hat{\sigma}}{\partial T} + \frac{\partial \hat{\sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial T} \right) \quad (157)$$

Substituting into the denominator of equation 139,

$$\begin{aligned} \left( 1 - \frac{x}{\bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial x} \right)^2 + T\bar{\sigma} \frac{\partial^2 \bar{\sigma}}{\partial x^2} - \frac{\bar{\sigma}^2 T^2}{4} \left( \frac{\partial \bar{\sigma}}{\partial x} \right)^2 &= \left( 1 - \frac{\hat{x}}{\hat{\sigma}} \frac{\partial \hat{\sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} \right)^2 \\ &\quad + T\hat{\sigma} \left( \frac{\partial^2 \hat{\sigma}}{\partial \hat{x}^2} \left( \frac{\partial \hat{x}}{\partial x} \right)^2 + \frac{\partial \hat{\sigma}}{\partial \hat{x}} \frac{\partial^2 \hat{x}}{\partial x^2} \right) \\ &\quad - \frac{\hat{\sigma}^2 T^2}{4} \left( \frac{\partial \hat{\sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} \right)^2 \end{aligned} \quad (158)$$

$$= \left( 1 - \frac{\hat{x}}{\hat{\sigma}} \frac{\partial \hat{\sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} \right)^2 + \left( \frac{\partial \hat{x}}{\partial x} \right)^2 \left( T\hat{\sigma} \frac{\partial^2 \hat{\sigma}}{\partial \hat{x}^2} - \frac{T^2 \hat{\sigma}^2}{4} \left( \frac{\partial \hat{\sigma}}{\partial \hat{x}} \right)^2 \right) + T\hat{\sigma} \frac{\partial^2 \hat{x}}{\partial x^2} \frac{\partial \hat{\sigma}}{\partial \hat{x}} \quad (159)$$

Combining equations 157 and 159,

$$\sigma_L^2(x, T) = \frac{\hat{\sigma}^2 T + 2T\hat{\sigma} \left( \frac{\partial \hat{\sigma}}{\partial T} + \frac{\partial \hat{\sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial T} \right)}{\left( 1 - \frac{\hat{x}}{\hat{\sigma}} \frac{\partial \hat{\sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} \right)^2 + \left( \frac{\partial \hat{x}}{\partial x} \right)^2 \left[ T\hat{\sigma} \frac{\partial^2 \hat{\sigma}}{\partial \hat{x}^2} - \frac{T^2 \hat{\sigma}^2}{4} \left( \frac{\partial \hat{\sigma}}{\partial \hat{x}} \right)^2 \right] + T\hat{\sigma} \frac{\partial^2 \hat{x}}{\partial x^2} \frac{\partial \hat{\sigma}}{\partial \hat{x}}} \quad (160)$$

## 6.2 Integrated Variance

Let  $\hat{x}$  be a natural parameter to model the integrated variance, for example, it could be the delta. Let  $\hat{\Sigma}$  be the implied volatility expressed in this natural parameter.

$$\hat{x} := \hat{x}(x, T) \quad (161)$$

$$\hat{\Sigma} := \hat{\Sigma}(\hat{x}, T) \quad (162)$$

$$\Sigma(x, t) := \hat{\Sigma}(\hat{x}, T) \quad (163)$$

$$\therefore \frac{\partial \Sigma}{\partial T} = \frac{\partial \hat{\Sigma}}{\partial T} + \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial T} \quad (164)$$

$$\therefore \frac{\partial \Sigma}{\partial x} = \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} \quad (165)$$

$$\therefore \frac{\partial^2 \Sigma}{\partial x^2} = \frac{\partial^2 \hat{\Sigma}}{\partial \hat{x}^2} \left( \frac{\partial \hat{x}}{\partial x} \right)^2 + \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \frac{\partial^2 \hat{x}}{\partial x^2} \quad (166)$$

Substituting into the denominator of equation 150,

$$\begin{aligned} & \left( 1 - \frac{x}{2\Sigma} \frac{\partial \Sigma}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial^2 \Sigma}{\partial x^2} - \left( \frac{\partial \Sigma}{\partial x} \right)^2 \left( \frac{1}{4\Sigma} + \frac{1}{16} \right) \\ &= \left( 1 - \frac{\hat{x}}{2\hat{\Sigma}} \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial^2 \hat{\Sigma}}{\partial \hat{x}^2} \left( \frac{\partial \hat{x}}{\partial x} \right)^2 + \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \frac{\partial^2 \hat{x}}{\partial x^2} \right) - \left( \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} \right)^2 \left( \frac{1}{4\hat{\Sigma}} + \frac{1}{16} \right) \end{aligned} \quad (167)$$

$$= \left( 1 - \frac{\hat{x}}{2\hat{\Sigma}} \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} \right)^2 + \left( \frac{\partial \hat{x}}{\partial x} \right)^2 \left[ \frac{1}{2} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{x}^2} - \left( \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \right)^2 \left( \frac{1}{16} + \frac{1}{4\hat{\Sigma}} \right) \right] + \frac{1}{2} \frac{\partial^2 \hat{x}}{\partial x^2} \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \quad (168)$$

Combining with equation 150,

$$\sigma_L^2(x, T) = \frac{\frac{\partial \hat{\Sigma}}{\partial T} + \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial T}}{\left( 1 - \frac{\hat{x}}{2\hat{\Sigma}} \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} \right)^2 + \left( \frac{\partial \hat{x}}{\partial x} \right)^2 \left[ \frac{1}{2} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{x}^2} - \left( \frac{\partial \hat{\Sigma}}{\partial \hat{x}} \right)^2 \left( \frac{1}{16} + \frac{1}{4\hat{\Sigma}} \right) \right] + \frac{1}{2} \frac{\partial^2 \hat{x}}{\partial x^2} \frac{\partial \hat{\Sigma}}{\partial \hat{x}}} \quad (169)$$