Q1. Bayes Nets: Representation

(a) Consider the Bayes' Net $B_1$ below, and fill in all the circles (or select None of the above) corresponding to the Bayes' Nets $G_1$ through $G_8$ that...

(i) ...are able to represent at least one distribution that $B_1$ is able to represent.

- $G_1$
- $G_5$
- None of the above.

Consider the fully independent joint $P(A, B, C, D) = P(A)P(B)P(C)P(D)$ with uniform distribution (that is, every table entry has probability $\frac{1}{16}$). This can be represented by any Bayes' net. Pick any conditional independence assumption and verify that it is satisfied with by this distribution.

(ii) ...are able to represent all distributions that $B_1$ is able to represent.

- $G_1$
- $G_5$
- None of the above.

To represent all of the distributions of $B_1$, a Bayes' net must not make further independence assumptions. A family of distributions that makes only a subset of the independences assumptions of $B_1$ can represent all of the same distributions as $B_1$.

(b) Consider the Bayes' Net $B_2$ below, and fill in all the circles (or select None of the above) corresponding to the Bayes' Nets $G_1$ through $G_8$ that...

(i) ...are able to represent at least one distribution that $B_2$ is able to represent.
(ii) ...are able to represent all distributions that $B_2$ is able to represent.

- $G_1$
- $G_2$
- $G_3$
- $G_4$
- $G_5$
- $G_6$
- $G_7$
- $G_8$

- None of the above.
Q2. Variable Elimination

For the Bayes’ net shown on the right, we are given the query \( P(B, D \mid +f) \). All variables have binary domains. Assume we run variable elimination to compute the answer to this query, with the following variable elimination ordering: \( A, C, E, G \).

(a) Complete the following description of the factors generated in this process:

After inserting evidence, we have the following factors to start out with:

\[
P(A), P(B \mid A), P(C \mid B), P(D \mid C), P(E \mid C, D), P(+f \mid C, E), P(G \mid C, +f)
\]

When eliminating \( A \) we generate a new factor \( f_1 \) as follows:

\[
f_1(B) = \sum_a P(a)P(B \mid a)
\]

This leaves us with the factors:

\[
P(C \mid B), P(D \mid C), P(E \mid C, D), P(+f \mid C, E), P(G \mid C, +f), f_1(B)
\]

(i) When eliminating \( C \) we generate a new factor \( f_2 \) as follows:

\[
f_2(B, D, E, +f, G) = \sum_c P(c \mid B)P(D \mid c)P(E \mid c, D)P(+f \mid c, E)P(G \mid c, +f)
\]

This leaves us with the factors:

\[
f_1(B), f_2(B, D, E, +f, G)
\]

(ii) When eliminating \( E \) we generate a new factor \( f_3 \) as follows:

\[
f_3(B, D, +f, G) = \sum_c f_2(B, D, E, +f, G)
\]

This leaves us with the factors:

\[
f_1(B), f_3(B, D, +f, G)
\]

(iii) When eliminating \( G \) we generate a new factor \( f_4 \) as follows:

\[
f_4(B, D, +f) = \sum_g f_3(B, D, +f, G)
\]

This leaves us with the factors:

\[
f_1(B), f_4(B, D, +f)
\]

(b) Explain in one sentence how \( P(B, D \mid +f) \) can be computed from the factors left in part (iii) of (a)?

Join \( f_1f_4 \) to obtain \( P(B, D \mid +f) \) and normalize it to get \( P(B, D \mid f) \) Concretely, \( P(b, d \mid +f) = \frac{f_1(b)f_4(b, d, +f)}{\sum_{b', d'} f_1(b')f_4(b', d', +f)} \).

(c) Among \( f_1, f_2, \ldots , f_4 \), which is the largest factor generated, and how large is it? Assume all variables have binary domains and measure the size of each factor by the number of rows in the table that would represent
the factor.

\( f_2(B, D, E, +f, G) \) is the largest factor generated. It has 4 variables, hence \( 2^4 = 16 \) entries.

For your convenience, the Bayes’ net from the previous page is shown again below.

![Bayes' net diagram]

**(d)** Find a variable elimination ordering for the same query, i.e., for \( P(B, D \mid +f) \), for which the maximum size factor generated along the way is smallest. Hint: the maximum size factor generated in your solution should have only 2 variables, for a table size of \( 2^2 = 4 \). Fill in the variable elimination ordering and the factors generated into the table below.

<table>
<thead>
<tr>
<th>Variable Eliminated</th>
<th>Factor Generated</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( f_1(B) )</td>
</tr>
<tr>
<td>( G )</td>
<td>( f_2(C, +f) )</td>
</tr>
<tr>
<td>( E )</td>
<td>( f_3(C, D) )</td>
</tr>
<tr>
<td>( C )</td>
<td>( f_4(B, D, +f) )</td>
</tr>
</tbody>
</table>

For example, in the naive ordering we used earlier, the first line in this table would have had the following two entries: \( A, f_1(B) \). For this question there is no need to include how each factor is computed, i.e., no need to include expressions of the type \( \sum_a P(a)P(B \mid a) \).

Note: multiple orderings are possible. In particular in this case all orderings with \( E \) and \( G \) before \( C \) are correct.