

Advanced Algebra II:

Conceptual Explanations

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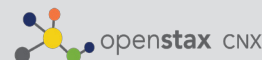
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Advanced Algebra II: Conceptual Explanations

By:
Kenny Felder

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Online:

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C O N N E X I O N S

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Chapter 1

Functions

1.1 Function Concepts¹

The unit on functions is the most important in the Algebra II course, because it provides a crucial transition point. Roughly speaking. . .

- Before Algebra I, math is about **numbers**.
- Starting in Algebra I, and continuing into Algebra II, math is about **variables**.
- Beginning with Algebra II, and continuing into Calculus, math is about **functions**.

Each step builds on the previous step. Each step expands the ability of mathematics to model behavior and solve problems. And, perhaps most crucially, each step can be frightening to a student. It can be very intimidating for a beginning Algebra student to see an entire page of mathematics that is covered with letters, with almost no numbers to be found!

Unfortunately, many students end up with a very vague idea of what variables are (“That’s when you use letters in math”) and an even more vague understanding of functions (“Those things that look like $f(x)$ or something”). If you leave yourself with this kind of vague understanding of the core concepts, the lessons will make less and less sense as you go on: you will be left with the feeling that “I just can’t do this stuff” without realizing that the problem was all the way back in the idea of a variable or function.

The good news is, variables and functions both have very specific meanings that are not difficult to understand.

1.2 What is a Variable?²

A variable is a letter that stands for a number you don’t know, or a number that can change.

A few examples:

Example 1.1: Good Examples of Variable Definitions

- “Let p be the number of people in a classroom.”
- “Let A be John’s age, measured in years.”
- “Let h be the number of hours that Susan has been working.”

¹This content is available online at <<http://cnx.org/content/m18192/1.2/>>.

²This content is available online at <<http://cnx.org/content/m18194/1.2/>>.

In each case, the letter stands for a very specific number. However, we use a letter instead of a number because we don't know the specific number. In the first example above, different classrooms will have different numbers of people (so p can be different numbers in different classes); in the second example, John's age is a specific and well-defined number, but we don't know what it is (at least not yet); and in the third example, h will actually change its value every hour. In all three cases, we have a good reason for using a letter: it represents a number, but we cannot use a specific number such as -3 or $4\frac{1}{2}$.

Example 1.2: Bad Examples of Variable Definitions

- "Let n be the nickels."
- "Let M be the number of minutes in an hour."

The first error is by far the most common. Remember that a variable always stands for a number. "The nickels" are not a number. Better definitions would be: "Let n be the number of nickels" or "Let n be the total value of the nickels, measured in cents" or "Let n be the total mass of the nickels, measured in grams."

The second example is better, because "number of minutes in an hour" is a number. But there is no reason to call it "The Mysterious Mr. M " because we already know what it is. Why use a letter when you just mean "60"?

Bad variable definitions are one of the most common reasons that students get stuck on word problems—or get the wrong answer. The first type of error illustrated above leads to variable confusion: n will end up being used for "number of nickels" in one equation and "total value of the nickels" in another, and you end up with the wrong answer. The second type of error is more harmless—it won't lead to wrong answers—but it won't help either. It usually indicates that the student is asking the wrong question ("What can I assign a variable to?") instead of the right question ("What numbers do I need to know?")

1.2.1 Variables aren't all called x . Get used to it.

Many students expect all variables to be named x , with possibly an occasional guest appearance by y . In fact, variables can be named with any practically any letter. Uppercase letters, lowercase letters, and even Greek letters are commonly used for variable names. Hence, a problem might start with "Let H be the home team's score and V be the visiting team's score."

If you attempt to call both of these variables x , it just won't work. You could in principle call one of them x and the other y , but that would make it more difficult to remember which variable goes with which team. It is important to become comfortable using a wide range of letters. (I do, however, recommend avoiding the letter o whenever possible, since it looks like the number 0.)

1.3 What is a Function?³

A function is neither a number nor a variable: it is a **process for turning one number into another**. For instance, "Double and then add 6" is a function. If you put a 4 into that function, it comes out with a 14. If you put a $\frac{1}{2}$ into that function, it comes out with a 7.

The traditional image of a function is a machine, with a slot on one side where numbers go in and a slot on the other side where numbers come out.

³This content is available online at <<http://cnx.org/content/m18189/1.2/>>.

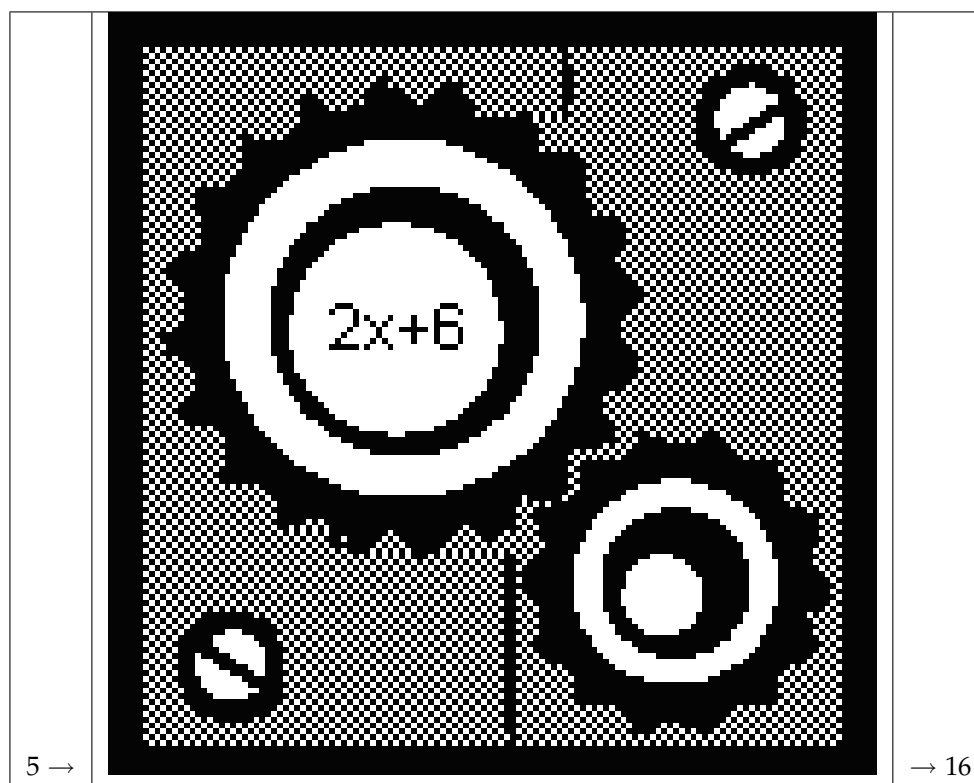


Table 1.1: A number goes in. A number comes out. The function is the machine, the process that turns 4 into 14 or 5 into 16 or 100 into 206.

The point of this image is that the function is not the numbers, but the machine itself—the **process**, not the results of the process.

The primary purpose of “The Function Game” that you play on Day 1 is to get across this idea of a numerical process. In this game, one student (the “leader”) is placed in the role of a function. “Whenever someone gives you a number, you double that number, add 6, and give back the result.” It should be very clear, as you perform this role, that you are **not** modeling a number, a variable, or even a list of numbers. You are instead modeling a process—or an algorithm, or a recipe—for turning numbers into other numbers. That is what a function is.

The function game also contains some more esoteric functions: “Respond with -3 no matter what number you are given,” or “Give back the lowest prime number that is greater than or equal to the number you were given.” Students playing the function game often ask “Can a function **do** that?” The answer is always yes (with one caveat mentioned below). So another purpose of the function game is to expand your idea of what a function can do. Any process that consistently turns numbers into other numbers, is a function.

By the way—having defined the word “function” I just want to say something about the word “equation.” An “equation” is when you “equate” two things—that is to say, set them equal. So $x^2 - 3$ is a function, but it is not an equation. $x^2 - 3 = 6$ is an equation. An “equation” always has an equal sign in it.

1.4 The Rule of Consistency⁴

There is only one limitation on what a function can do: **a function must be consistent.**

For instance, the function in the above drawing is given a 5, and gives back a 16. That means this particular function turns 5 into 16—always. That particular function can never take in a 5 and give back a 14. This “rule of consistency” is a very important constraint on the nature of functions.

NOTE: **This rule does not treat the inputs and outputs the same!**

For instance, consider the function $y = x^2$. This function takes both 3 and -3 and turns them into 9 (two different inputs, same output). That is allowed. However, it is not reversible! If you take a 9 and turn it into both a 3 and a -3 (two different outputs, same input), you are not a function.

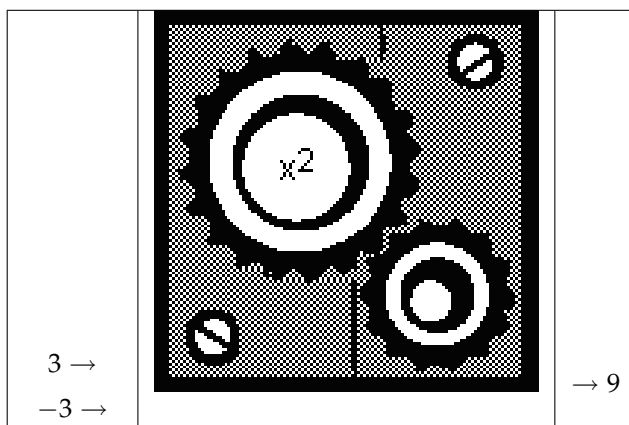


Table 1.2: If 3 goes in, 9 comes out. If -3 goes in, 9 also comes out. No problem: x^2 is a function.

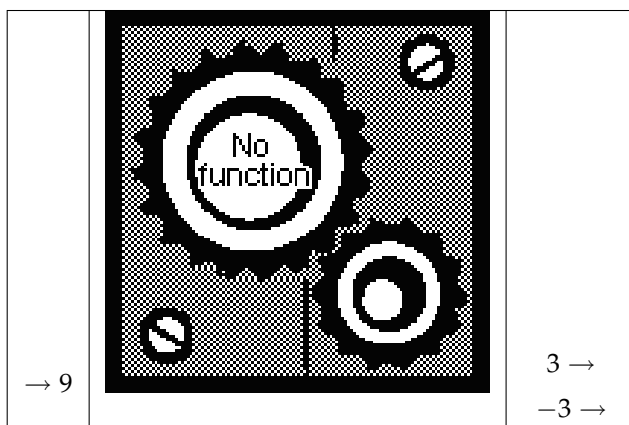


Table 1.3: If 9 goes in, both -3 and 3 come out. This **violates the rule of consistency**: no function can do this

This asymmetry has the potential to cause a great deal of confusion, but it is a very important aspect of functions.

⁴This content is available online at <<http://cnx.org/content/m18190/1.3/>>.

1.5 Four Ways to Represent a Function⁵

Modern Calculus texts emphasize that a function can be expressed in four different ways.

1. **Verbal** - This is the first way functions are presented in the function game: “Double and add six.”
2. **Algebraic** - This is the most common, most concise, and most powerful representation: $2x + 6$. Note that in an algebraic representation, the input number is represented as a variable (in this case, an x).
3. **Numerical** - This can be done as a list of value pairs, as $(4, 14)$ — meaning that if a 4 goes in, a 14 comes out. (You may recognize this as (x, y) points used in graphing.)
4. **Graphical** - This is discussed in detail in the section on graphing.

These are **not** four different types of functions: they are four different views of the same function. One of the most important skills in Algebra is converting a function between these different forms, and this theme will recur in different forms throughout the text.

1.6 Domain and Range⁶

Consider the function $y = \sqrt{x}$. If this function is given a 9 it hands back a 3. If this function is given a 2 it hands back...well, it hands back $\sqrt{2}$, which is approximately 1.4. The answer cannot be specified exactly as a fraction or decimal, but it is a perfectly good answer nonetheless.

On the other hand, what if this function is handed -4 ? There is no $\sqrt{-4}$, so the function has no number to hand back. If our function is a computer or calculator, it responds with an error message. So we see that this function is able to respond to the numbers 9 and 2, but it is not able to respond in any way to the number -4 . Mathematically, we express this by saying that 9 and 2 are in the “domain” of the square root function, and -4 is **not** in the domain of this function.

Definition 1.1: Domain

The **domain** of a function is all the numbers that it can successfully act on. Put another way, it is all the numbers that can go into the function.

A square root cannot successfully act on a negative number. We say that “The domain of $\sqrt{x}$ is all numbers **Image not finished** such that **Image not finished**” meaning that if you give this function zero or a positive number, it can act on it; if you give this function a negative number, it cannot.

A subtler example is the function $y = \sqrt{x+7}$. Does this function have the same domain as the previous function? No, it does not. If you hand this function a -4 it successfully hands back $\sqrt{3}$ (about 1.7). -4 is in the domain of this function. On the other hand, if you hand this function a -8 it attempts to take $\sqrt{-1}$ and fails; -8 is **not** in the domain of this function. If you play with a few more numbers, you should be able to convince yourself that the domain of this function is all numbers x such that $x \geq -7$.

You are probably familiar with two mathematical operations that are **not allowed**. The first is, you are not allowed to take the square root of a negative number. As we have seen, this leads to restrictions on the domain of any function that includes square roots.

The second restriction is, you are not allowed to divide by zero. This can also restrict the domain of functions. For instance, the function $y = \frac{1}{x^2-4}$ has as its domain all numbers **except** $x = 2$ and $x = -2$. These two numbers both cause the function to attempt to divide by 0, and hence fail. If you ask a calculator to plug $x = 2$ into this function, you will get an error message.

⁵This content is available online at <<http://cnx.org/content/m18195/1.2/>>.

⁶This content is available online at <<http://cnx.org/content/m18191/1.2/>>.

So: if you are given a function, how can you find its domain? Look for any number that puts a negative number under the square root; these numbers are **not** in the domain. Look for any number that causes the function to divide by zero; these numbers are not in the domain. All other numbers **are** in the domain.

Function	Domain	Comments
$\sqrt{x}$	$x \geq 0$	You can take the square root of 0, or of any positive number, but you cannot take the square root of a negative number.
$\sqrt{x+7}$	$x \geq -7$	If you plug in any number greater than or equal to -7 , you will be taking a legal square root. If you plug in a number less than -7 , you will be taking the square root of a negative number. This domain can also be understood graphically: the graph $y = \sqrt{x}$ has been moved 7 units to the left. See “horizontal permutations” below.
$\frac{1}{x}$	$x \neq 0$	In other words, the domain is “all numbers except 0.” You are not allowed to divide by 0. You are allowed to divide by anything else.
$\frac{1}{x-3}$	$x \neq 3$	If $x = 3$ then you are dividing by 0, which is not allowed. If $x = 0$ you are dividing by -3 , which is allowed. So be careful! The rule is not “when you are dividing, x cannot be 0.” The rule is “ x can never be any value that would put a 0 in the denominator.”
continued on next page		

$\frac{1}{x^2-4}$	$x \neq \pm 2$	Or, “ x can be any number except 2 or -2 .” Either of these x values will put a 0 in the denominator, so neither one is allowed.
$2^x + x^2 - 3x + 4$	All numbers	You can plug any x value into this function and it will come back with a number.
$\frac{\sqrt{x-3}}{x-5}$	$x \geq 3$ $x \neq 5$	In words, the domain is all numbers greater than or equal to 3, except the number 5 . Numbers less than 3 put negative numbers under the square root; 5 causes a division by 0.

Table 1.4

You can confirm all these results with your calculator; try plugging numbers into these functions, and see when you get errors!

A related concept is **range**.

Definition 1.2: Range

The **range** of a function is all the numbers that it may possibly produce. Put another way, it is all the numbers that can come **out** of the function.

To illustrate this example, let us return to the function $y = \sqrt{x+7}$. Recall that we said the domain of this function was all numbers x such that $x \geq -7$; in other words, you are allowed to put any number greater than or equal to -7 into this function.

What numbers might come out of this function? If you put in a -7 you get out a 0. ($\sqrt{0} = 0$) If you put in a -6 you get out $\sqrt{1} = 1$. As you increase the x value, the y values also increase. However, if you put in $x = -8$ nothing comes out at all. Hence, the range of this function is all numbers y such that $y \geq 0$. That is, this function is capable of handing back 0 or any positive number, but it will never hand back a negative number.

It’s easy to get the words domain and range confused—and it’s important to keep them distinct, because although they are **related** concepts, they are **different** from each other. One trick that sometimes helps is to remember that, in everyday useage, “your domain” is your home, your land—it is where you begin. A function begins in its own domain. It ends up somewhere out on the range.

1.6.1 A different notation for domain and range

Domains and ranges above are sometimes expressed as intervals, using the following rules:

- Parentheses $()$ mean “an interval starting or ending here, but not including this number”
- Square brackets $[]$ mean “an interval starting or ending here, including this number”

This is easiest to explain with examples.

This notation...	...means this...	...or in other words
$(-3, 5)$	All numbers between -3 and 5 , not including -3 and 5 .	$-3 < x < 5$
$[-3, 5]$	All numbers between -3 and 5 , including -3 and 5 .	$-3 \leq x \leq 5$
$[-3, 5)$	All numbers between -3 and 5 , including -3 but not 5 .	$-3 \leq x < 5$
$(-\infty, 10]$	All numbers less than or equal to 10 .	$x \leq 10$
$(23, \infty)$	All numbers greater than 23 .	$x > 23$
$(-\infty, 4) \cup (4, \infty)$	All numbers less than 4 , and all numbers greater than 4 . In other words, all numbers except 4 .	$x \neq 4$

Table 1.5

1.7 Functions in the Real World⁷

Why are functions so important that they form the heart of math from Algebra II onward?

Functions are used whenever **one variable depends on another variable**. This relationship between two variables is the most important in mathematics. It is a way of saying "If you tell me what x is, I can tell you what y is." We say that y "depends on" x , or y "is a function of" x .

A few examples:

Example 1.3: Function Concepts – Functions in the Real World

- "The area of a circle depends on its radius."
- "The amount of money Alice makes depends on the number of hours she works."
- "Max threw a ball. The height of the ball depends on how many seconds it has been in the air."

In each case, there are two variables. Given enough information about the scenario, you could assert that **if you tell me this variable, I will tell you that one**. For instance, suppose you know that Alice makes \$100 per day. Then we could make a chart like this.

If Alice works this many days...	...she makes this many dollars
0	0
1	100
$1\frac{1}{2}$	150
8	800

Table 1.6

⁷This content is available online at <<http://cnx.org/content/m18193/1.2/>>.

If you tell me how long she has worked, I will tell you how much money she has made. Her earnings “depend on” how long she works.

The two variables are referred to as the **dependent variable** and the **independent variable**. The dependent variable is said to “depend on” or “be a function of” the independent variable. “The height of the ball is a function of the time.”

Example 1.4: Bad Examples of Functional Relationships

- “The number of Trojan soldiers depends on the number of Greek soldiers.”
- “The time depends on the height of the ball.”

The first of these two examples is by far the most common. It is simply not true. There may be a relationship between these two quantities—for instance, the **sum** of these two variables might be the total number of soldiers, and the **difference** between these two quantities might suggest whether the battle will be a fair one. But there is no **dependency** relationship—that is, no way to say “If you tell me the number of Greek soldiers, I will tell you the number of Trojan soldiers”—so this is not a function.

The second example is subtler: it confuses the **dependent** and the **independent** variables. The height depends on the time, not the other way around. More on this in the discussion of “Inverse Functions”.

1.8 Function Notation⁸

1.8.1 Function Notation

Functions are represented in math by parentheses. When you write $f(x)$ you indicate that the variable f is a function of—or depends on—the variable x .

For instance, suppose $f(x) = x^2 + 3x$. This means that f is a function that takes whatever you give it, and squares it, and multiplies it by 3, and adds those two quantities.

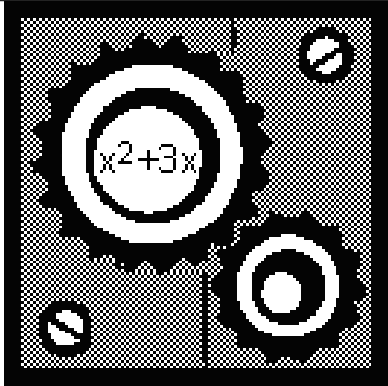
		
7 →		→ $f(7) = 7^2 + 3(7) = 70$
10 →		→ $f(10) = 10^2 + 3(10) = 130$
x →		→ $f(x) = x^2 + 3x$
y →		→ $f(y) = y^2 + 3y$
a dog →		→ $f(\text{dog}) = (\text{dog})^2 + 3(\text{dog})$ (*not in the domain)

Table 1.7

⁸This content is available online at <<http://cnx.org/content/m18188/1.3/>>.

The notation $f(7)$ means “plug the number 7 into the function f .” It does **not** indicate that you are multiplying f times 7. To evaluate $f(7)$ you take the function $f(x)$ and replace all occurrences of the variable x with the number 7. If this function is given a 7 it will come out with a 70.

If we write $f(y) = y^2 + 3y$ we have **not specified a different function**. Remember, the function is not the variables or the numbers, it is the process. $f(y) = y^2 + 3y$ also means “whatever number comes in, square it, multiply it by 3, and add those two quantities.” So it is a different way of writing the same function.

Just as many students expect all variables to be named x , many students—and an unfortunate number of parents—expect all functions to be named f . The correct rule is that—whenever possible—**functions, like variables, should be named descriptively**. For instance, if Alice makes \$100/day, we might write:

- Let m equal the amount of money Alice has made (measured in dollars)
- Let t equal the amount of time Alice has worked (measured in days)
- Then, $m(t) = 100t$

This last equation should be read “ m is a function of t (or m depends on t). Given any value of the variable t , you can multiply it by 100 to find the corresponding value of the variable m .”

Of course, this is a very simple function! While simple examples are helpful to illustrate the concept, it is important to realize that very complicated functions are also used to model real world relationships. For instance, in Einstein’s Special Theory of Relativity, if an object is going very fast, its mass is multiplied by $\frac{1}{\sqrt{1 - \frac{v^2}{9 \cdot 10^{16}}}}$. While this can look extremely intimidating, it is just another function. The speed v is the

independent variable, and the mass m is dependent. Given any speed v you can determine how much the mass m is multiplied by.

1.9 Algebraic Generalizations⁹

When you have a “generalization,” you have one **broad fact** that allows you to assume many **specific** facts as examples.

Example 1.5

Generalization: “Things fall down when you drop them.”

Specific facts, or examples:

- Leaves fall down when you drop them
- Bricks fall down when you drop them
- Tennis balls fall down when you drop them

If **any one** of the individual statements does not work, the generalization is invalid. (This generalization became problematic with the invention of the helium balloon.)

Scientists tend to work **empirically**, meaning they start with the specific facts and work their way back to the generalization. Generalizations are valued in science because they bring order to apparently disconnected facts, and that order in turn suggests underlying theories.

Mathematicians also spend a great deal of time looking for generalizations. When you have an “algebraic generalization” you have one **algebraic** fact that allows you to assume many **numerical** facts as examples.

Consider, for instance, the first two functions in the function game.

⁹This content is available online at <<http://cnx.org/content/m18186/1.3/>>.

1. Double the number, then add six.
2. Add three to the number, then double.

These are very different “recipes.” However, their inclusion in the function game is a bit unfair, because—here comes the generalization—**these two functions will always give the same answer.** Whether the input is positive or negative, integer or fraction, small or large, these two functions will mimic each other perfectly. We can express this generalization in words.

Example 1.6

Generalization: If you plug a number into the function **double and add six**, and plug the same number into the function **add three and double**, the two operations will give the same answer.

Specific facts, or examples:

- If you double -5 and add six; or, if you add -5 to 3 and then double; you end up with the same answer.
- If you double 13 and add six; or, if you add 13 to 3 and then double; you end up with the same answer.

There is literally an infinite number of specific claims that fit this pattern. We don’t need to prove or test each of these claims individually: once we have proven the generalization, we know that **all** these facts must be true.

We can express this same generalization pictorially by showing two “function machines” that always do the same thing.

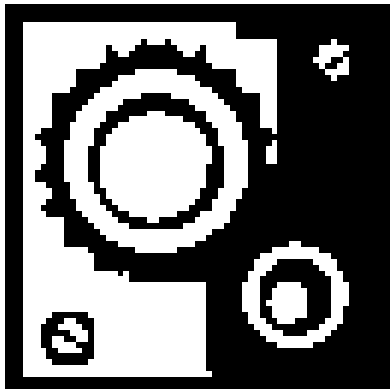
	$2x + 6$	
		
$-5 \rightarrow$		$\rightarrow 2(-5) + 6 = -4$
$0 \rightarrow$		$\rightarrow 2(0) + 6 = 6$
$13 \rightarrow$		$\rightarrow 2(13) + 6 = 32$

Table 1.8

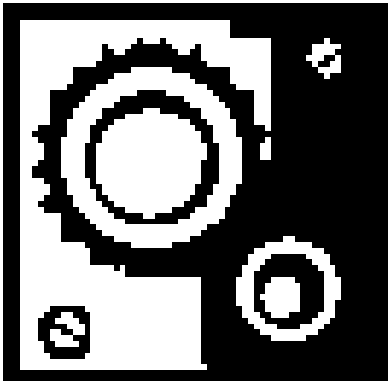
	$2(x+3)$	
		
$-5 \rightarrow$		$\rightarrow 2(-5) + 6 = -4$
$0 \rightarrow$		$\rightarrow 2(0) + 6 = 6$
$13 \rightarrow$		$\rightarrow 2(13) + 6 = 32$

Table 1.9

But the most common way to express this generalization is algebraically, by asserting that these two functions **equal** each other.

$$2x + 6 = 2(x + 3) \quad (1.1)$$

Many beginning Algebra II students will recognize this as the distributive property. Given $2(x + 3)$ they can correctly turn it into $2x + 6$. **But they often fail to realize what this equality means—that given the same input, the two functions will always yield the same output.**

Example 1.7

Generalization: $2x + 6 = 2(x + 3)$

Specific facts, or examples:

- $(2 \times -5) + 6 = 2 \times (-5 + 3)$
- $(2 \times 0) + 6 = 2 \times (0 + 3)$
- $(2 \times 13) + 6 = 2 \times (13 + 3)$

It's worth stopping for a moment here to think about the $=$ symbol. Whenever it is used, $=$ indicates that two things are the same. However, the following two equations use the $=$ in very different ways.

$$2x^2 + 5x = 3 \quad (1.2)$$

$$\frac{2x^2 - 18}{x + 3} = 2x - 6 \quad (1.3)$$

In the first equation, the $=$ challenges you to **solve for x** . “Find all the x values that make this equation true.” The answers in this case are $x = \frac{1}{2}$ and $x = -3$. If you plug in either of these two x -values, you get a true equation; for any other x -value, you get a false equation.

The second equation cannot be solved for x ; the $=$ sign in this case is asserting an equality that is true for **any** x -value. Let's try a few.

Example 1.8

Generalization: $\frac{2x^2 - 18}{x + 3} = 2x - 6$

Specific facts, or examples:

$x = 3$	$\frac{2(3)^2-18}{(3)+3} = \frac{18-18}{6} = 0$	$2(3) - 6 = 0$ [U+2713]
$x = -2$	$\frac{2(-2)^2-18}{(-2)+3} = \frac{8-18}{1} = -10$	$2(-2) - 6 = -10$ [U+2713]
$x = 0$	$\frac{2(0)^2-18}{(0)+3} = \frac{0-18}{3} = -6$	$2(0) - 6 = -6$ [U+2713]
$x = \frac{1}{2}$	$\frac{2(\frac{1}{2})^2-18}{(\frac{1}{2})+3} = \frac{\frac{1}{2}-18}{\frac{7}{2}} = \left(\frac{-35}{2}\right)\left(\frac{2}{7}\right) = -5$	$2\left(\frac{1}{2}\right) - 6 = -5$ [U+2713]

Table 1.10

With a calculator, you can attempt more difficult values such as $x = -26$ or $x = \pi$; in every case, the two formulas will give the same answer. When we assert that two very different functions will always produce the same answers, we are making a very powerful generalization.

Exception: $x = -3$ is outside the domain of one of these two functions. In this important sense, the two functions are not in fact equal. Take a moment to make sure you understand why this is true!

Such generalizations are very important because they allow us to **simplify**.

Suppose that you were told “I am going to give you a hundred numbers. For each number I give you, square it, then double the answer, then subtract eighteen, then divide by the original number plus three.” This kind of operation comes up all the time. But you would be quite relieved to discover that you can accomplish the same task by simply doubling each number and subtracting 6! The generalization in this case is $\frac{2x^2-18}{x+3} = 2x - 6$; you will be creating exactly this sort of generalization in the chapter on Rational Expressions.

1.10 Graphing¹⁰

Graphing, like algebraic generalizations, is a difficult topic because many students know **how to do it** but are not sure **what it means**.

For instance, consider the following graph:

¹⁰This content is available online at <<http://cnx.org/content/m18196/1.2/>>.

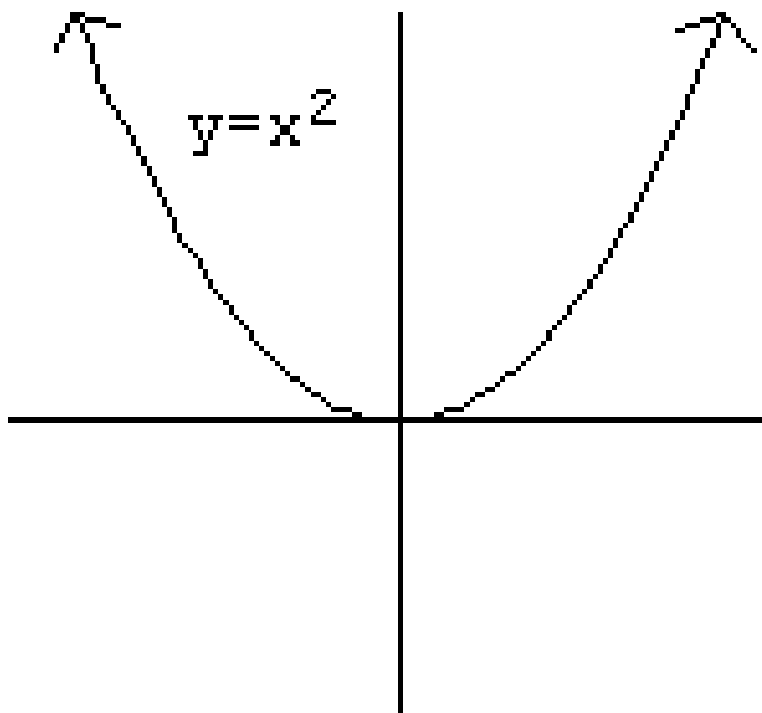


Figure 1.1

If I asked you “Draw the graph of $y = x^2$ ” you would probably remember how to plot points and draw the shape.

But suppose I asked you this instead: “Here’s a function, $y = x^2$. And here’s a shape, that sort of looks like a U. What do they actually have to do with each other?” This is a harder question! What does it **mean** to graph a function?

The answer is simple, but it has important implications for a proper understanding of functions. Recall that every point on the plane is designated by a unique (x, y) pair of coordinates: for instance, one point is $(5, 3)$. We say that its x -value is 5 and its y -value is 3.

A few of these points have the particular property that their y -values are the square of their x -values. For instance, the points $(0, 0)$, $(3, 9)$, and $(5, -25)$ all have that property. $(5, 3)$ and $(-2, -4)$ do not.

The graph shown—the pseudo-U shape—is **all the points in the plane that have this property**. Any point whose y -value is the square of its x -value is on this shape; any point whose y -value is not the square of its x -value is not on this shape. Hence, glancing at this shape gives us a complete visual picture of the function $y = x^2$ **if we know how to interpret it correctly**.

1.10.1 Graphing Functions

Remember that every function specifies a relationship between two variables. When we graph a function, we put the **independent variable** on the x -axis, and the **dependent variable** on the y -axis.

For instance, recall the function that describes Alice's money as a function of her hours worked. Since Alice makes \$12/hour, her financial function is $m(t) = 12t$. We can graph it like this.

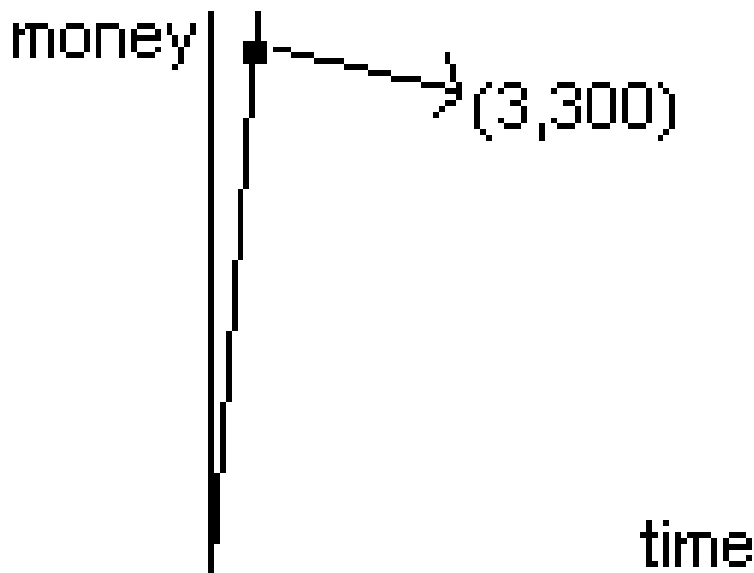


Figure 1.2

This simple graph has a great deal to tell us about Alice's job, if we read it correctly.

- The graph contains the point (3,300). What does that tell us? That after Alice has worked for three hours, she has made \$300.
- The graph goes through the origin (the point (0,0)). What does that tell us? That when she works 0 hours, Alice makes no money.
- The graph exists only in the first quadrant. What does that tell us? On the mathematical level, it indicates the **domain** of the function ($t \geq 0$) and the **range** of the function ($m \geq 0$). In terms of the situation, it tells us that Alice cannot work negative hours or make negative money.
- The graph is a straight line. What does that tell us? That Alice makes the same amount of money every day: every day, her money goes up by \$100. (\$100/day is the **slope** of the line—more on this in the section on linear functions.)

Consider now the following, more complicated graph, which represents **Alice's hair length as a function of time** (where time is now measured in weeks instead of hours).

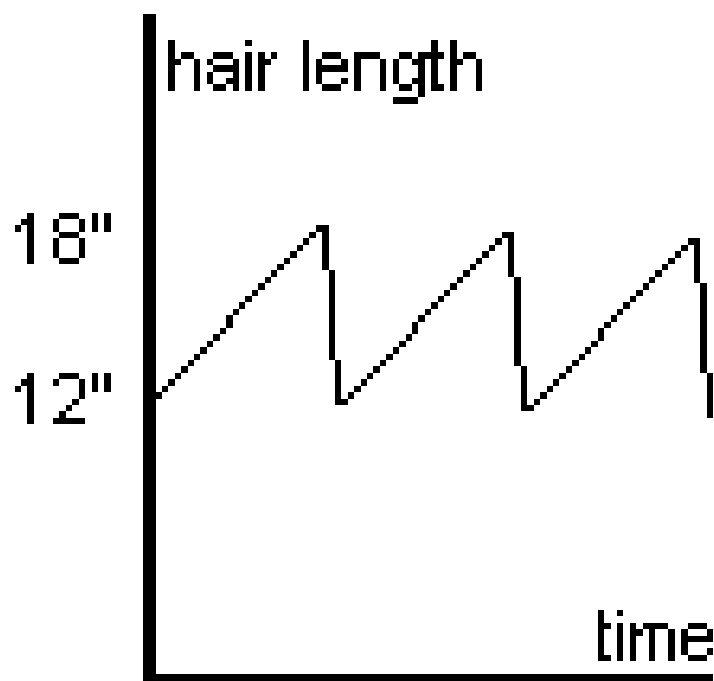


Figure 1.3

What does this graph $h(t)$ tell us? We can start with the same sort of simple analysis.

- The graph goes through the point $(0, 12)$. This tells us that at time $(t = 0)$, Alice's hair is 12" long.
- The range of this graph appears to be $12 \leq h \leq 18$. Alice never allows her hair to be shorter than 12" or longer than 18".

But what about the shape of the graph? The graph shows a gradual incline up to 18", and then a precipitous drop back down to 12"; and this pattern repeats throughout the shown time. The most likely explanation is that Alice's hair grows slowly until it reaches 18", at which point she goes to the hair stylist and has it cut down, within a very short time (an hour or so), to 12". Then the gradual growth begins again.

1.10.2 The rule of consistency, graphically

Consider the following graph.

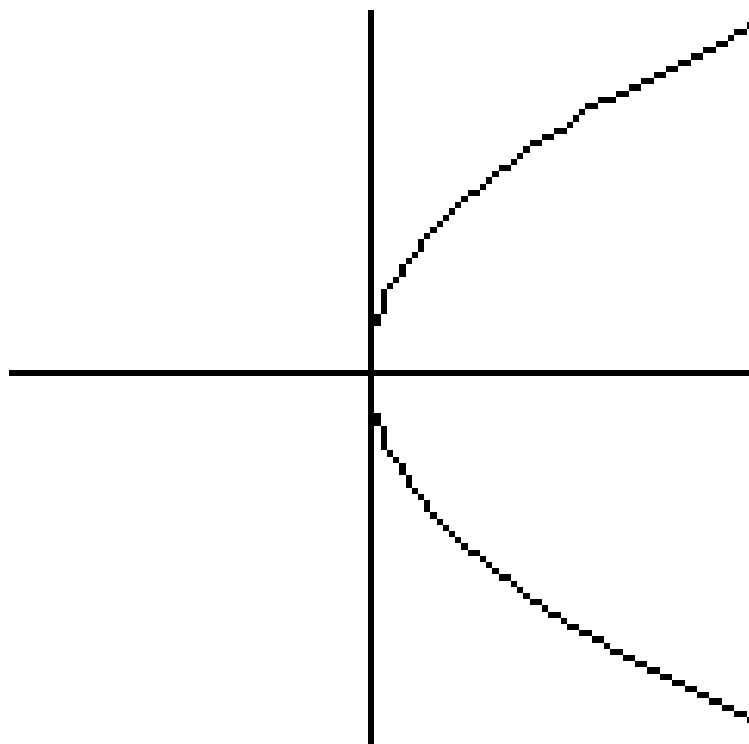


Figure 1.4

This is our earlier “U” shaped graph ($y = x^2$) turned on its side. This might seem like a small change. But ask this question: what is y when $x = 3$? This question has two answers. This graph contains the points $(3, -9)$ and $(3, 9)$. So when $x = 3$, y is both 9 and -9 on this graph.

This violates the only restriction on functions—the rule of consistency. Remember that the x -axis is the independent variable, the y -axis the dependent. In this case, one “input” value (3) is leading to two different “output” values $(-9, 9)$. We can therefore conclude that this graph **does not represent a function at all**. No function, no matter how simple or complicated, could produce this graph.

This idea leads us to the “vertical line test,” the graphical analog of the rule of consistency.

Definition 1.3: The Vertical Line Test

If you can draw **any** vertical line that touches a graph in two places, then that graph violates the rule of consistency and therefore does not represent any function.

It is important to understand that the vertical line test is not a new rule! It is the graphical version of the rule of consistency. If any vertical line touches a graph in two places, then the graph has two different y -values for the same x -value, and this is the only thing that functions are not allowed to do.

1.10.3 What happens to the graph, when you add 2 to a function?

Suppose the following is the graph of the function $y = f(x)$.

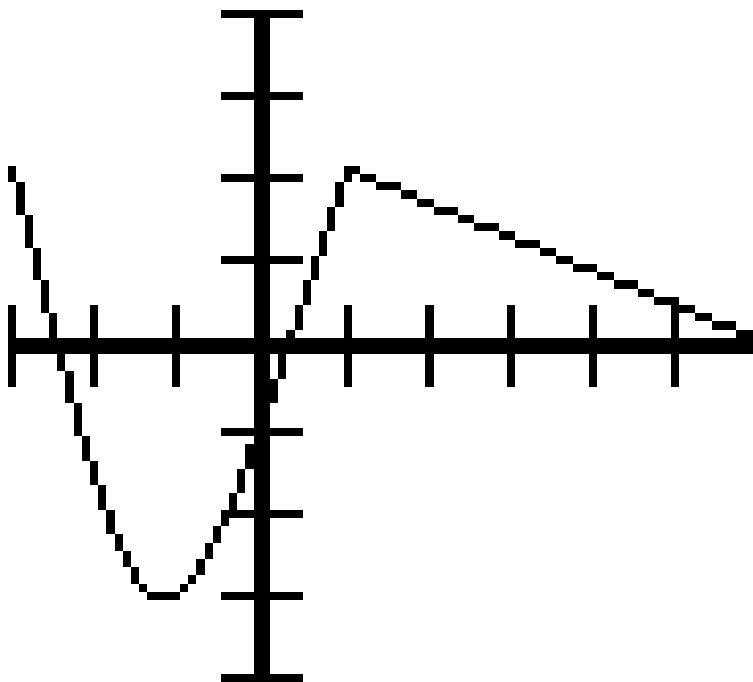


Figure 1.5: $y = f(x)$; Contains the following points (among others): $(-3, 2)$, $(-1, -3)$, $(1, 2)$, $(6, 0)$

We can see from the graph that the domain of the graph is $-3 \leq x \leq 6$ and the range is $-3 \leq y \leq 2$.

Question: **What does the graph of $y = f(x) + 2$ look like?**

This might seem an impossible question, since we do not even know what the function $f(x)$ is. But we don't need to know that in order to plot a few points.

x	$f(x)$	$f(x+2)$	so $y = f(x)$ contains this point	and $y = f(x) + 2$ contains this point
-3	2	4	$(-3, 2)$	$(-3, 4)$
-1	-3	-1	$(-1, -3)$	$(-1, -1)$
1	2	4	$(1, 2)$	$(1, 4)$
6	0	2	$(6, 0)$	$(6, 2)$

Table 1.11

If you plot these points on a graph, the pattern should become clear. Each point on the graph is **moving up by two**. This comes as no surprise: since you added 2 to each y-value, and adding 2 to a y-value moves any point up by 2. So the new graph will look identical to the old, only moved up by 2.

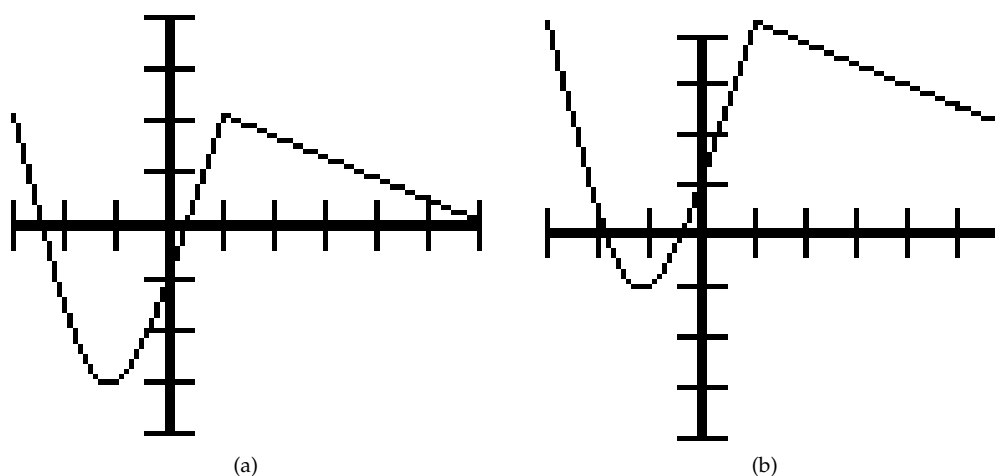
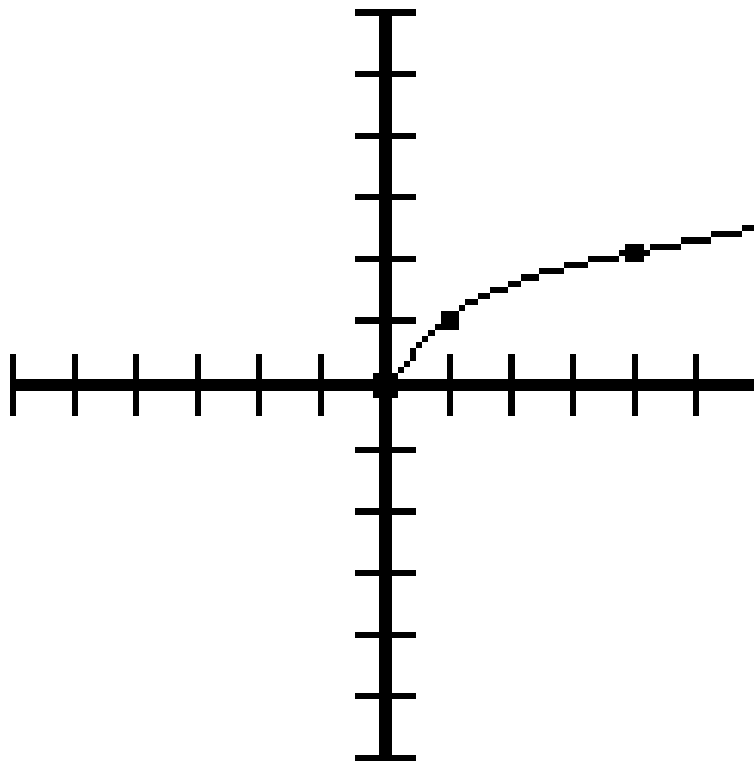


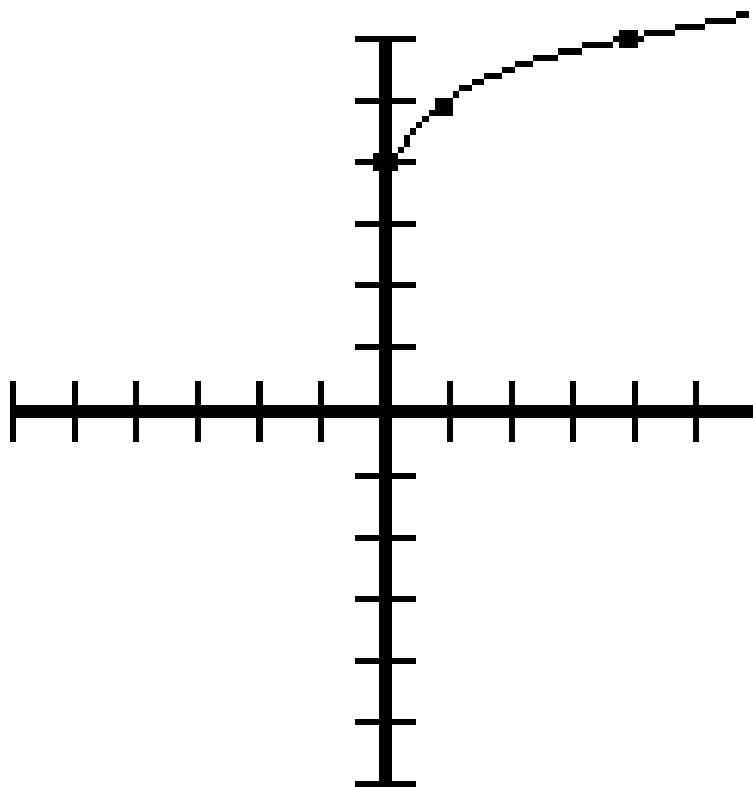
Figure 1.6: (a) $y = f(x)$ (b) $y = f(x) + 2$; All y -values are 2 higher

In a similar way, it should be obvious that if you **subtract** 10 from a function, the graph moves **down** by 10. Note that, in either case, the **domain** of the function is the same, but the **range** has changed.

These permutations work for **any function**. Hence, given the graph of the function $y = \sqrt{x}$ below (which you could generate by plotting points), you can produce the other two graphs **without** plotting points, simply by moving the first graph up and down.



(a)



(b)

1.10.4 Other vertical permutations

Adding or subtracting a constant from $f(x)$, as described above, is one example of a **vertical permutation**: it moves the graph up and down. There are other examples of vertical permutations.

For instance, what does **doubling** a function do to a graph? Let's return to our original function:

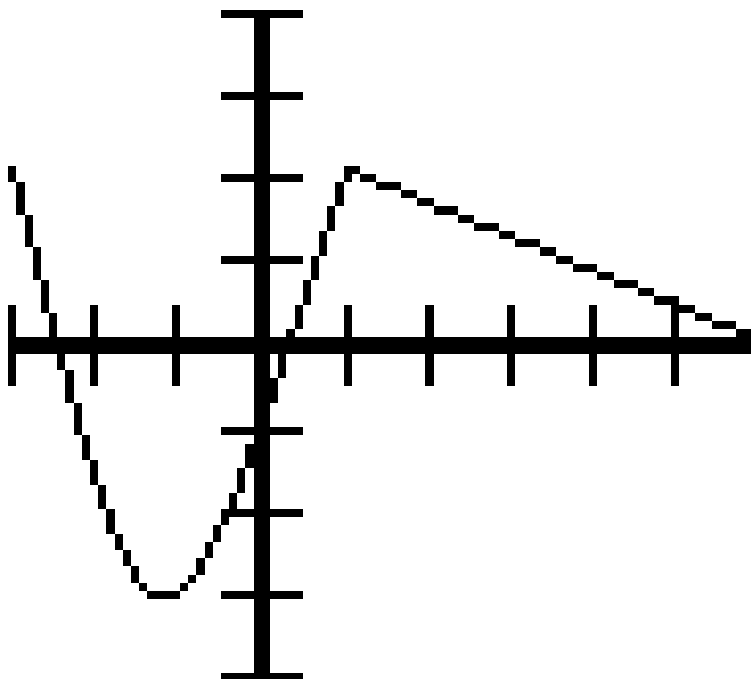


Figure 1.8: $y = f(x)$

What does the graph $y = 2f(x)$ look like? We can make a table similar to the one we made before.

x	$f(x)$	$2f(x)$	so $y = 2f(x)$ contains this point
-3	2	4	$(-3, 4)$
-1	-3	-6	$(-1, -6)$
1	2	4	$(1, 4)$
6	0	0	$(6, 0)$

Table 1.12

In general, the high points move higher; the low points move lower. The entire graph is **vertically stretched**, with each point moving farther away from the x-axis.

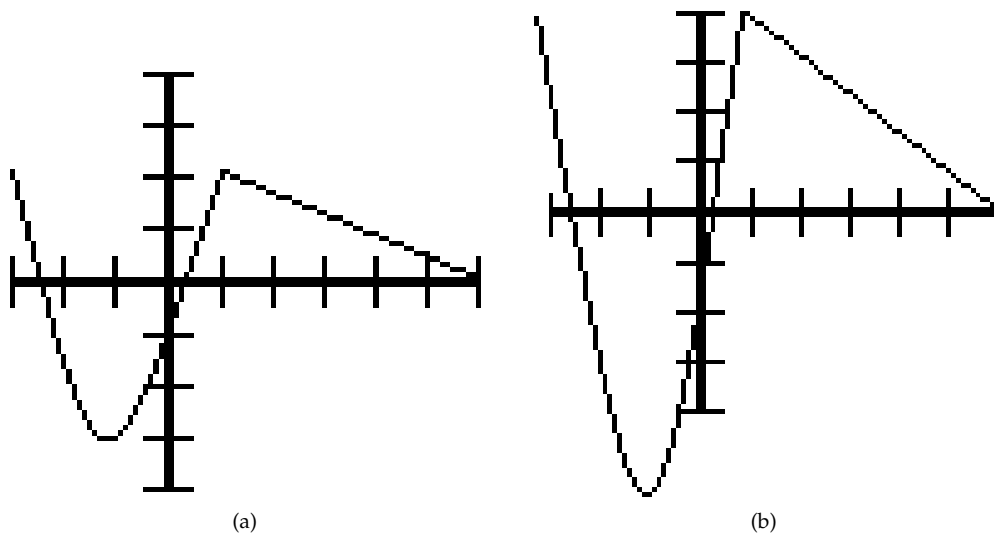


Figure 1.9: (a) $y = f(x)$ (b) $y = 2f(x)$; All y -values are doubled

Similarly, $y = \frac{1}{2}f(x)$ yields a graph that is vertically compressed, with each point moving toward the x -axis.

Finally, what does $y = -f(x)$ look like? All the positive values become negative, and the negative values become positive. So, point by point, the entire graph flips over the x -axis.

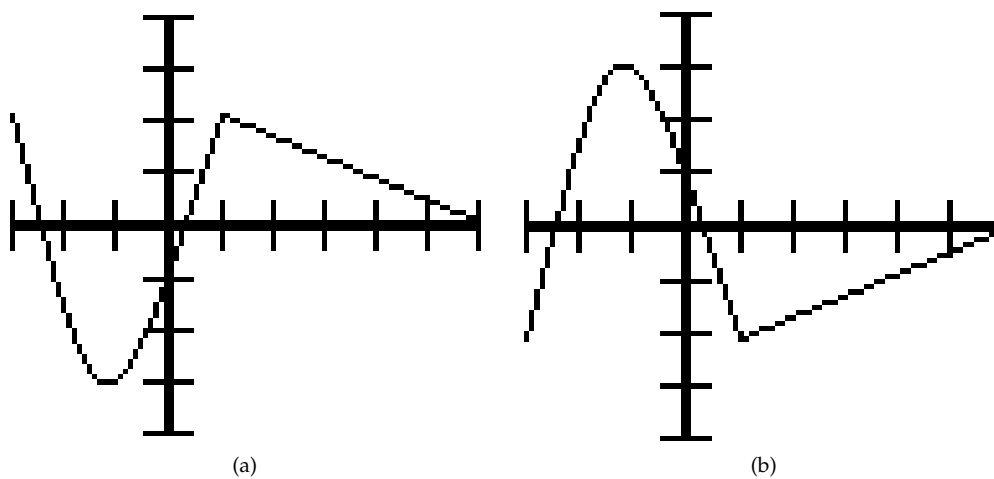


Figure 1.10: (a) $y = f(x)$ (b) $y = -f(x)$; All y -values change sign

1.10.5 What happens to the graph, when you add 2 to the x value?

Vertical permutations affect the y-value; that is, the output, or the function itself. Horizontal permutations affect the x-value; that is, the numbers that come in. They often do the opposite of what it naturally seems they should.

Let's return to our original function $y = f(x)$.

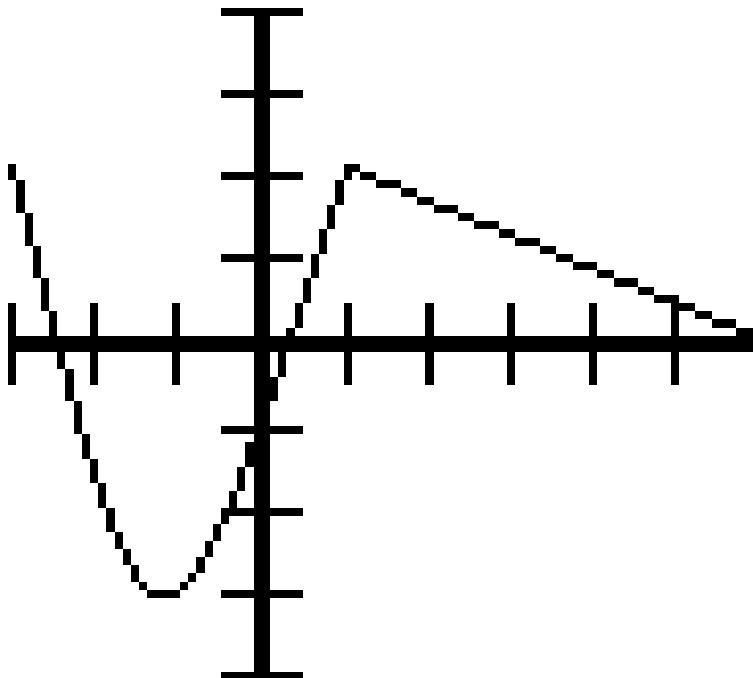


Figure 1.11: $y = f(x)$; Contains the following points (among others): $(-3, 2)$, $(-1, -3)$, $(1, 2)$, $(6, 0)$

Suppose you were asked to graph $y = f(x + 2)$. Note that this is **not the same** as $f(x) + 2$! The latter is an instruction to run the function, and then add 2 to all results. But $y = f(x + 2)$ is an instruction to add 2 to every x-value **before plugging it into the function**.

- $f(x) + 2$ changes y , and therefore shifts the graph vertically
- $f(x + 2)$ changes x , and therefore shifts the graph horizontally.

But which way? In analogy to the vertical permutations, you might expect that adding two would shift the graph to the right. But let's make a table of values again.

x	$x + 2$	$f(x + 2)$	so $y = f(x + 2)$ contains this point
-5	-3	$f(-3)=2$	$(-5, 2)$
-3	-1	$f(-1)=-3$	$(-3, -3)$
-1	1	$f(1)=2$	$(-1, 2)$
4	6	$f(6)=0$	$(4, 0)$

Table 1.13

This is a very subtle, very important point—please follow it closely and carefully! First of all, make sure you understand where all the numbers in that table came from. Then look what happened to the original graph.

NOTE: The original graph $f(x)$ contains the point $(6, 0)$; therefore, $f(x + 2)$ contains the point $(4, 0)$. The point has moved two spaces to the left.

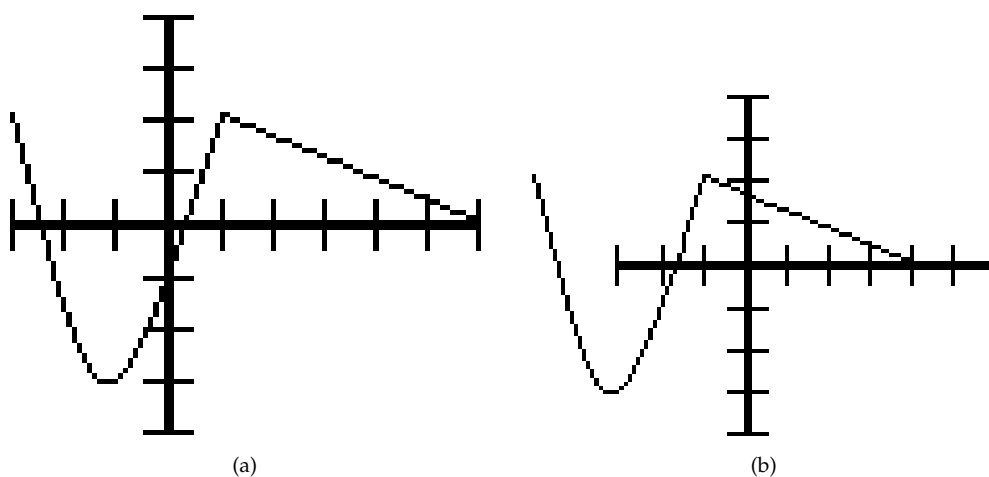


Figure 1.12: (a) $y = f(x)$ (b) $y = f(x + 2)$; Each point is shifted to the left

You see what I mean when I say horizontal permutations “often do the opposite of what it naturally seems they should”? **Adding** two moves the graph to the **left**.

Why does it work that way? Here is my favorite way of thinking about it. $f(x - 2)$ is an instruction that says to each point, “look two spaces to your left, and copy what the original function is doing **there**.” At $x = 5$ it does what $f(x)$ does at $x = 3$. At $x = 10$, it copies $f(8)$. And so on. Because it is always copying $f(x)$ to its **left**, this graph ends up being a copy of $f(x)$ moved to the **right**. If you understand this way of looking at it, all the rest of the horizontal permutations will make sense.

Of course, as you might expect, subtraction has the opposite effect: $f(x - 6)$ takes the original graph and moves it 6 units to the **right**. In either case, these horizontal permutations affect the **domain** of the original function, but not its **range**.

1.10.6 Other horizontal permutations

Recall that $y = 2f(x)$ **vertically stretches** a graph; $y = \frac{1}{2}f(x)$ **vertically compresses**. Just as with addition and subtraction, we will find that the horizontal equivalents work backward.

x	$2x$	$f(2x)$	so $y = 2f(x)$ contains this point
$-1\frac{1}{2}$	-3	2	$(-1\frac{1}{2}, 2)$
$-\frac{1}{2}$	-1	-3	$(-\frac{1}{2}, -3)$
$\frac{1}{2}$	1	2	$(\frac{1}{2}, 2)$
3	6	0	$(3, 0)$

Table 1.14

The original graph $f(x)$ contains the point $(6, 0)$; therefore, $f(2x)$ contains the point $(3, 0)$. Similarly, $(-1; -3)$ becomes $(-\frac{1}{2}; -3)$. Each point is closer to the y -axis; the graph has horizontally **compressed**.

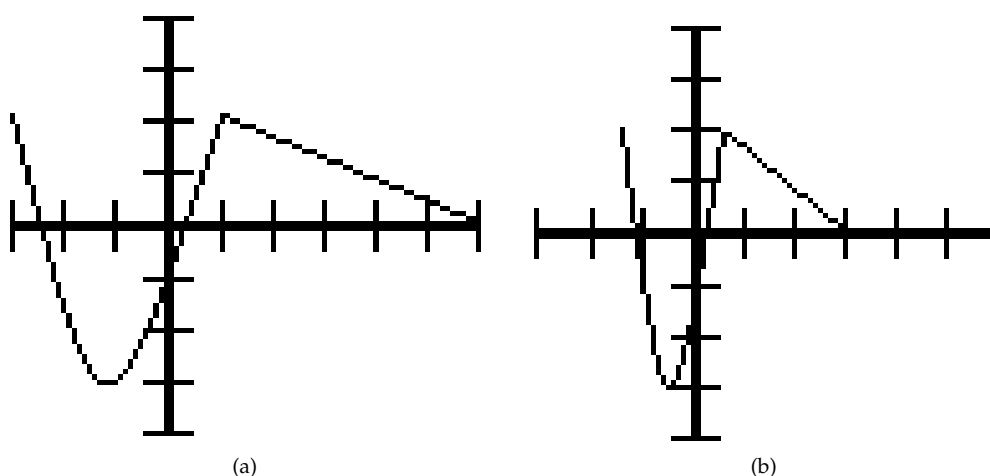


Figure 1.13: (a) $y = f(x)$ (b) $y = f(2x)$; Each point is twice as close to the y -axis

We can explain this the same way we explained $f(x - 2)$. In this case, $f(2x)$ is an instruction that says to each point, “Look outward, at the x -value that is double yours, and copy what the original function is doing **there**.” At $x = 5$ it does what $f(x)$ does at $x = 10$. At $x = -3$, it copies $f(-6)$. And so on. Because it is always copying $f(x)$ **outside** itself, this graph ends up being a copy of $f(x)$ moved **inward**; ie a compression. Similarly, $f(\frac{1}{2}x)$ causes each point to look inward toward the y -axis, so it winds up being a **horizontally stretched** version of the original.

Finally, $y = f(-x)$ does precisely what you would expect: it flips the graph around the y -axis. $f(-2)$ is the old $f(2)$ and vice-versa.

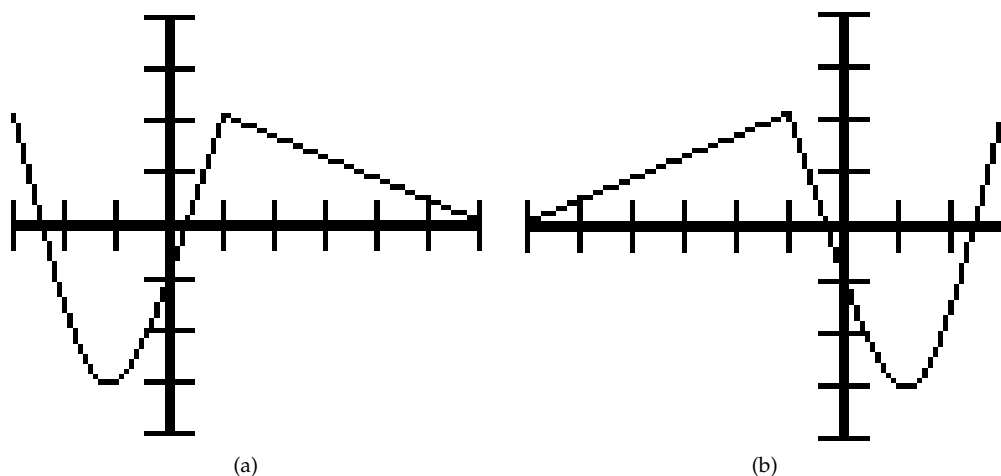
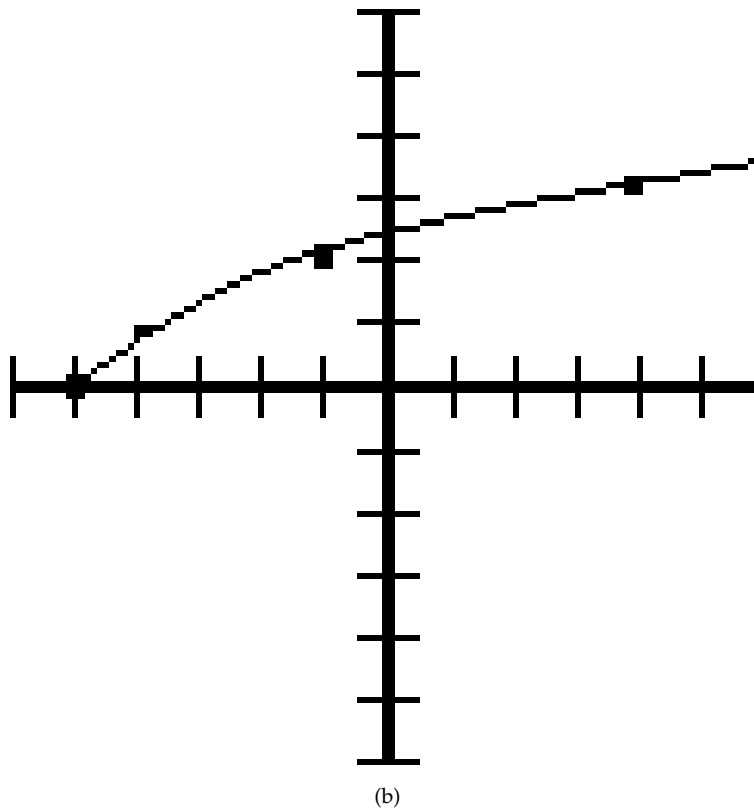
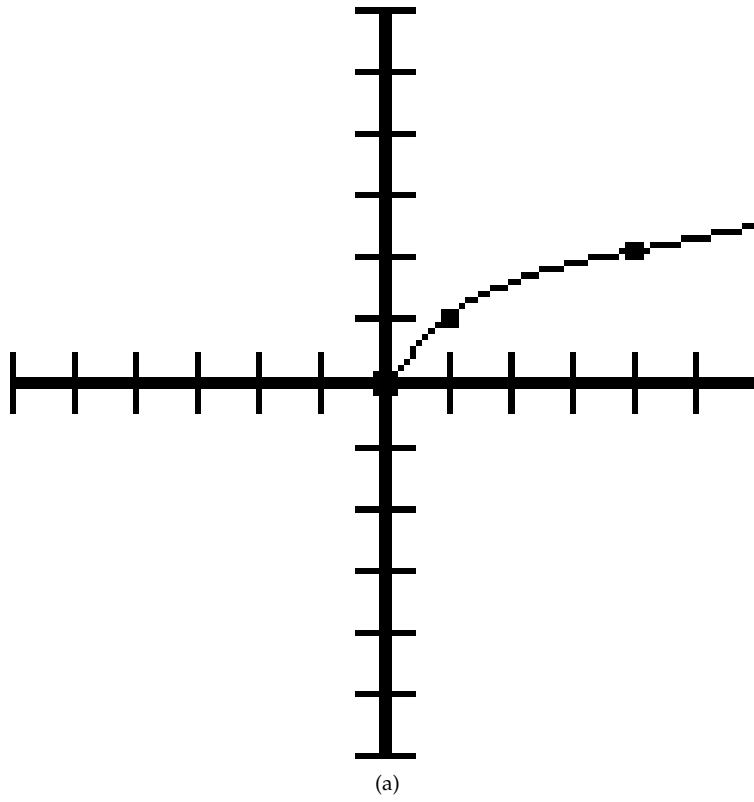


Figure 1.14: (a) $y = f(x)$ (b) $y = f(-x)$; Each point flips around the y -axis

All of these permutations do not need to be memorized: only the general principles need to be understood. But once they are properly understood, even a complex graph such as $y = -2(x + 3)^2 + 5$ can be easily graphed. You take the (known) graph of $y = x^2$, flip it over the x -axis (because of the negative sign), stretch it vertically (the 2), move it to the left by 3, and move it up 5.

With a good understanding of permutations, and a very simple list of known graphs, it becomes possible to graph a wide variety of important functions. To complete our look at permutations, let's return to the graph of $y = \sqrt{x}$ in a variety of flavors.



1.11 Lines¹¹

Most students entering Algebra II are already familiar with the basic mechanics of graphing lines. Recap-
ping very briefly: the equation for a line is $y = mx + b$ where b is the y -intercept (the place where the line
crosses the y -axis) and m is the slope. If a linear equation is given in another form (for instance, $4x + 2y = 5$),
the easiest way to graph it is to rewrite it in $y = mx + b$ form (in this case, $y = -2x + 2\frac{1}{2}$).

There are two purposes of reintroducing this material in Algebra II. The first is to frame the discussion as
linear functions modeling behavior. The second is to deepen your understanding of the important concept
of slope.

Consider the following examples. Sam is a salesman—he earns a commission for each sale. Alice is a
technical support representative—she earns \$100 each day. The chart below shows their bank accounts
over the week.

After this many days (t)	Sam's bank account (S)	Alice's bank account (A)
0 (*what they started with)	\$75	\$750
1	\$275	\$850
2	\$375	\$950
3	\$450	\$1,050
4	\$480	\$1,150
5	\$530	\$1,250

Table 1.15

Sam has some extremely good days (such as the first day, when he made \$200) and some extremely bad
days (such as the second day, when he made nothing). Alice makes exactly \$100 every day.

Let d be the number of days, S be the number of dollars Sam has made, and A be the number of dollars
Alice has made. Both S and A are functions of time. But $s(t)$ is **not a linear function**, and $A(t)$ is a **linear
function**.

Definition 1.4: Linear Function

A function is said to be “linear” if **every time the independent variable increases by 1, the de-
pendent variable increases or decreases by the same amount**.

Once you know that Alice's bank account function is linear, there are only two things you need to know
before you can predict her bank account on any given day.

- How much money she started with (\$750 in this example). This is called the **y -intercept**.
- How much she makes each day (\$100 in this example). This is called the **slope**.

y -intercept is relatively easy to understand. Verbally, it is where the function starts; graphically, it is where
the line crosses the y -axis.

But what about slope? One of the best ways to understand the idea of slope is to convince yourself that all
of the following definitions of slope are actually the same.

¹¹This content is available online at <<http://cnx.org/content/m18197/1.2/>>.

Definitions of Slope		
In our example	In general	On a graph
Each day, Alice's bank account increases by 100. So the slope is 100.	Each time the independent variable increases by 1, the dependent variable increases by the slope.	Each time you move to the right by 1, the graph goes up by the slope.
Between days 2 and 5, Alice earns \$300 in 3 days. $300/3=100$. Between days 1 and 3, she earns \$200 in 2 days. $200/2=100$.	Take any two points. The change in the dependent variable, divided by the change in the independent variable, is the slope.	Take any two points. The change in y divided by the change in x is the slope. This is often written as $\frac{\Delta y}{\Delta x}$, or as $\frac{\text{rise}}{\text{run}}$
The higher the slope, the faster Alice is making money.	The higher the slope, the faster the dependent variable increases.	The higher the slope, the faster the graph rises as you move to the right.

Table 1.16

So slope does not tell you where a graph is, but how quickly it is rising. Looking at a graph, you can get an approximate feeling for its slope without any numbers. Examples are given below.

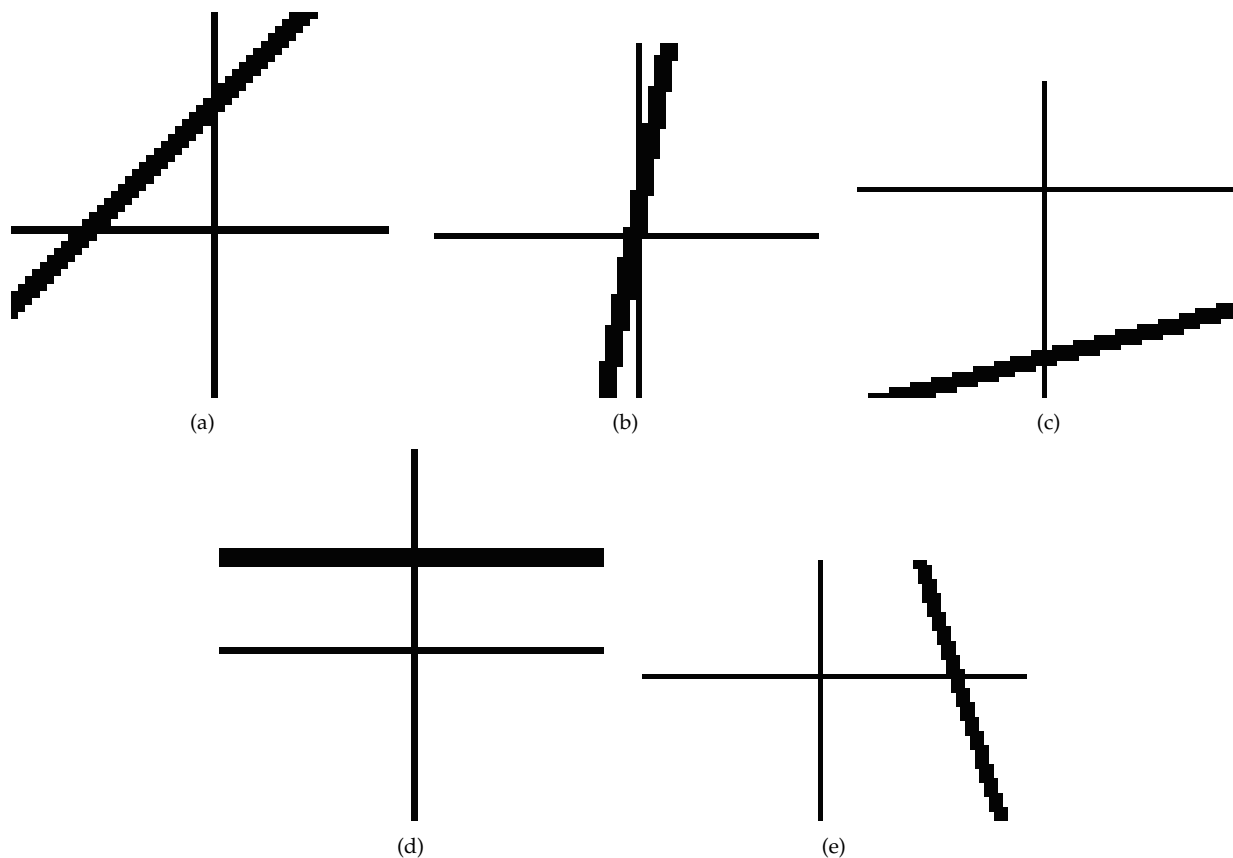


Figure 1.16: (a) A slope of 1: each time you go **over** 1, you also go **up** 1 (b) A steep slope of perhaps 3 or 4 (c) A gentle slope of perhaps $\frac{1}{2}$. (d) A horizontal line has a slope of 0: each time you go **over** 1, you don't go **up** at all! (e) This goes **down** as you move left to right. So the slope is negative. It is steep: maybe a -2 .

1.12 Composite Functions¹²

You are working in the school cafeteria, making peanut butter sandwiches for today's lunch.

- The more classes the school has, the more children there are.
- The more children there are, the more sandwiches you have to make.
- The more sandwiches you have to make, the more pounds (lbs) of peanut butter you will use.
- The more peanut butter you use, the more money you need to budget for peanut butter.

...and so on. Each sentence in this little story is a function. Mathematically, if c is the number of classes and h is the number of children, then the first sentence asserts the existence of a function $h(c)$.

The principal walks up to you at the beginning of the year and says "We're considering expanding the school. If we expand to 70 classes, how much money do we need to budget? What if we expand to 75?

¹²This content is available online at <http://cnx.org/content/m18187/1.2/>.

How about 80?" For each of these numbers, you have to calculate each number from the previous one, until you find the final budget number.

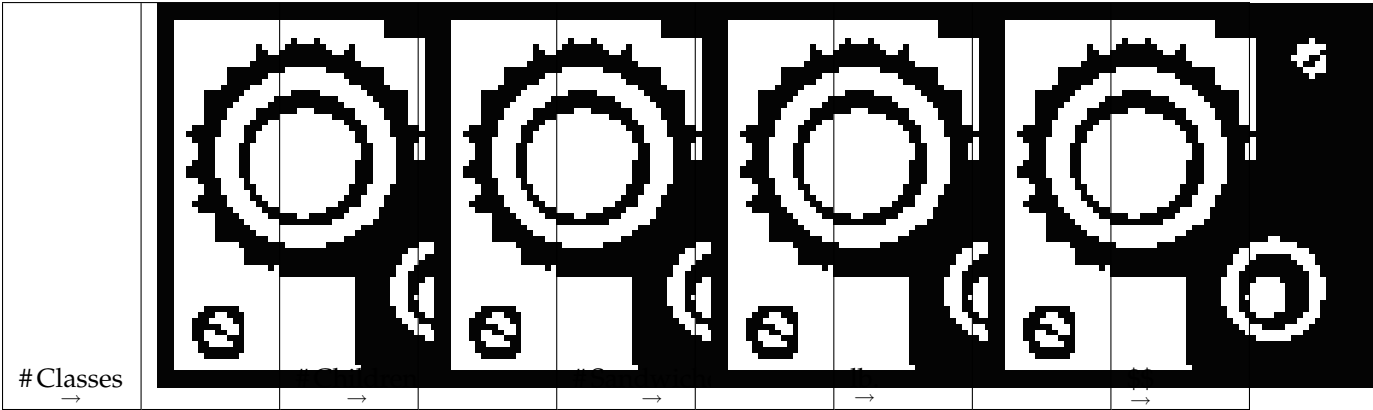


Table 1.17

But going through this process each time is tedious. What you want is one function that puts the entire chain together: "You tell me the number of classes, and I will tell you the budget."

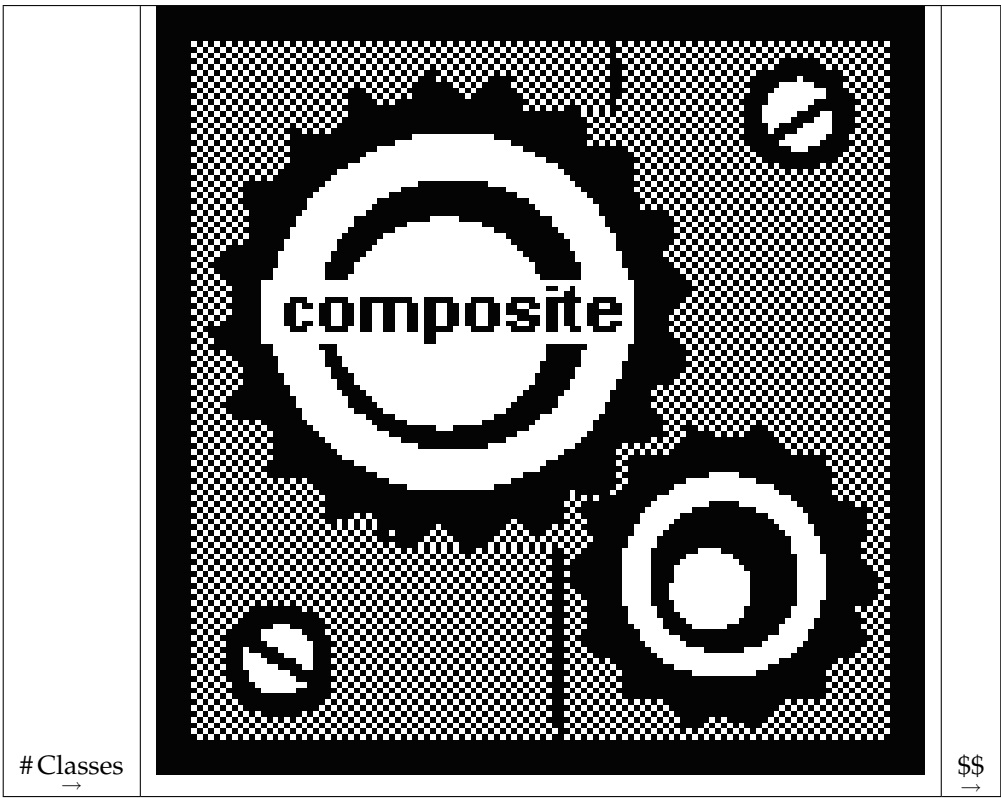


Table 1.18

This is a **composite function**—a function that represents in one function, the results of an entire **chain of dependent functions**. Since such chains are very common in real life, finding composite functions is a very important skill.

1.12.1 How do you make a composite Function?

We can consider how to build composite functions into the function game that we played on the first day. Suppose Susan takes any number you give her, quadruples it, and adds 6. Al takes any number you give him and divides it by 2. Mathematically, we can represent the two functions like this:

$$S(x) = 4x + 6 \quad (1.4)$$

$$A(x) = \frac{x}{2} \quad (1.5)$$

To create a chain like the one above, we give a number to Susan; she acts on it, and gives the resulting number to Al; and he then acts on it and hands back a third number.

$$3 \rightarrow \text{Susan} \rightarrow S(3) = 18 \rightarrow \text{Al} \rightarrow A(18) = 9$$

In this example, we are plugging $S(3)$ —in other words, 18—into Al's function. In general, for any x that comes in, we are plugging $S(x)$ **into** $A(x)$. So we could represent the entire process as $A(S(x))$. This notation for composite functions is really nothing new: it means that you are plugging $S(x)$ into the A function.

But in this case, recall that $S(x) = 4x + 6$. So we can write:

$$A(S(x)) = \frac{S(x)}{2} = \frac{4x + 6}{2} = 2x + 3 \quad (1.6)$$

What happened? We've just discovered a shortcut for the entire process. When you perform the operation $A(S(x))$ —that is, when you perform the Al function on the result of the Susan function—you are, in effect, doubling and adding 3. For instance, we saw earlier that when we started with a 3, we ended with a 9. Our composite function does this in one step:

$$3 \rightarrow 2x + 3 \rightarrow 9$$

Understanding the meaning of composite functions requires real thought. It requires understanding the idea that this variable depends on that variable, which in turn depends on the other variable; and how that idea is translated into mathematics. **Finding** composite functions, on the other hand, is a purely mechanical process—it requires practice, but no creativity. Whenever you are asked for $f(g(x))$, just plug the $g(x)$ function into the $f(x)$ function and then simplify.

Example 1.9: Building and Testing a Composite Function

$$f(x) = x^2 - 4x$$

$$g(x) = x + 2$$

What is $f(g(x))$?

- To find the composite, plug $g(x)$ into $f(x)$, just as you would with any number.

$$f(g(x)) = (x + 2)^2 - 4(x + 2)$$

- Then simplify.

$$f(g(x)) = (x^2 + 4x + 4) - (4x + 8)$$

$$f(g(x)) = x^2 - 4$$

- Let's test it. $f(g(x))$ means do g , then f . What happens if we start with $x = 9$?

$$7 \rightarrow g(x) \rightarrow 7 + 2 = 9 \rightarrow f(x) \rightarrow (9)^2 - 4(9) = 45$$

- So, if it worked, our **composite function** should do all of that in one step.

$$7 \rightarrow x^2 - 4 = (7)^2 - 4 = 45 \text{ [U+2713] It worked!}$$

There is a different notation that is sometimes used for composite functions. This book will consistently use $f(g(x))$ which very naturally conveys the idea of “plugging $g(x)$ into $f(x)$.” However, you will sometimes see the same thing written as $f \circ g(x)$, which more naturally conveys the idea of “doing one function, and then the other, in sequence.” The two notations mean the same thing.

1.13 Inverse Functions¹³

Let’s go back to Alice, who makes \$100/day. We know how to answer questions such as “After 3 days, how much money has she made?” We use the function $m(t) = 100t$.

But suppose I want to ask the reverse question: “If Alice has made \$300, how many hours has she worked?” This is the job of an inverse function. It gives the same relationship, but reverses the dependent and independent variables. $t(m) = m/100$. Given any amount of money, divide it by 100 to find how many days she has worked.

- **If a function answers the question:** “Alice worked this long, how much money has she made?” **then its inverse answers the question:** “Alice made this much money, how long did she work?”
- **If a function answers the question:** “I have this many spoons, how much do they weigh?” **then its inverse answers the question:** “My spoons weigh this much, how many do I have?”
- **If a function answers the question:** “How many hours of music fit on 12 CDs?” **then its inverse answers the question:** “How many CDs do you need for 3 hours of music?”

1.13.1 How do you recognize an inverse function?

Let’s look at the two functions above:

$$m(t) = 100t \tag{1.7}$$

$$t(m) = m/100 \tag{1.8}$$

Mathematically, you can recognize these as inverse functions because they **reverse the inputs and the outputs**.

$3 \rightarrow m(t) = 100t \rightarrow 300$
$300 \rightarrow t(m) = m/100 \rightarrow 3$
[U+2713] Inverse functions

Table 1.19

Of course, this makes logical sense. The first line above says that “If Alice works 3 hours, she makes \$300.” The second line says “If Alice made \$300, she worked 3 hours.” It’s the same statement, made in two different ways.

¹³This content is available online at <<http://cnx.org/content/m18198/1.2/>>.

But this “reversal” property gives us a way to test any two functions to see if they are inverses. For instance, consider the two functions:

$$f(x) = 3x + 7 \quad (1.9)$$

$$g(x) = \frac{1}{3}x - 7 \quad (1.10)$$

They **look** like inverses, don’t they? But let’s test and find out.

$2 \rightarrow 3x + 7 \rightarrow 13$
$2 \rightarrow 3x + 7 \rightarrow 13$
$\times$ Not inverse functions

Table 1.20

The first function turns a 2 into a 13. But the second function does **not** turn 13 into 2. So these are not inverses.

On the other hand, consider:

$$f(x) = 3x + 7 \quad (1.11)$$

$$g(x) = \frac{1}{3}(x - 7) \quad (1.12)$$

Let’s run our test of inverses on these two functions.

$2 \rightarrow 3x + 7 \rightarrow 13$
$13 \rightarrow \frac{1}{3}(x - 7) \rightarrow 2$
[U+2713] Inverse functions

Table 1.21

So we can see that these functions do, in fact, reverse each other: they are inverses.

A common example is the Celsius-to-Fahrenheit conversion:

$$F(C) = \left(\frac{9}{5}\right)C + 32 \quad (1.13)$$

$$C(F) = \left(\frac{5}{9}\right)(F - 32) \quad (1.14)$$

where C is the Celsius temperature and F the Fahrenheit. If you plug 100°C into the first equation, you find that it is 212°F . If you ask the second equation about 212°F , it of course converts that back into 100°C .

1.13.2 The notation and definition of an inverse function

The notation for the inverse function of $f(x)$ is $f^{-1}(x)$. This notation can cause considerable confusion, because it **looks like** an exponent, but it isn't. $f^{-1}(x)$ simply means "the inverse function of $f(x)$." It is defined formally by the fact that if you plug any number x into one function, and then plug the result into the other function, you get back where you started. (Take a moment to convince yourself that this is the same definition I gave above more informally.) We can represent this as a composition function by saying that $f(f^{-1}(x)) = x$.

Definition 1.5: Inverse Function

$f^{-1}(x)$ is defined as the **inverse function** of $f(x)$ if it consistently reverses the $f(x)$ process. That is, if $f(x)$ turns a into b , then $f^{-1}(x)$ must turn b into a . More concisely and formally, $f^{-1}(x)$ is the inverse function of $f(x)$ if $f(f^{-1}(x)) = x$.

1.13.3 Finding an inverse function

In examples above, we saw that if $f(x) = 3x + 7$, then $f^{-1}(x) = \frac{1}{3}(x - 7)$. We also saw that the function $\frac{1}{3}x - 7$, which may have looked just as likely, did **not** work as an inverse function. So in general, given a function, how do you find its inverse function?

Remember that an inverse function reverses the inputs and outputs. When we graph functions, we always represent the incoming number as x and the outgoing number as y . So to find the inverse function, **switch the x and y values, and then solve for y** .

Example 1.10: Building and Testing an Inverse Function

1. Find the inverse function of $f(x) = \frac{2x-3}{5}$
 - a. - Write the function as $y = \frac{2x-3}{5}$
 - b. - Switch the x and y variables. $x = \frac{2y-3}{5}$
 - c. - Solve for y . $5x = 2y - 3$. $5x + 3 = 2y$. $\frac{5x+3}{2} = y$. So $f^{-1}(x) = \frac{5x+3}{2}$.
2. Test to make sure this solution fills the definition of an inverse function.
 - a. - Pick a number, and plug it into the original function. $9 \rightarrow f(x) \rightarrow 3$.
 - b. - See if the inverse function reverses this process. $3 \rightarrow f^{-1}(x) \rightarrow 9$. [U+2713] It worked!

Were you surprised by the answer? At first glance, it seems that the numbers in the original function (the 2, 3, and 5) have been rearranged almost at random.

But with more thought, the solution becomes very intuitive. The original function $f(x)$ described the following process: **double a number, then subtract 3, then divide by 5**. To reverse this process, we need to reverse each step in order: **multiply by 5, then add 3, then divide by 2**. This is just what the inverse function does.

1.13.4 Some functions have no inverse function

Some functions have no inverse function. The reason is the rule of consistency.

For instance, consider the function $y = x^2$. This function takes both 3 and -3 and turns them into 9. No problem: a function is allowed to turn different **inputs** into the same **output**. However, what does that say about the inverse of this particular function? In order to fulfill the requirement of an inverse function, it would have to take 9, and turn it into both 3 and -3—which is the one and only thing that functions are **not** allowed to do. Hence, the inverse of this function would not be a function at all!

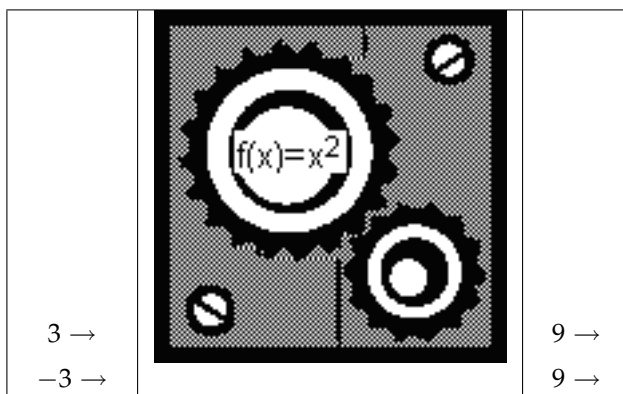


Table 1.22: If 3 goes in, 9 comes out. If -3 goes in, 9 also comes out. No problem:

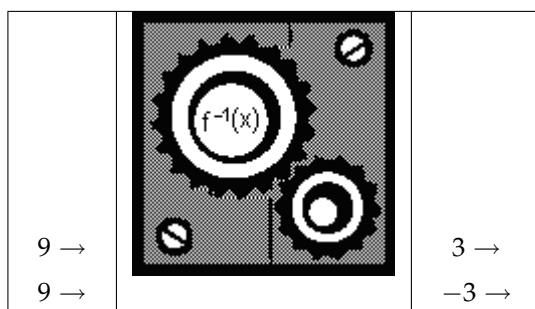


Table 1.23: But its inverse would have to turn 9 into both 3 and -3. No function can do this, so there is no inverse.

In general, any function that turns multiple inputs into the same output, does not have an inverse function.

What does that mean in the real world? If we can convert Fahrenheit to Celsius, we must be able to convert Celsius to Fahrenheit. If we can ask “How much money did Alice make in 3 days?” we must surely be able to ask “How long did it take Alice to make \$500?” When would you have a function that cannot be inverted?

Let’s go back to this example:

Recall the example that was used earlier: “Max threw a ball. The height of the ball depends on how many seconds it has been in the air.” The two variables here are h (the height of the ball) and t (the number of seconds it has been in the air). The function $h(t)$ enables us to answer questions such as “After 3 seconds, where is the ball?”

The inverse question would be “At what time was the ball 10 feet in the air?” The problem with that question is, it may well have **two answers**!

The ball is here...	...after this much time has elapsed
10 ft	2 seconds (*on the way up)
10 ft	5 seconds (*on the way back down)

Table 1.24

So what does that mean? Does it mean we can't ask that question? Of course not. We can ask that question, and we can expect to mathematically find the answer, or answers—and we will do so in the quadratic chapter. However, it does mean that **time is not a function of height** because such a “function” would not be consistent: one question would produce multiple answers.

Chapter 2

Inequalities and Absolute Values

2.1 Inequalities¹

The symbols for inequalities are familiar:

- $x < 7$ “ x is less than 7”
- $x > 7$ “ x is greater than 7”
- $x \leq 7$ “ x is less than or equal to 7”
- $x \geq 7$ “ x is greater than or equal to 7”

If you have trouble remembering which is which, it may be helpful to remember that the larger side of the $<$ symbol always goes with the larger number. Hence, when you write $x < 7$ you can see that the 7 is the larger of the two numbers. Some people think of the $<$ symbol as an alligator’s mouth, which always opens toward the largest available meal!

Visually, we can represent these inequalities on a number line. An open circle [U+25CB] [U+25CB] is used to indicate a boundary that is not a part of the set; a closed circle [U+25CF] is used for a boundary that is a part of the set.

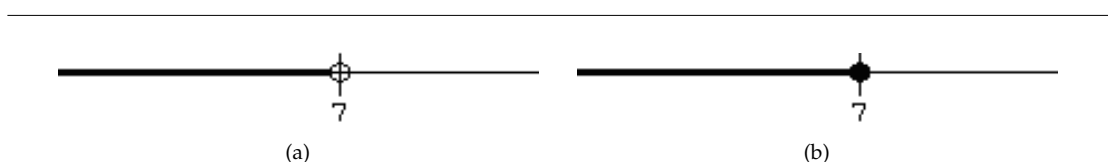


Figure 2.1: (a) Includes all numbers less than 7, but not 7; $x < 7$; $(-\infty, 7)$ (b) Includes all numbers less than 7, and 7 itself; $x \leq 7$; $(-\infty, 7]$

2.1.1 AND and OR

More complicated intervals can be represented by combining these symbols with the logical operators **AND** and **OR**.

¹This content is available online at <<http://cnx.org/content/m18205/1.2/>>.

For instance, “ $x \geq 3$ AND $x < 6$ ” indicates that x must be **both** greater-than-or-equal-to 3, **and** less-than 6. A number only belongs in this set if it meets **both** conditions. Let’s try a few numbers and see if they fit.

Sample number	$x \geq 3$	$x < 6$	$x \geq 3$ AND $x < 6$ (both true)
$x = 8$	Yes	No	No
$x = 0$	No	Yes	No
$x = 4$	Yes	Yes	Yes

Table 2.1

We can see that a number must be **between** 3 and 6 in order to meet this AND condition.



Figure 2.2: $x \geq 3$ AND $x < 6$ All numbers that are greater-than-or-equal-to 3, **and are also** less than 6; $3 \leq x < 6$

This type of set is sometimes represented concisely as $3 \leq x < 6$, which visually communicates the idea that x is **between** 3 and 6. This notation always indicates an AND relationship.

“ $x \geq 3$ OR $x < 6$ ” is the exact opposite. It indicates that x must be **either** less-than 3, **or** greater-than-or-equal-to 6. Meeting both conditions is OK, but it is not necessary.

Sample number	$x \geq 3$	$x < 6$	$x \geq 3$ OR $x < 6$ (either one or both true)
$x = 8$	No	Yes	Yes
$x = 0$	Yes	No	Yes
$x = 4$	No	No	No

Table 2.2

Visually, we can represent this set as follows:

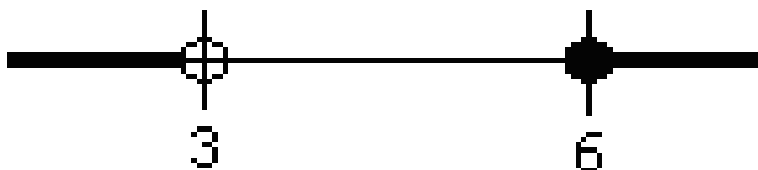


Figure 2.3: All numbers that are either less than 3, or greater-than-or-equal-to 6; $x < 3$ OR $x \geq 6$

Both of the above examples are meaningful ways to represent useful sets. It is possible to put together many combinations that are perfectly logical, but are not meaningful or useful. See if you can figure out simpler ways to write each of the following conditions.

1. $x \geq 3$ AND $x < 6$
2. $x \geq 3$ OR $x < 6$
3. $x < 3$ AND $x > 6$
4. $x > 3$ OR $x < 6$

If you are not sure what these mean, try making tables of numbers like the ones I made above. Try a number below 3, a number between 3 and 6, and a number above 6. See when each condition is true. You should be able to convince yourself of the following:

1. The first condition above is filled by any number greater than 6; it is just a big complicated way of writing $x > 6$.
2. Similarly, the second condition is the same as $x \geq 3$.
3. The third condition is **never** true.
4. The fourth condition is **always** true.

I have to pause here for a brief philosophical digression. The biggest difference between a good math student, and a poor or average math student, is that the good math student works to **understand** things; the poor student tries to **memorize rules** that will lead to the right answer, without actually understanding them.

The reason this unit (Inequalities and Absolute Values) is right here at the beginning of the book is because it distinguishes sharply between these two kinds of students. Students who try to understand things will follow the previous discussion of AND and OR and will think about it until it makes sense. When approaching a new problem, they will try to make logical sense of the problem and its solution set.

But many students will attempt to learn a set of mechanical rules for solving inequalities. These students will often end up producing nonsensical answers such as the four listed above. Instead of thinking about what their answers mean, they will move forward, comfortable because “it looks sort of like the problem the teacher did on the board.”

If you have been accustomed to looking for mechanical rules to follow, now is the time to begin changing your whole approach to math. It’s not too late!!! Re-read the previous section carefully, line by line, and make sure each sentence makes sense. Then, as you work problems, think them through in the same way: **not** “whenever I see this kind of problem the answer is an **and**” but instead “What does AND mean? What does OR mean? Which one correctly describes this problem?”

All that being said, there are still a few hard-and-fast rules that I will point out as I go. These rules are useful—but they do not relieve you of the burden of thinking.

One special kind of OR is the symbol $\pm$. Just as $\geq$ means “greater than OR equal to,” $\pm$ means “plus OR minus.” Hence, if $x^2 = 9$, we might say that $x = \pm 3$; that is, x can be either 3, or -3 .

Another classic sign of “blind rule-following” is using this symbol with inequalities. What does it mean to say $x < \pm 3$? If it means anything at all, it must mean “ $x < 3$ OR $x > -3$ ”; which, as we have already seen, is just a sloppy shorthand for $x < 3$. If you find yourself using an inequality with a $\pm$ sign, go back to think again about the problem.

NOTE: Inequalities and the $\pm$ symbol don’t mix.

2.1.2 Solving Inequalities

Inequalities are solved just like equations, with one key exception.

NOTE: Whenever you **multiply or divide by a negative number**, the sign changes.

You can see how this rule affects the solution of a typical inequality problem:

$3x + 4 > 5x + 10$	An "inequality" problem
$-2x + 4 > 10$	subtract 5 x from both sides
$-2x > 6$	subtract 4 from both sides
$x < -3$	divide both sides by -2 , and change sign!

Table 2.3

As always, being able to solve the problem is important, but even more important is knowing **what the solution means**. In this case, we have concluded that any number less than -3 will satisfy the original equation, $3x + 4 > 5x + 10$. Let's test that.

$x = -4$:	$3(-4) + 4 > 5(-4) + 10$	$-8 > -10$	Yes.
$x = -2$:	$3(-2) + 4 > 5(-2) + 10$	$-2 > 0$	No.

Table 2.4

As expected, $x = -4$ (which is less than -3) works; $x = -2$ (which is not) does not work.

Why do you reverse the inequality when multiplying or dividing by a negative number? Because negative numbers are backward! 5 is greater than 3 , but -5 is **less than** -3 . Multiplying or dividing by negative numbers moves you to the other side of the number line, where everything is backward!

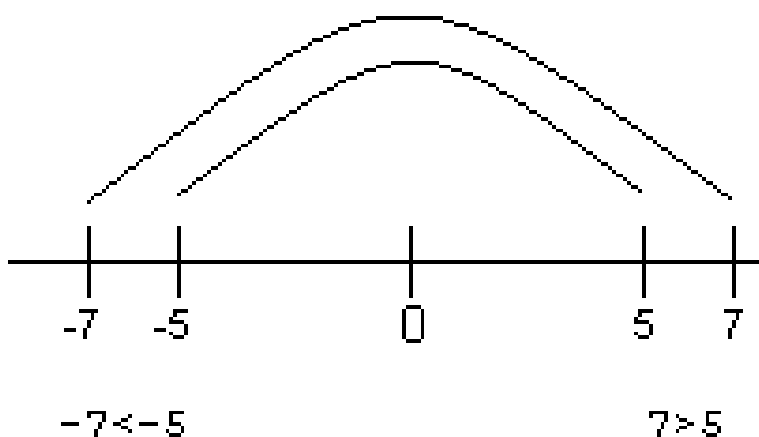


Figure 2.4: Multiplying by -1 moves you "over the rainbow" to the land where everything is backward!

2.2 Absolute Value Equations²

Absolute value is one of the simplest functions—and paradoxically, one of the most problematic.

On the face of it, nothing could be simpler: it just means “whatever comes in, a positive number comes out.”

$$|5| = 5 \quad (2.1)$$

$$|-5| = 5 \quad (2.2)$$

Absolute values seem to give us permission to ignore the whole nasty world of negative numbers and return to the second grade when all numbers were positive.

But consider these three equations. They look very similar—only the number changes—but the solutions are completely different.

Three Simple Absolute Value Equations

$ x = 10$	$ x = -10$	$ x = 0$
$x = 10$ works.	$x = 10$ doesn't work.	$x = 0$ is the only solution.
Hey, so does $x = -10$!	Neither does $x = -10$.	
Concisely, $x = \pm 10$.	Hey... absolute values are never negative!	

Table 2.5

We see that the first problem has **two solutions**, the second problem has **no solutions**, and the third problem has **one solution**. This gives you an example of how things can get confusing with absolute values—and how you can solve things if you **think** more easily than with **memorized rules**.

For more complicated problems, follow a three-step approach.

1. Do the algebra to isolate the absolute value.
2. Then, **think it through** like the simpler problems above.
3. Finally, do more algebra to isolate x .

In my experience, most problems with this type of equation do not occur in the first and third step. And they do not occur because students try to think it through (second step) and don't think it through correctly. They occur because students try to take “shortcuts” to avoid the second step entirely.

Example 2.1: Absolute Value Equation (No Variable on the Other Side)

$$3|2x + 1| - 7 = 5$$

Step 1:: Algebraically isolate the absolute value

- $3|2x + 1| = 12$
- $|2x + 1| = 4$

Step 2:: Think!

- For the moment, forget about the quantity $2x + 1$; just think of it as **something**. The absolute value of “something” is 4. So, in analogy to what we did before, the “something” can either be 4, or -4 . So that gives us two possibilities...

²This content is available online at <<http://cnx.org/content/m18201/1.3/>>.

- $2x + 1 = 4$
- $2x + 1 = -4$

Step 3:: Algebraically solve (both equations) for x

- $2x = 3$ or $2x = -5$
- $x = \frac{3}{2}$ or $x = -\frac{5}{2}$

So this problem has two answers: $x = \frac{3}{2}$ and $x = -\frac{5}{2}$

Example 2.2: Absolute Value Equation (No Variable on the Other Side)

$$\frac{6|x-2|}{5} + 7 = 4$$

Step 1: Algebraically isolate the absolute value:

$$\frac{6|x-2|}{5} = -3 \quad (2.3)$$

$$6|x-2| = -15 \quad (2.4)$$

$$|x-2| = \frac{-15}{6} = \frac{-5}{2} \quad (2.5)$$

Step 2: Think!: The absolute value of “something” is $-2\frac{1}{2}$. But wait—absolute values are **never** negative! It can’t happen! So we don’t even need a third step in this case: the equation is impossible.

No solution.

The “think” step in the above examples was relatively straightforward, because there were no variables on the right side of the equation. When there are variables on the right side, you temporarily “pretend” that the right side of the equation is a positive number, and break the equation up accordingly. However, there is a price to be paid for this slight of hand: you have to check your answers, because **they may not work even if you do your math correctly.**

Example 2.3: Absolute Value Equation with Variables on Both Sides

$$|2x + 3| = -11x + 42$$

We begin by approaching this in analogy to the first problem above, $|x| = 10$. We saw that x could be either 10, or -10 . So we will assume in this case that $2x + 3$ can be either $-11x + 42$, or the negative of that, and solve both equations.

Problem One

$$2x + 3 = -11x + 42 \quad (2.6)$$

$$13x + 3 = 42 \quad (2.7)$$

$$13x = 39 \quad (2.8)$$

$$x = 3 \quad (2.9)$$

Problem Two

$$2x + 3 = -(-11x + 42) \quad (2.10)$$

$$2x + 3 = 11x - 42 \quad (2.11)$$

$$-9x + 3 = -42 \quad (2.12)$$

$$-9x = -45 \quad (2.13)$$

$$x = 5 \quad (2.14)$$

So we have two solutions: $x = 3$ and $x = 5$. Do they both work? Let's try them both.

Problem One

1. $|2(3) + 3| = -11(3) + 42$.
 - $|9| = 9$. [U+2713]

Problem Two

1. $|2(5) + 3| = -11(5) + 42$.
 - $|13| = -13$. ×

We see in this case that the first solution, $x = 3$, worked; the second, $x = 5$, did not. So the only solution to this problem is $x = 3$.

However, there was no way of knowing that in advance. For such problems, the only approach is to solve them twice, and then **test both answers**. In some cases, both will work; in some cases, neither will work. In some cases, as in this one, one will work and the other will not.

HARD AND FAST RULE: Whenever an absolute value equation has **variables on both sides**, you have to check your answer(s). Even if you do all the math perfectly, your answer(s) may not work.

OK, why is that? Why can you do all the math right and still get a wrong answer?

Remember that the problem $|x| = 10$ has two solutions, and $|x| = -10$ has none. We started with the problem $|2x + 3| = -11x + 42$. OK, which is that like? Is the right side of the equation like 10 or -10? If you think about it, you can convince yourself that **it depends on what x is**. After you solve, you may wind up with an x -value that makes the right side positive; that will work. Or, you may wind up with an x -value that makes the right side negative; that won't work. But you can't know until you get there.

2.3 Absolute Value Equations³

Here's one of my favorite problems:

$$|x| < 10$$

Having seen that the solution to $|x| = 10$ is $|x| = \pm 10$, many students answer this question $|x| < \pm 10$. However, this is not only wrong; it is, as discussed above, relatively meaningless. In order to approach this question you have to—you guessed it!—step back and think.

Here are two different, perfectly correct ways to look at this problem.

³This content is available online at <<http://cnx.org/content/m18207/1.2/>>.

1. What numbers work? 4 works. -4 does too. 0 works. 13 doesn't work. How about -13? No: if $x = -13$ then $|x| = 13$, which is not less than 10. By playing with numbers in this way, you should be able to convince yourself that the numbers that work must be **somewhere between -10 and 10**. This is one way to approach the answer.
2. The other way is to think of absolute value as representing **distance from 0**. $|5|$ and $|-5|$ are both 5 because both numbers are 5 away from 0. In this case, $|x| < 10$ means "the distance between x and 0 is less than 10"—in other words, you are within 10 units of zero in either direction. Once again, we conclude that the answer must be **between -10 and 10**.



Figure 2.5: All numbers whose **absolute value** is less than 10; $-10 < x < 10$

It is not necessary to use both of these methods; use whichever method is easier for you to understand.

More complicated absolute value problems should be approached in the same three steps as the equations discussed above: algebraically isolate the absolute value, then **think**, then algebraically solve for x . However, as illustrated above, the **think** step is a bit more complicated with inequalities than with equations.

Example 2.4: Absolute Value Inequality

$$-3(|2x + 3| - 8) < -15$$

Step 1: - Algebraically isolate the absolute value

- $|2x + 3| - 8 > 5$ (don't forget to switch the inequality when dividing by -3!)
- $|2x + 3| > 13$

Step 2: - Think!

- As always, forget the $2x + 3$ in this step. The absolute value of **something** is greater than 13. What could the **something** be?
- We can approach this in two ways, just as the previous absolute value inequality. The first method is trying numbers. We discover that all numbers greater than 13 work (such as 14, 15, 16)—their absolute values are greater than 13. Numbers less than -13 (such as -14, -15, -16) also have absolute values greater than 13. But in-between numbers, such as -12, 0, or 12, do not work.
- The other approach is to think of absolute value as representing distance to 0. The distance between **something** and 0 is greater than 13. So the **something** is more than 13 away from 0—in either direction.
- Either way, we conclude that the **something** must be anything greater than 13, OR less than -13!



Figure 2.6: The absolute value of something is greater than 13; $\text{something} < -13$ OR $\text{something} > 13$

Step 3: - Algebraically solve (both inequalities) for x

$$\begin{array}{rcl} 2x + 3 < -13 & \text{OR} & 2x + 3 > 13 \\ 2x < -16 & \text{OR} & 2x > 10 \\ x < -8 & \text{OR} & x > 5 \end{array} \quad (2.15)$$

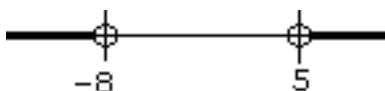


Figure 2.7: Any x -value which is less than -8 **or** greater than 5 will make the original inequality true;
 $x < -8$ OR $x > 5$

Many students will still resist the **think** step, attempting to figure out “the rules” that will always lead from the question to the answer. At first, it seems that memorizing a few rules won’t be too hard: “greater-than problems always lead to **OR** answers” and that kind of thing. But those rules will fail you when you hit a problem like the next one.

Example 2.5: Absolute Value Inequality

$$|x - 3| + 10 > 7$$

Step 1: - Algebraically isolate the absolute value

- $|x - 3| > -3$

Step 2: - Think!

- The absolute value of **something** is greater than -3 . What could the **something** be? 2 works. -2 also works. And 0 . And 7 . And -10 . And...hey! Absolute values are always positive, so the absolute value of anything is greater than -3 !

All numbers work

2.4 Graphing Absolute Values⁴

You can graph $y = |x|$ easily enough by plotting points. The characteristic V shape is illustrated below, with a couple of sample points highlighted.

⁴This content is available online at <<http://cnx.org/content/m18199/1.2/>>.

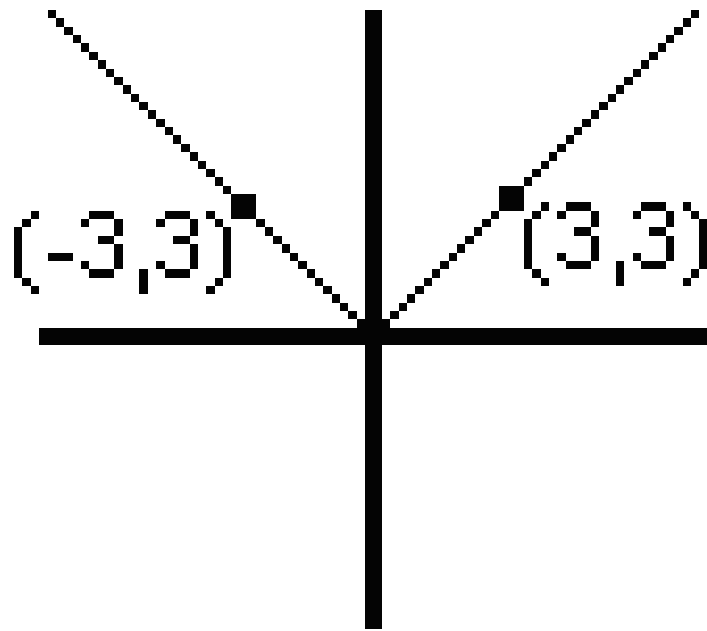
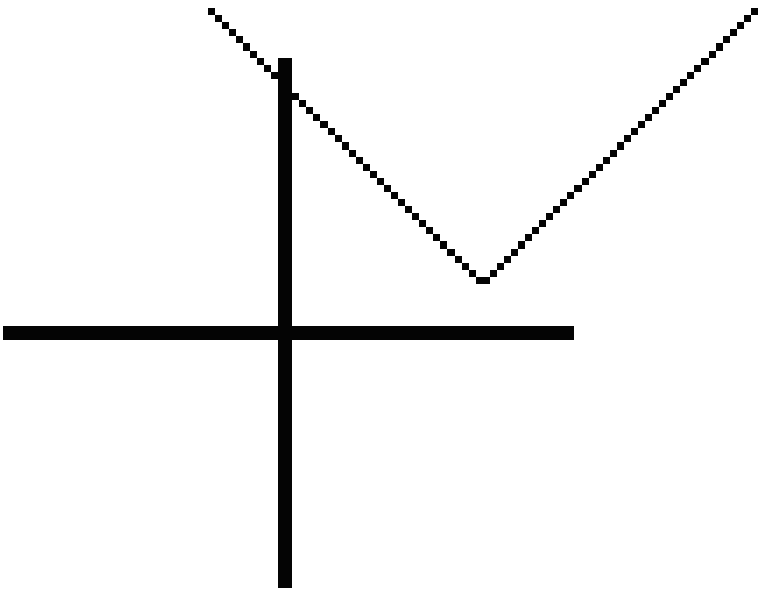
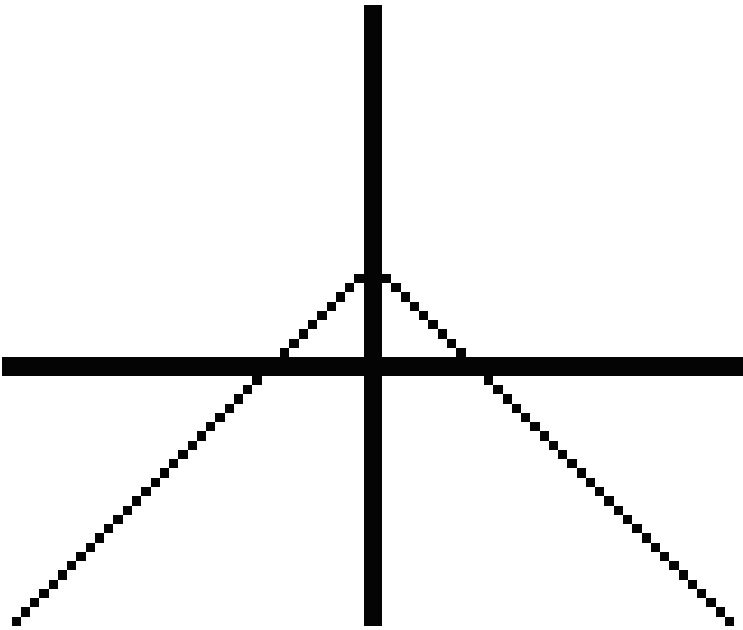


Figure 2.8: $y = |x|$

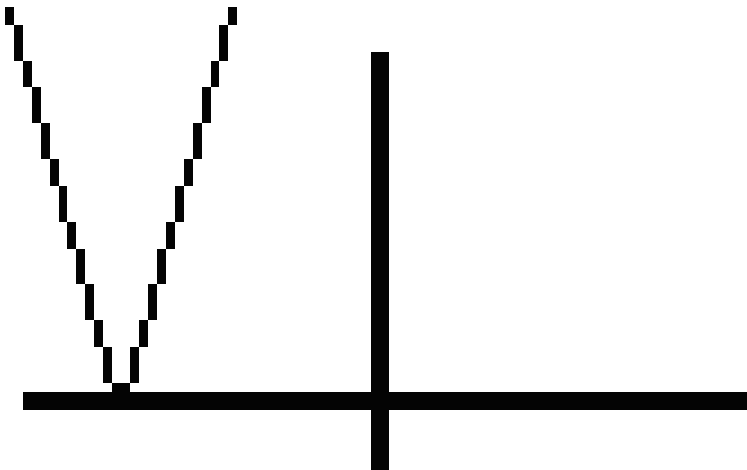
Of course, this shape is subject to the same permutations as any other function! A few examples are given below.



(a)



(b)



2.5 Graphing Inequalities⁵

In general, the graph of an inequality is a **shaded area**.

Consider the graph $y = |x|$ shown above. Every point on that V-shape has the property that **its y -value is the absolute value of its x -value**. For instance, the point $(-3, 3)$ is on the graph because 3 is the absolute value of -3 .

The inequality $y < |x|$ means **the y -value is less than the absolute value of the x -value**. This will occur anywhere **underneath** the above graph. For instance, the point $(-3, 1)$ meets this criterion; the point $(-3, 4)$ does not. If you think about it, you should be able to convince yourself that all points **below** the above graph fit this criterion.

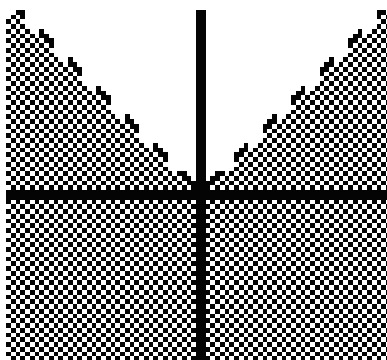


Figure 2.10: $y < |x|$

The dotted line indicates that the graph $y = |x|$ is not actually a part of our set. If we were graphing $y \leq |x|$ the line would be complete, indicating that those points would be part of the set.

2.6 "Piecewise Functions" and Absolute Value⁶

What do you get if you put a positive number into an absolute value? Answer: you get that same number back. $|5| = 5$. $|\pi| = \pi$. And so on. We can say, as a generalization, that $|x| = x$; but **only if x is positive**.

OK, so, what happens if you put a **negative** number into an absolute value? Answer: you get that same number back, but made positive. OK, how do you **make** a negative number positive? Mathematically, you **multiply it by -1** . $|-5| = -(-5) = 5$. $|\pi| = -(-\pi) = \pi$. We can say, as a generalization, that $|x| = -x$; but **only if x is negative**.

So the absolute value function can be defined like this.

The "Piecewise" Definition of Absolute Value

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \quad (2.16)$$

⁵This content is available online at <<http://cnx.org/content/m18208/1.2/>>.

⁶This content is available online at <<http://cnx.org/content/m18200/1.2/>>.

If you've never seen this before, it looks extremely odd. If you try to pin that feeling down, I think you'll find this looks odd for some combination of these three reasons.

1. The whole idea of a "piecewise function"—that is, a function which is defined differently on different domains—may be unfamiliar. Think about it in terms of the function game. Imagine getting a card that says "If you are given a positive number or 0, respond with the same number you were given. If you are given a negative number, multiply it by -1 and give that back." This is one of those "can a function **do that**?" moments. Yes, it can—and, in fact, functions defined in this "piecewise manner" are more common than you might think.
2. The $-x$ looks suspicious. "I thought an absolute value could never **be** negative!" Well, that's right. But if x is negative, then $-x$ is positive. Instead of thinking of the $-x$ as "negative x " it may help to think of it as "change the sign of x ."
3. Even if you get past those objections, you may feel that we have taken a perfectly ordinary, easy to understand function, and redefined it in a terribly complicated way. Why bother?

Surprisingly, the piecewise definition makes many problems **easier**. Let's consider a few graphing problems.

You already know how to graph $y = |x|$. But you can explain the V shape very easily with the piecewise definition. On the right side of the graph (where $x \geq 0$), it is the graph of $y = x$. On the **left** side of the graph (where $x < 0$), it is the graph of $y = -x$.

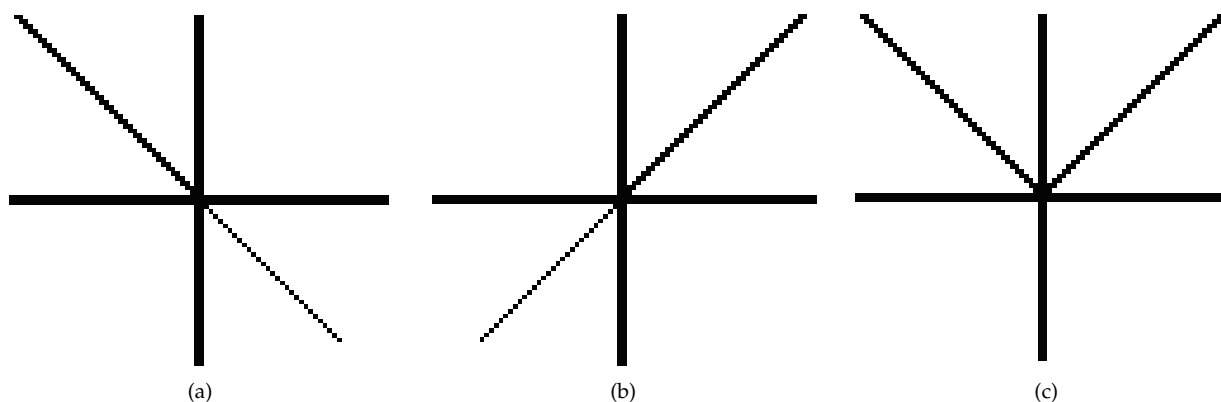


Figure 2.11: (a) $y = -x$ The whole graph is shown, but the only part we care about is on the left, where $x < 0$ (b) $y = x$ The whole graph is shown, but the only part we care about is on the right, where $x \geq 0$ (c) $y = |x|$ Created by putting together the relevant parts of the other two graphs.

Still, that's just a new way of graphing something that we already knew how to graph, right? But now consider this problem: graph $y = x + |x|$. How do we approach that? With the piecewise definition, it becomes a snap.

$$x + |x| = \begin{cases} x + x = 2x & x \geq 0 \\ x + (-x) = 0 & x < 0 \end{cases} \quad (2.17)$$

So we graph $y = 2x$ on the right, and $y = 0$ on the left. (You may want to try doing this in three separate drawings, as I did above.)

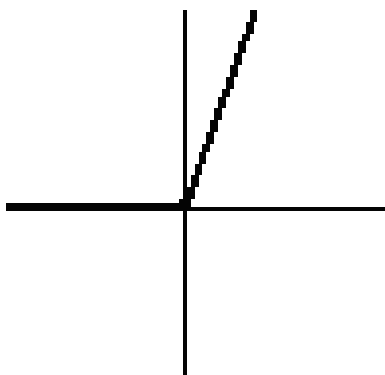


Figure 2.12: $y = x + |x|$

Our final example requires us to use the piecewise definition of the absolute value for both x and y .

Example 2.6: Graph $|x| + |y| = 4$

We saw that in order to graph $|x|$ we had to view the left and right sides separately. Similarly, $|y|$ divides the graph **vertically**.

- On top, where $y \geq 0$, $|y| = y$.
- Where $y < 0$, on the bottom, $|y| = -y$.

Since this equation has **both** variables under absolute values, we have to divide the graph both horizontally and vertically, which means we look at **each quadrant separately**. $|x| + |y| = 4$

Second Quadrant	First Quadrant
$x \geq 0$, so $ x = -x$	$x \geq 0$, so $ x = x$
$y \geq 0$, so $ y = y$	$y \geq 0$, so $ y = y$
$(-x) + y = 4$	$x + y = 4$
$y = x + 4$	$y = -x + 4$
Third Quadrant	Fourth Quadrant
$x \leq 0$, so $ x = -x$	$x \leq 0$, so $ x = x$
$y \leq 0$, so $ y = -y$	$y \leq 0$, so $ y = -y$
$(-x) + (-y) = 4$	$x + (-y) = 4$
$y = -x - 4$	$y = x - 4$

Table 2.6

Now we graph each line, but only in its respective quadrant. For instance, in the fourth quadrant, we are graphing the line $y = x - 4$. So we draw the line, but use only the part of it that is in the fourth quadrant.

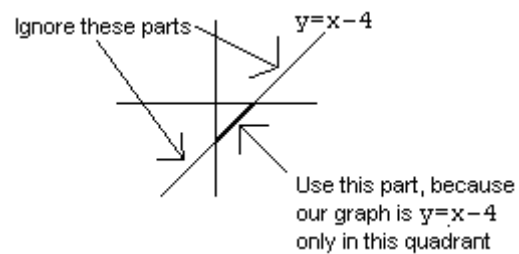
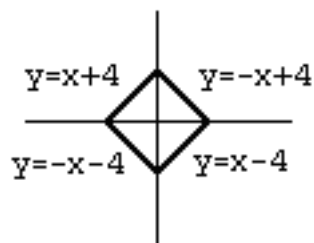


Figure 2.13

Repeating this process in all four quadrants, we arrive at the proper graph.

Figure 2.14: $|x| + |y| = 4$

Chapter 3

Simultaneous Equations

3.1 Distance, Rate and Time¹

If you travel 30 miles per hour for 4 hours, how far do you go? A little common sense will tell you that the answer is 120 miles.

This relationship is captured in the following equation:

$d = rt$ where...

- d is distance traveled (sometimes the letter x is used instead, for position)
- r is the rate, or speed (sometimes the letter v is used, for velocity)
- t is the time

This is presented here because it forms the basis for many common simultaneous equations problems.

3.2 Simultaneous Equations by Graphing²

Consider the equation $y = 2\sqrt{x}$. How many (x, y) pairs are there that satisfy this equation? Answer: $(0, 0)$, $(1, 2)$, $(4, 4)$, and $(9, 6)$ are all solutions; and there is an **infinite number of other solutions**. (And don't forget non-integer solutions, such as $(, 1)$!)

Now, consider the equation $y = x + \frac{1}{2}$. How many pairs satisfy **this** equation? Once again, an infinite number. Most equations that relate two variables have an infinite number of solutions.

To consider these two equations “simultaneously” is to ask the question: what (x, y) pairs make **both equations true**? To express the same question in terms of functions: what values can you hand the functions $2\sqrt{x}$ and $x + \frac{1}{2}$ that will make these two functions produce the **same answer**?

¹This content is available online at <<http://cnx.org/content/m18211/1.2/>>.

²This content is available online at <<http://cnx.org/content/m18209/1.2/>>.

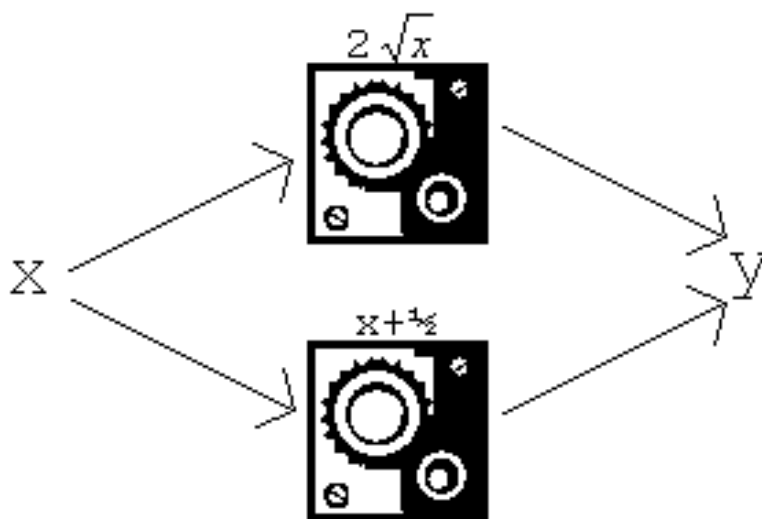


Figure 3.1: What number goes into **both functions** and makes them give the **same answer**? Is there even such a number? Is there more than one such number?

At first glance, it is not obvious how to approach such a question— it is not even obvious how many answers there will be.

One way to answer such a question is by graphing. Remember, the graph of $y = 2\sqrt{x}$ is the set of all points that satisfy that relationship; and the graph of $y = x + \frac{1}{2}$ is the set of all points that satisfy that relationship. So **the intersection(s) of these two graphs is the set of all points that satisfy both relationships.**

How can we graph these two? The second one is easy: it is a line, already in $y = mx + b$ format. The y -intercept is $\frac{1}{2}$ and the slope is 1. We can graph the first equation by plotting points; or, if you happen to know what the graph of $y = \sqrt{x}$ looks like, you can stretch the graph vertically to get $y = 2\sqrt{x}$, since all the y -values will double. Either way, you wind up with something like this:

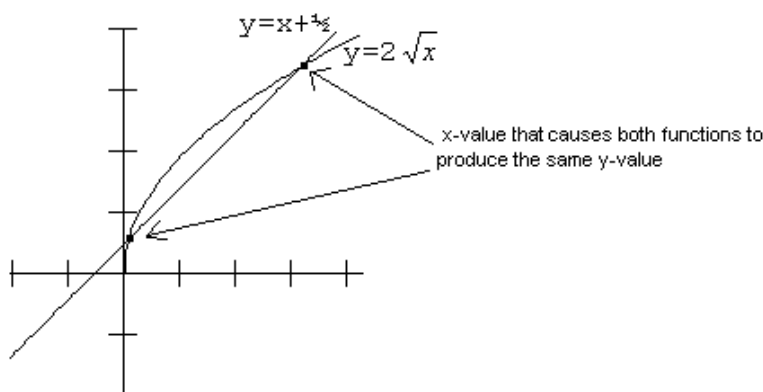


Figure 3.2

We can see that there are two points of intersection. One occurs when x is barely greater than 0 (say, $x = 0.1$), and the other occurs at approximately $x = 3$. There will be no more points of intersection after this, because the line will rise faster than the curve.

Exercise 3.1*(Solution on p. 64.)*

$$y = 2\sqrt{x}$$

$$y = x + \frac{1}{2}$$

Graphing has three distinct advantages as a method for solving simultaneous equations.

1. It works on any type of equations.
2. It tells you **how many** solutions there are, as well as what the solutions are.
3. It can help give you an **intuitive feel** for why the solutions came out the way they did.

However, graphing also has two **disadvantages**.

1. It is time-consuming.
2. It often yields solutions that are **approximate**, not exact—because you find the solutions by simply “eyeballing” the graph to see where the two curves meet.

For instance, if you plug the number 3 into both of these functions, will you get the same answer?

$$3 \rightarrow 2\sqrt{x} \rightarrow 2\sqrt{3} \approx 3.46$$

$$3 \rightarrow x + \frac{1}{2} \rightarrow 3.5$$

Pretty close! Similarly, $2\sqrt{.1} \approx 0.632$, which is quite close to 0.6. But if we want more exact answers, we will need to draw a much more exact graph, which becomes **very** time-consuming. (Rounded to three decimal places, the actual answers are 0.086 and 2.914.)

For more exact answers, we use analytic methods. Two such methods will be discussed in this chapter: **substitution** and **elimination**. A third method will be discussed in the section on Matrices.

3.3 Substitution³

Here is the algorithm for substitution.

1. Solve one of the equations for one variable.
2. Plug this variable into the other equation.
3. Solve the second equation, which now has only one variable.
4. Finally, use the equation you found in step (1) to find the other variable.

Example 3.1: Solving Simultaneous Equations by Substitution

$$3x + 4y = 1$$

$$2x - y = 8$$

1. The easiest variable to solve for here is the y in the second equation.
 - $-y = -2x + 8$
 - $y = 2x - 8$
2. Now, we plug that into the **other** equation:
 - $3x + 4(2x - 8) = 1$
3. We now have an equation with only x in it, so we can solve for x .
 - $3x + 8x - 32 = 1$
 - $11x = 33$
 - $x = 3$
4. Finally, we take the equation from step (1), $y = 2x - 8$, and use it to find y .
 - $y = 2(3) - 8 = -2$

So $(3, -2)$ is the solution. You can confirm this by plugging this pair into both of the original equations.

Why does substitution work?

We found in the first step that $y = 2x - 8$. This means that y and $2x - 8$ are **equal** in the sense that we discussed in the first chapter on functions—they will always be the same number, in these equations—they are the **same**. This gives us permission to simply replace one with the other, which is what we do in the second (“substitution”) step.

3.4 Elimination⁴

Here is the algorithm for elimination.

1. Multiply one equation (or in some cases both) by some number, so that the two equations have the **same coefficient** for one of the variables.
2. Add or subtract the two equations to make that variable go away.
3. Solve the resulting equation, which now has only one variable.
4. Finally, plug back in to find the other variable.

Example 3.2: Solving Simultaneous Equations by Elimination

$$3x + 4y = 1$$

$$2x - y = 8$$

³This content is available online at <<http://cnx.org/content/m18212/1.2/>>.

⁴This content is available online at <<http://cnx.org/content/m18215/1.2/>>.

- 1 - The first question is: how do we get one of these variables to have the **same coefficient in both equations**? To get the x coefficients to be the same, we would have to multiply the top equation by 2 and the bottom by 3. It is much easier with y ; if we simply multiply the bottom equation by 4, then the two y values will both be multiplied by 4.
 - $3x + 4y = 1$
 - $8x - 4y = 32$
- 2 - Now we either **add** or **subtract** the two equations. In this case, we have $4y$ on top, and $-4y$ on the bottom; so if we add them, they will cancel out. (If the bottom had $+4y$ we would have to subtract the two equations to get the " y "s to cancel.)
 - $11x + 0y = 33$
- 3-4 - Once again, we are left with only one variable. We can solve this equation to find that $x = 3$ and then plug back in to either of the original equations to find $y = -2$ as before.

Why does elimination work?

As you know, you are always allowed to do the **same thing** to both sides of an equation. If an equation is true, it will still be true if you add 4 to both sides, multiply both sides by 6, or take the square root of both sides.

Now—consider, in the second step above, what we did to the equation $3x + 4y = 1$. We **added** something to both sides of this equation. What did we add? On the left, we added $8x - 4y$; on the right, we added 32. It seems that we have done something **different** to the two sides.

However, the second equation gives us a guarantee that these two quantities, $8x - 4y$ and 32, are in fact **the same as each other**. So by adding $8x - 4y$ to the left, and 32 to the right, we really have done exactly the same thing to both sides of the equation $3x + 4y = 1$.

3.5 Special Cases⁵

Consider the two equations:

$$2x + 3y = 8 \tag{3.1}$$

$$4x + 6y = 3 \tag{3.2}$$

Suppose we attempt to solve these two equations by elimination. So, we double the first equation and subtract, and the result is:

$$\begin{array}{rcl} 4x + 6y & = & 16 \\ 4x + 6y & = & 3 \\ \hline 0 & = & 13 \end{array} \tag{3.3}$$

Hey, what happened? 0 does not equal 13, no matter what x is. Mathematically, we see that these two equations have **no** simultaneous solution. You asked the question "When will both of these equations be true?" And the math answered, "Hey, buddy, not until 0 equals 13."

No solution.

⁵This content is available online at <<http://cnx.org/content/m18213/1.4/>>.

Now, consider these equations:

$$\begin{aligned} 2x + 3y &= 8 \\ 4x + 6y &= 16 \end{aligned} \tag{3.4}$$

Once again, we attempt elimination, but the result is different:

$$\begin{aligned} 2x + 3y &= 8 \\ 4x + 6y &= 16 \\ 0 &= 0 \end{aligned} \tag{3.5}$$

What happened **that** time? $0 = 0$ no matter what x is. Instead of an equation that is always false, we have an equation that is always true. Does that mean these equations work for **any** x and y ? Clearly not: for instance, $(1, 1)$ does not make either equation true. What this means is that the two equations are the same: any pair that solves one will also solve the other. There is an infinite number of solutions.

Infinite number of solutions.

All of this is much easier to understand graphically! Remember that one way to solve simultaneous equations is by graphing them and looking for the intersection. In the first case, we see that original equations represented two **parallel lines**. There is no point of intersection, so there is no simultaneous equation.

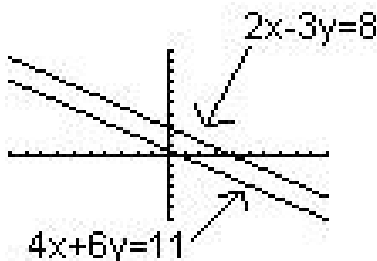


Figure 3.3

In the second case, we see that the original equations represented the **same line, in two different forms**. Any point on the line is a solution to both equations.

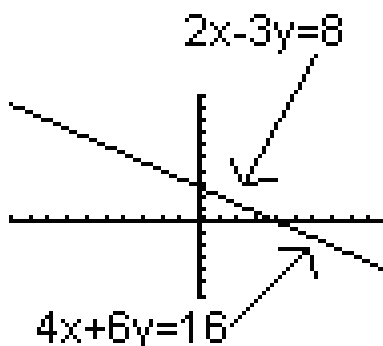


Figure 3.4

GENERAL RULE: If you solve an equation and get a mathematical impossibility such as $0 = 13$, there is no solution. If you get a mathematical tautology such as $0 = 0$, there is an infinite number of solutions.

3.6 Word Problems⁶

Many students approach math with the attitude that “I can do the equations, but I’m just not a ‘word problems’ person.” No offense, but that’s like saying “I’m pretty good at handling a tennis racket, as long as there’s no ball involved.” The only point of handling the tennis racket is to hit the ball. The only point of math equations is to solve problems. So if you find yourself in that category, try this sentence instead: “I’ve never been good at word problems. There must be something about them I don’t understand, so I’ll try to learn it.”

Actually, many of the key problems with word problems were discussed in the very beginning of the “Functions” unit, in the discussion of variable descriptions. So this might be a good time to quickly re-read that section. If you can correctly identify the variables, you’re half-way through the hard part of a word problem. The other half is **translating the sentences of the problem into equations** that use those variables.

Let’s work through an example, very carefully.

Example 3.3: Simultaneous Equation Word Problem

A roll of dimes and a roll of quarters lie on the table in front of you. There are three more quarters than dimes. But the quarters are worth three times the amount that the dimes are worth. How many of each do you have?

1. Identify and label the variables.
 - There are actually two different, valid ways to approach this problem. You could make a variable that represents the number of dimes; or you could have a variable that represents the value of the dimes. Either way will lead you to the right answer. However, it is vital to know which one you’re doing! If you get confused half-way through the problem, you will end up with the wrong answer.

⁶This content is available online at <<http://cnx.org/content/m18210/1.2/>>.

Let's try it this way:
d is the number of dimes
q is the number of quarters

Table 3.1

2. Translate the sentences in the problem into equations.
 - "There are three more quarters than dimes" $\rightarrow q = d + 3$
 - "The quarters are worth three times the amount that the dimes are worth" $\rightarrow 25q = 3(10d)$
 - This second equation relies on the fact that if you have q quarters, they are worth a total of $25q$ cents.
3. Solve.
 - We can do this by elimination or substitution. Since the first equation is already solved for q , I will substitute that into the second equation and then solve.

$25(d + 3) = 3(10d)$
$25d + 75 = 30d$
$75 = 5d$
$d = 15$
$q = 18$

Table 3.2

So, did it work? The surest check is to go all the way back to the original problem—not the equations, but the words. We have concluded that there are 15 **dimes** and 18 quarters.

"There are three more quarters than dimes." [U+2713]

"The quarters are worth three times the amount that the dimes are worth." $\rightarrow$ Well, the quarters are worth $18 \cdot 25 = \$4.50$. The dimes are worth $15 \cdot 10 = \$1.50$. [U+2713]

3.7 Using Letters as Numbers⁷

Toward the end of this chapter, there are some problems in substitution and elimination where **letters are used in place of numbers**. For instance, consider the following problem:

$$2y - ax = 7 \tag{3.6}$$

$$4y + 3ax = 9 \tag{3.7}$$

What do we do with those "a"s? Like any other variable, they simply represent an unknown number. As we solve for x , we will simply leave a as a variable.

⁷This content is available online at <<http://cnx.org/content/m18214/1.2/>>.

This problem lends itself more naturally to elimination than to substitution, so I will double the top equation and then subtract the two equations and solve.

$$\begin{array}{r} 4y - 2ax = 14 \\ - (4y + 3ax = 9) \\ \hline 0y - 5ax = 5 \end{array} \quad (3.8)$$

$$x = \frac{5}{-5a} = \frac{-1}{a} \quad (3.9)$$

As always, we can solve for the second variable by plugging into either of our original equations.

$$2y - a \left(\frac{-1}{a} \right) = 7 \quad (3.10)$$

$$2y + 1 = 7 \quad (3.11)$$

$$y = 3 \quad (3.12)$$

There is no new math here, just elimination. The real trick is not to be spooked by the a , and do the math just like you did before.

And what does that **mean**? It means we have found a solution that works for those two equations, regardless of a . We can now solve the following three problems (and an infinite number of others) without going through the hard work.

If $a = 5$,	If $a = 10$,	If $a = -3$,
The original equations become:	The original equations become:	The original equations become:
$2y - 5x = 7$ $4y + 15x = 9$	$2y - 10x = 7$ $4y + 30x = 9$	$2y - 3x = 7$ $4y + 9x = 9$
And the solution is:	And the solution is:	And the solution is:
$x = \frac{-1}{5}, y = 3$	$x = \frac{-1}{10}, y = 3$	$x = \frac{1}{3}, y = 3$

Table 3.3

The whole point is that I did not have to **solve** those three problems—by elimination, substitution, or anything else. All I had to do was plug a into the general answer I had already found previously. If I had to solve a hundred such problems, I would have saved myself a great deal of time by going through the hard work once to find a general solution!

Mathematicians use this trick all the time. If they are faced with many similar problems, they will attempt to find a **general** problem that encompasses all the **specific** problems, by using variables to replace the numbers that change. You will do this in an even more general way in the text, when you solve the “general” simultaneous equations where **all** the numbers are variables. Then you will have a formula that you can plug any pair of simultaneous equations into to find the answer at once. This formula would also make it very easy, for instance, to program a computer to solve simultaneous equations (computers are terrible at figuring things out, but they’re great at formulas).

Solutions to Exercises in Chapter 3

Solution to Exercise 3.1 (p. 57)

From graphing...

$$x = 0.1, x = 3$$

Chapter 4

Quadratic Functions

4.1 Multiplying Binomials¹

The following three formulae should be memorized.

$$(x + a)^2 = x^2 + 2ax + a^2 \quad (4.1)$$

$$(x - a)^2 = x^2 - 2ax + a^2 \quad (4.2)$$

$$(x + a)(x - a) = x^2 - a^2 \quad (4.3)$$

It is important to have these three formulae on the top of your head. It is also nice to be able to show **why** these formulae work, for instance by using FOIL. But the most important thing of all is knowing **what these three formulae mean, and how to use them**.

These three are all “algebraic generalizations,” as discussed in the first unit on functions. That is, they are equations that hold true for **any** values of x and a . It may help if you think of the second equation above as standing for:

(Anything – Anything Else)² = Anything² – 2(Anything Else)² For instance, suppose the **Anything** (or x) is 5, and the **Anything Else** (or a) is 3.

Example 4.1

$(x - a)^2 = x^2 - 2ax + a^2$, **when** $x = 5, a = 3$.

$$5 - 3^2 \stackrel{?}{=} 5^2 - 2(3)(5) + 3^2$$

$$2^2 \stackrel{?}{=} 25 - 30 + 9$$

$$4 = 4$$

It worked! Now, let’s leave the **Anything** as x , but play with different values of a .

¹This content is available online at <<http://cnx.org/content/m18224/1.3/>>.

Example 4.2

More examples of $(x - a)^2 = x^2 - 2ax + a^2$

$$\begin{aligned}
 a = 1 : \quad (x - 1)^2 &= x^2 - 2x + 1 \\
 a = 2 : \quad (x - 2)^2 &= x^2 - 4x + 4 \\
 a = 3 : \quad (x - 3)^2 &= x^2 - 6x + 9 \\
 a = 5 : \quad (x - 5)^2 &= x^2 - 10x + 25 \\
 a = 10 : \quad (x - 10)^2 &= x^2 - 20x + 100
 \end{aligned} \tag{4.4}$$

Once you've seen a few of these, the pattern becomes evident: the number **doubles** to create the middle term (the coefficient of x), and **squares** to create the final term (the number).

The hardest thing about this formula is **remembering to use it**. For instance, suppose you are asked to expand:

$$(2y - 6)^2 \tag{4.5}$$

There are three ways you can approach this.

$(2y - 6)^2$, computed three different ways		
Square each term	FOIL	Using the formula above
$(2y - 6)^2$ $(2y)^2 - 2(6)(2y) + 6^2$ $4y^2 - 24y + 36$	$(2y - 6)(2y - 6)$ $(2y)(2y) - (2y)6 - (2y)6 + 36$ $4y^2 - 12y - 12y + 36$ $4y^2 - 24y + 36$	$(2y - 6)^2$ $(2y)^2 - 2(6)(2y) + 6^2$ $4y^2 - 24y + 36$

Table 4.1

Did it work? If a formula is true, it should work for any y -value; let's test each one with $y = 5$. (Note that the second two methods got the same answer, so we only need to test that once.)

$(2y - 6)^2 \stackrel{?}{=} 4y^2 - 36$	$(2y - 6)^2 \stackrel{?}{=} 4y^2 - 24y + 36$
$(2 \cdot 5 - 6)^2 \stackrel{?}{=} 4y^2 - 36$	$(2 \cdot 5 - 6)^2 \stackrel{?}{=} 4(5)^2 - 24 \cdot 5 + 36$
$(10 - 6)^2 \stackrel{?}{=} 100 - 36$	$(10 - 6)^2 \stackrel{?}{=} 100 - 120 + 36$
$4^2 \stackrel{?}{=} 64 \times$	$4^2 \stackrel{?}{=} 16 \text{ [U+2713]}$

Table 4.2

We conclude that squaring each term individually does not work. The other two methods both give the same answer, which works.

The first method is the easiest, of course. And it **looks** good. $(2y)^2$ is indeed $4y^2$. And 6^2 is indeed 36. But as you can see, it led us to a **false answer**—an algebraic generalization that did not hold up.

I just can't stress this point enough. It sounds like a detail, but it causes errors all through Algebra II and beyond. **When you're adding or subtracting things, and then squaring them, you can't just square**

them one at a time. Mathematically, $(x + a)^2 \neq x^2 + a^2$. You can confirm this with numbers all day. $(7 + 3)^2 = 100$, but $7^2 + 3^2 = 58$. They're not the same.

So that leaves the other two methods. FOIL will never lead you astray. But the third approach, the formula, has three distinct advantages.

1. The formula is faster than FOIL.
2. Using these formulae is a specific case of the vital mathematical skill of using **any** formula—learning how to plug numbers and variables into some equation that you've been given, and therefore understanding the abstraction that formulae represent.
3. Before this unit is done, we will be completing the square, which requires running that particular formula **backward**—which you cannot do with FOIL.

4.2 Factoring²

When we multiply, we put things together: when we factor, we pull things apart. Factoring is a critical skill in simplifying functions and solving equations.

There are four basic types of factoring. In each case, I will start by showing a multiplication problem—then I will show how to use factoring to **reverse** the results of that multiplication.

4.2.1 “Pulling Out” Common Factors

This type of factoring is based on the **distributive property**, which (as you know) tells us that:

$$2x(4x^2 - 7x + 3) = 8x^3 - 14x^2 + 6x \quad (4.6)$$

When we factor, we do that in reverse. So we would start with an expression such as $8x^3 - 14x^2 + 6x$ and say “Hey, **every one** of those terms is divisible by 2. Also, **every one** of those terms is divisible by x . So we “factor out,” or “pull out,” a $2x$.

$$8x^3 - 14x^2 + 6x = 2x(__ - __ + __) \quad (4.7)$$

For each term, we see what happens when we **divide** that term by $2x$. For instance, if we divide $8x^3$ by $2x$ the answer is $4x^2$. Doing this process for each term, we end up with:

$$8x^3 - 14x^2 + 6x = 2x(4x^2 - 7x + 3) \quad (4.8)$$

As you can see, this is just what we started with, but in reverse. However, for many types of problems, this factored form is easier to work with.

As another example, consider $6x + 3$. The common factor in this case is 3. When we factor a 3 out of the $6x$, we are left with $2x$. When we factor a 3 out of the 3, we are left with...what? Nothing? No, we are left with 1, since we are **dividing** by 3.

$$6x + 3 = 3(2x + 1) \quad (4.9)$$

There are two key points to take away about this kind of factoring.

1. This is the simplest kind of factoring. Whenever you are trying to factor a complicated expression, **always begin** by looking for **common factors** that you can pull out.

²This content is available online at <<http://cnx.org/content/m18227/1.2/>>.

2. A common factor must be common to **all** the terms. For instance, $8x^3 - 14x^2 + 6x + 7$ has no common factor, since the last term is not divisible by either 2 or x .

4.2.2 Factoring Perfect Squares

The second type of factoring is based on the “squaring” formulae that we started with:

$$(x + a)^2 = x^2 + 2ax + a^2 \quad (4.10)$$

$$(x - a)^2 = x^2 - 2ax + a^2 \quad (4.11)$$

For instance, if we see $x^2 + 6x + 9$, we may recognize the signature of the first formula: the middle term is three **doubled**, and the last term is three **squared**. So this is $(x + 3)^2$. Once you get used to looking for this pattern, it is easy to spot.

$$x^2 + 10x + 25 = (x + 5)^2 \quad (4.12)$$

$$x^2 + 2x + 1 = (x + 1)^2 \quad (4.13)$$

And so on. If the middle term is **negative**, then we have the second formula:

$$x^2 - 8x + 16 = (x - 4)^2 \quad (4.14)$$

$$x^2 - 14x + 49 = (x - 7)^2 \quad (4.15)$$

This type of factoring only works if you have **exactly this case**: the middle number is something **doubled**, and the last number is that same something **squared**. Furthermore, although the **middle** term can be either positive or negative (as we have seen), the **last** term cannot be negative.

All this may make it seem like such a special case that it is not even worth bothering about. But as you will see with “completing the square” later in this unit, this method is very general, because even if an expression does not look like a perfect square, you can usually **make** it look like one if you want to—and if you know how to spot the pattern.

4.2.3 The Difference Between Two Squares

The third type of factoring is based on the third of our basic formulae:

$$(x + a)(x - a) = x^2 - a^2 \quad (4.16)$$

You can run this formula in reverse whenever you are **subtracting two perfect squares**. For instance, if we see $x^2 - 25$, we recognize that both x^2 and 25 are perfect squares. We can therefore factor it as $(x + 5)(x - 5)$. Other examples include:

Example 4.3

- $x^2 - 64 = (x + 8)(x - 8)$
- $16y^2 - 49 = (4y + 7)(4y - 7)$
- $2x^2 - 18 = 2(x^2 - 9) = 2(x + 3)(x - 3)$

And so on. Note that, in the last example, we begin by **pulling out** a 2, and we are then left with two perfect squares. This is an example of the rule that you should always **begin by pulling out common factors** before you try anything else!

It is also important to note that you cannot factor the sum of two squares. $x^2 + 4$ is a perfectly good function, but it cannot be factored.

4.2.4 Brute Force, Old-Fashioned, Bare-Knuckle, No-Holds-Barred Factoring

In this case, the multiplication that we are reversing is just FOIL. For instance, consider:

$$(x + 3)(x + 7) = x^2 + 3x + 7x + 21 = x^2 + 10x + 21 \quad (4.17)$$

What happened? The 3 and 7 **added** to yield the middle term (10), and **multiplied** to yield the final term (21). We can generalize this as: $(x + a)(x + b) = x^2 + (a + b)x + ab$.

The point is, if you are given a problem such as $x^2 + 10x + 21$ to factor, you look for two numbers that **add** up to 10, and **multiply** to 21. And how do you find them? There are a lot of pairs of numbers that add up to 10, but relatively few that multiply to 21. So you start by looking for factors of 21.

Below is a series of examples. Each example showcases a different aspect of the factoring process, so I would encourage you not to skip over any of them: try each problem yourself, then take a look at what I did.

If you are uncomfortable with factoring, the best practice you can get is to multiply things out. In each case, look at the final answer I arrive at, and multiply it with FOIL. See that you get the problem I started with. Then look back at the steps I took and see how they led me to that answer. The steps will make a lot more sense if you have done the multiplication already.

Exercise 4.1

(Solution on p. 85.)

Factor $x^2 + 11x + 18$

$$(x + \underline{\quad})(x + \underline{\quad})$$

NOTE: Start by listing all factors of the third term. Then see which ones add to give you the middle term you want.

Exercise 4.2

(Solution on p. 85.)

Factor $x^2 - 13x + 12$

$$(x + \underline{\quad})(x + \underline{\quad})$$

NOTE: If the middle term is negative, it doesn't change much: it just makes both numbers negative. If this had been $x^2 + 13x + 12$, the process would have been the same, and the answer would have been $(x + 1)(x + 12)$.

Exercise 4.3

(Solution on p. 85.)

Factor $x^2 + 12x + 24$

$$(x + \underline{\quad})(x + \underline{\quad})$$

NOTE: Some things can't be factored. Many students spend a long time fighting with such problems, but it really doesn't have to take long. Try all the possibilities, and if none of them works, it can't be factored.

Exercise 4.4*(Solution on p. 85.)*Factor $x^2 + 2x - 15$

$$(x + \underline{\quad})(x + \underline{\quad})$$

NOTE: If the **last** term is negative, that changes things! In order to **multiply** to -15 , the two numbers will have to have different signs—one negative, one positive—which means they will **subtract** to give the middle term. Note that if the **middle** term were negative, that wouldn't change the process: the final answer would be reversed, $(x + 5)(x - 3)$. This fits the rule that we saw earlier—changing the sign of the middle term changes the answer a bit, but not the process.

Exercise 4.5*(Solution on p. 85.)*Factor $2x^2 + 24x + 72$

NOTE: Never forget, **always** start by looking for common factors to pull out. Then look to see if it fits one of our formulae. Only after trying all that do you begin the FOIL approach.

Exercise 4.6*(Solution on p. 85.)*Factor $3x^2 + 14x + 16$

$$(3x + \underline{\quad})(x + \underline{\quad})$$

NOTE: If the x^2 has a coefficient, and if you can't pull it out, the problem is trickier. In this case, we know that the factored form will look like $(3x + \underline{\quad})(x + \underline{\quad})$ so we can see that, when we multiply it back, one of those numbers—the one on the right—will be tripled, before they add up to the middle term! So you have to check the number pairs to see if any work that way.

4.2.5 Checking Your Answers

There are two different ways to check your answer after factoring: multiplying back, and trying numbers.

Example 4.4

- Problem:** Factor $40x^3 - 250x$
 - $10x(4x - 25)$ **First, pull out the common factor**
 - $10x(2x + 5)(2x - 5)$ **Difference between two squares**
- So, does $40x^3 - 250x = 10x(2x + 5)(2x - 5)$? First let's check by multiplying back.**
 - $10x(2x + 5)(2x - 5)$
 - $= (20x^2 + 50x)(2x - 5)$ **Distributive property**
 - $= 40x^3 - 100x^2 + 100x^2 - 250x$ **FOIL**
 - $= 40x^3 - 250x$ [U+2713]
- Check by trying a number. This should work for any number. I'll use $x = 7$ and a calculator.**
 - $40x^3 - 250x \stackrel{?}{=} 10x(2x + 5)(2x - 5)$
 - $40(7)^3 - 250(7) \stackrel{?}{=} 10(7)(2 \cdot 7 + 5)(2 \cdot 7 - 5)$
 - $11970 = 11970$ [U+2713]

I stress these methods of checking answers, not just because checking answers is a generally good idea, but because they reinforce key concepts. The first method reinforces the idea that **factoring is multiplication done backward**. The second method reinforces the idea of algebraic generalizations.

4.3 Solving Quadratic Equations by Factoring³

Consider the equation $4x^2 + 14x - 60 = 0$. This is **not** an algebraic generalization, but an equation to be solved for x : that is, it asks the question “What x value, or values, will make this equation true?” We will be solving such equations in three different ways. The fastest and easiest is by factoring.

Using the techniques discussed above, we can rewrite this problem as follows. (Try it for yourself!)

$$4x^2 + 14x - 60 = 0 \text{ Original form}$$

$$2(2x - 5)(x + 6) = 0 \text{ Factored form}$$

The second form may look more complicated than what we started with. But consider what this equation says. There are three numbers: 2, $2x - 5$, and $x + 6$. The equation says that when you multiply these three numbers, you get 0. Ask yourself this crucial question: **How can you multiply numbers and get the answer 0?**

The only way it can happen is if one of the numbers is 0. Take a moment to convince yourself of this: if several numbers **multiply** to give 0, one of those numbers must be 0.

So we have three possibilities.

$2 = 0$	$2x - 5 = 0$	$x + 6 = 0$
(it just isn't)	$x = 2\frac{1}{2}$	$x = -6$

Table 4.3

The moral of the story is: when a quadratic equation is factored, it can be solved easily. In this case, the equation $4x^2 + 14x - 60 = 0$ has two valid solutions, $x = 2\frac{1}{2}$ and $x = -6$.

Consider this example:

$$x^2 - 9x + 20 = 6 \tag{4.18}$$

A common mistake is to solve it like this.

Example 4.5

$x^2 - 9x + 20 = 6$, solved **incorrectly**

- $(x - 4)(x - 5) = 6$
- $(x - 4) = 6$
 - $x = 10 \times$
- $(x - 5) = 6$
 - $x = 11 \times$

All looks good, doesn't it? The factoring was correct. But if you try $x = 10$ or $x = 11$ in the original equation, you will find that neither one works. What went wrong?

The factoring was correct, but the next step was wrong. Just because $(x - 4)(x - 5) = 6$ does not mean that either $(x - 4)$ or $(x - 5)$ has to be 6. There are lots of ways for two numbers to multiply to give 6. This trick only works for 0!

Example 4.6

$x^2 - 9x + 20 = 6$, solved **correctly**

³This content is available online at <<http://cnx.org/content/m18222/1.2/>>.

- $x^2 - 9x + 14 = 0$
- $(x - 7)(x - 2) = 0$
- $(x - 7) = 0$
 - $x = 7$ [U+2713]
- $(x - 2) = 0$
 - $x = 2$ [U+2713]

You may want to confirm for yourself that these are the correct solutions.

Moral: When solving quadratic equations, always begin by moving **everything to one side of the equation**, leaving only a 0 on the other side. This is true regardless of which of the three methods you use.

Example 4.7

$$x^2 + 14x + 49 = 0$$

- $(x + 7)^2 = 0$
- $x = -7$

Moral: If the left side factors as a perfect square, the quadratic equation has only one solution.

Not all quadratic functions can be factored. This does not mean they have no solutions! If the function cannot be factored, we must use other means to find the solutions.

4.4 Solving Quadratic Equations by Completing the Square⁴

Consider the equation:

$$(x + 3)^2 = 16 \tag{4.19}$$

We can solve this by analogy to the way that we approached absolute value problems. **something** squared is 16. So what could the **something** be? It could be 4. It could also be -4 . So the solution is:

- $x + 3 = 4$
 - $x = 1$
- $x + 3 = -4$
 - $x = -7$

These are the two solutions.

This simple problem leads to a completely general way of solving quadratic equations—because any quadratic equation can be put in a form like the above equation. The key is **completing the square** which, in turn, is based on our original two formulae:

$$(x + a)^2 = x^2 + 2ax + a^2 \tag{4.20}$$

$$(x - a)^2 = x^2 - 2ax + a^2 \tag{4.21}$$

As an example, consider the equation $x^2 + 10x + 21 = 0$. In order to make it fit one of the patterns above, we must replace the 21 with the correct number: a number such that $x^2 + 10x + \underline{\hspace{1cm}}$ is a perfect square. What

⁴This content is available online at <<http://cnx.org/content/m18217/1.2/>>.

number goes there? If you are familiar with the pattern, you know the answer right away. 10 is 5 **doubled**, so the number there must be 5 **squared**, or 25.

But how do we turn a 21 into a 25? We add 4, of course. And if we add 4 to **one** side of the equation, we have to add 4 to the **other** side of the equation. So the entire problem is worked out as follows:

Example 4.8

Solving by Completing the Square (quick-and-dirty version)		
Solve $x^2 + 10x + 21 = 0$		The problem.
$x^2 + 10x + 25 = 4$		Add 4 to both sides, so that the left side becomes a perfect square.
$(x + 5)^2 = 4$		Rewrite the perfect square.
$x + 5 = 2$ $x = -3$	$x + 5 = -2$ $x = -7$	If something-squared is 4, the something can be 2, or -2. Solve both possibilities to find the two answers.

Table 4.4

Thus, we have our two solutions.

Completing the square is more time-consuming than factoring: so whenever a quadratic equation can be factored, factoring is the preferred method. (In this case, we would have factored the original equation as $(x + 2)(x + 8)$ and gotten straight to the answer.) However, completing the square can be used on **any** quadratic equation. In the example below, completing the square is used to find two answers that would not have been found by factoring.

Example 4.9

Solving by Completing the Square (showing all the steps more carefully)		
Solve $9x^2 - 54x + 80 = 0$		The problem.
$9x^2 - 54x = -80$		Put all the x terms on one side, and the number on the other
$x^2 - 6x = -\frac{80}{9}$		Divide both sides by the coefficient of x^2 (*see below)
$x^2 - 6x + 9 = -\frac{80}{9} + 9$		Add the same number (*see below) to both sides, so that the left side becomes a perfect square.
<i>continued on next page</i>		

$(x - 3)^2 = -\frac{80}{9} + \frac{81}{9} = \frac{1}{9}$		Rewrite the perfect square.
$x - 3 = \frac{1}{3}$ $x = 3\frac{1}{3}$	$x - 3 = -\frac{1}{3}$ $x = 2\frac{2}{3}$	If something-squared is $\frac{1}{9}$, the something can be $\frac{1}{3}$, or $-\frac{1}{3}$. Solve both possibilities to find the two answers.

Table 4.5

Two steps in particular should be pointed out here.

In the third step, we divide both sides by 9. When completing the square, you do not want to have any coefficient in front of the term; if there is a number there, you divide it out. Fractions, on the other hand (such as the $-\frac{80}{9}$ in this case) do not present a problem. This is in marked contrast to factoring, where a coefficient in front of the x^2 can be left alone, but fractions make things nearly impossible.

The step after that is where we actually complete the square. $x^2 + 6x + \underline{\hspace{1cm}}$ will be our perfect square. How do we find what number we want? Start with the coefficient of x^2 (in this case, 6). **Take half of it, and square the result.** Half of 6 is 3, squared is 9. So we want a 9 there to create $x^2 + 6x + 9$ which can be simplified to $(x + 3)^2$.

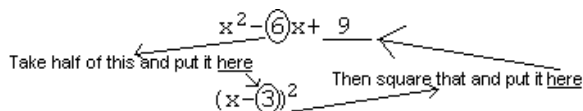


Figure 4.1

If the coefficient of x is an **odd number**, the problem becomes a little uglier, but the principle is the same. For instance, faced with:

$$x^2 + 5x + \underline{\hspace{1cm}} \quad (4.22)$$

You would begin by taking half of 5 (which is $\frac{5}{2}$) and then squaring it:

$$x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$$

Another “completing the square” example, in which you cannot get rid of the square root at all, is presented in the worksheet “The Generic Quadratic Equation.”

One final note on completing the square: there are three different possible outcomes.

- If you end up with something like $(x - 3)^2 = 16$ you will find two solutions, since $x - 3$ can be either 4, or -4 . You will always have two solutions if the right side of the equation, after completing the square, is positive.
- If you end up with $(x - 3)^2 = 0$ then there is only one solution: x must be 3 in this example. If the right side of the equation is 0 after completing the square, there is only one solution.
- Finally, what about a negative number on the right, such as $(x - 3)^2 = -16$? Nothing squared can give a negative answer, so there is no solution.

4.5 The Quadratic Formula⁵

In "Solving Quadratic Equations by Completing the Square" I talked about the common mathematical trick of solving a problem once, **using letters instead of numbers**, and then solving specific problems by plugging numbers into a general solution.

In the text, you go through this process for quadratic equations in general. The definition of a quadratic equation is any equation that can be written in the form:

$$ax^2 + bx + c = 0 \quad (4.23)$$

where $a \neq 0$. By completing the square on this generic equation, you arrive at the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (4.24)$$

This formula can then be used to solve any quadratic equation, without having to complete the square each time. To see how this formula works, let us return to the previous problem:

$$9x^2 - 54x + 80 = 0 \quad (4.25)$$

In this case, $a = 9$, $b = -54$, and $c = 80$. So the quadratic formula tells us that the answers are:

$$x = \frac{-(-54) \pm \sqrt{(-54)^2 - 4(9)(80)}}{2(9)} \quad (4.26)$$

We'll use a calculator here rather than squaring 54 by hand....

$$x = \frac{54 \pm \sqrt{2916 - 2880}}{18} = \frac{54 \pm \sqrt{36}}{18} = \frac{54 \pm 6}{18} = \frac{9 \pm 1}{3} \quad (4.27)$$

So we find that the two answers are $\frac{10}{3}$ and $\frac{8}{3}$, which are the same answers we got by completing the square.

Using the quadratic formula is usually faster than completing the square, though still slower than factoring. So, in general, try to factor first: if you cannot factor, use the quadratic formula.

So why do we learn completing the square? Two reasons. First, completing the square is how you **derive** the quadratic formula. Second, completing the square is vital to **graphing** quadratic functions, as you will see a little further on in the chapter.

4.6 Different Types of Solutions to Quadratic Equations⁶

The heart of the quadratic formula is the part under the square root: $b^2 - 4ac$. This part is so important that it is given its own name: the **discriminant**. It is called this because it **discriminates** the different types of solutions that a quadratic equation can have.

Common Error

Students often think that the discriminant is $\sqrt{b^2 - 4ac}$. But the discriminant is **not the square root**, it is the part that is **under** the square root:

$$\text{Discriminant} = b^2 - 4ac \quad (4.28)$$

It can often be computed quickly and easily without a calculator.

⁵This content is available online at <<http://cnx.org/content/m18231/1.2/>>.

⁶This content is available online at <<http://cnx.org/content/m18216/1.2/>>.

Why is this quantity so important? Consider the above example, where the discriminant was 36. This means that we wound up with ± 6 in the numerator. So the problem had two different, rational answers: $2\frac{2}{3}$ and $3\frac{1}{3}$.

Now, consider $x^2 + 3x + 1 = 0$. In this case, the discriminant is $3^2 - 4(1)(1) = 5$. We will end up with $\pm\sqrt{5}$ in the numerator. There will still be two answers, but they will be **irrational**—they will be impossible to express as a fraction without a square root.

$4x^2 - 20x + 25 = 0$. Now the discriminant is $20^2 - 4(4)(25) = 400 - 400 = 0$. We will end up with ± 0 in the numerator. But it makes no difference if you add or subtract 0; you get the same answer. So this problem will have only one answer.

And finally, $3x^2 + 5x + 4$. Now, $b^2 - 4ac = 5^2 - 4(3)(4) = 25 - 48 = -23$. So in the numerator we will have $\sqrt{-23}$. Since you cannot take the square root of a negative number, there will be no solutions!

Summary: The Discriminant

- If the discriminant is a perfect square, you will have two rational solutions.
- If the discriminant is a positive number that is not a perfect square, you will have two irrational solutions (ie they will have square roots in them).
- If the discriminant is 0, you will have one solution.
- If the discriminant is negative, you will have no solutions.

These rules do not have to be memorized: you can see them very quickly by understanding the quadratic formula (which **does** have to be memorized—if all else fails, try singing it to the tune of Frère Jacques).

Why is it that quadratic equations can have 2 solutions, 1 solution, or no solutions? This is easy to understand by looking at the following graphs. Remember that in each case the quadratic equation asks when the function is 0—that is to say, when it crosses the x -axis.

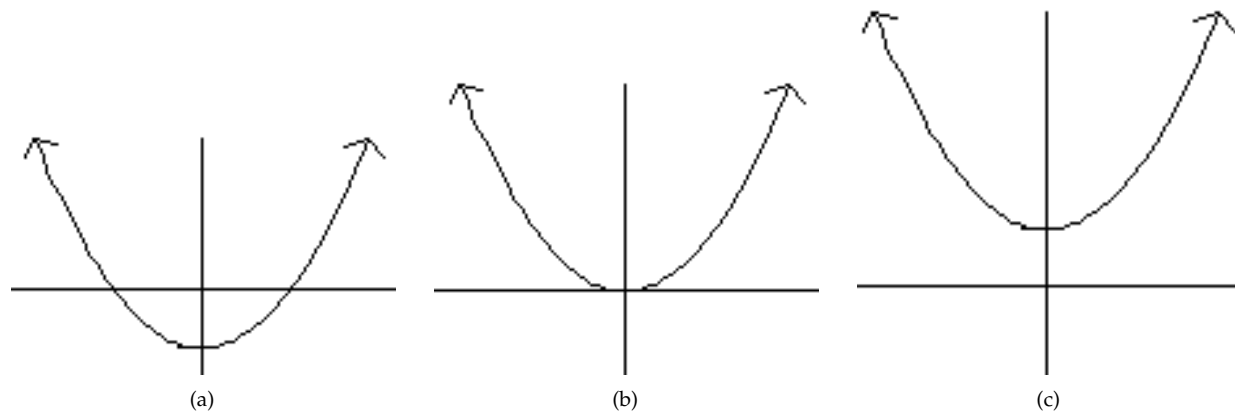


Figure 4.2: (a) $y = x^2 - 2$ Equals 0 two times (b) $y = x^2$ Equals 0 once (c) $y = x^2 + 2$. Never equals 0

More on how to generate these graphs is given below. For the moment, the point is that you can visually see why a quadratic function can equal 0 twice, or one time, or never. It can **not** equal 0 three or more times.

4.7 Graphing Quadratic Equations⁷

The graph of the simplest quadratic function, $y = x^2$, looks like this:

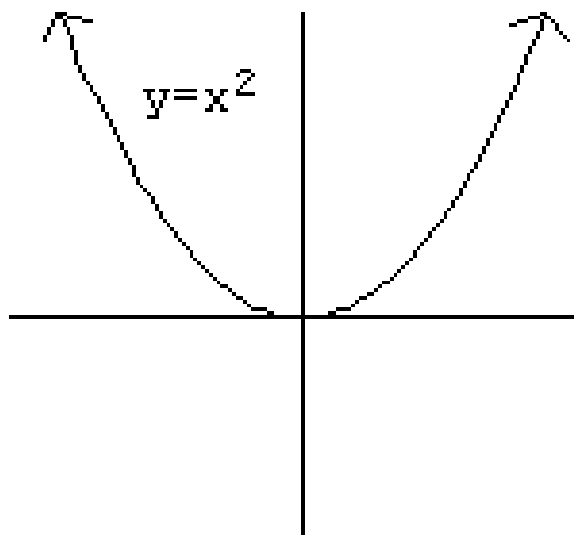


Figure 4.3

(You can confirm this by plotting points.) The point at the bottom of the U-shaped curve is known as the “vertex.”

Now consider the function $y = -3(x + 2)^2 + 1$. It’s an intimidating function, but we have all the tools we need to graph it, based on the **permutations** we learned in the first unit. Let’s step through them one by one.

- What does the $--$ sign do? It multiplies all y -values by -1 ; positive values become negative, and vice-versa. So we are going to get an **upside-down** U-shape. We say that $y = x^2$ “opens up” and $y = -x^2$ “opens down.”
- What does the 3 do? It multiplies all y -values by 3; positive values become more positive, and negative values become more negative. So it **vertical stretches** the function.
- What does the $+1$ at the end do? It adds 1 to all y -values, so it moves the function **up** by 1.
- Finally, what does the $+2$ do? This is a horizontal modification: if we plug in $x = 10$, we will be evaluating the function at $x = 12$. In general, we will always be copying the original x^2 function to our **right**; so we will be 2 units to the **left** of it.

So what does the graph look like? It has moved 2 to the left and 1 up, so the vertex moves from the origin $(0,0)$ to the point $(-2,1)$. The graph has also flipped upside-down, and stretched out vertically.

⁷This content is available online at <<http://cnx.org/content/m18228/1.2/>>.

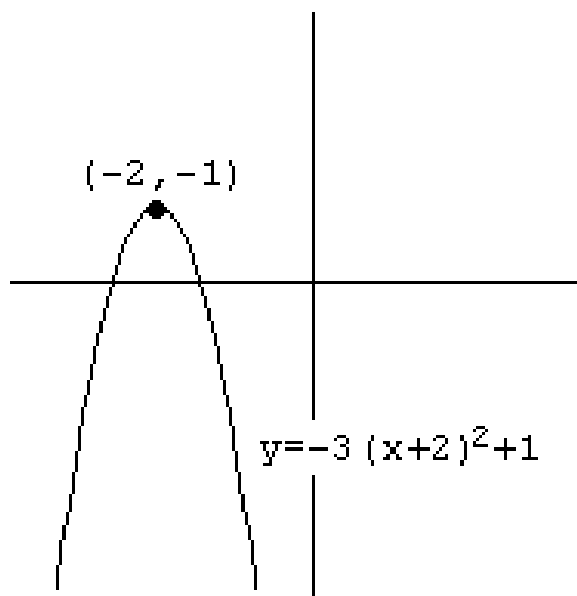


Figure 4.4

So graphing quadratic functions is easy, no matter how complex they are, if you understand permutations—and if the functions are written in the form $y = a(x - h)^2 + k$, as that one was.

Graphing Quadratic Functions

The graph of a quadratic function is always a vertical parabola. If the function is written in the form $y = a(x - h)^2 + k$ then the vertex is at (h, k) . If a is positive, the parabola opens up; if a is negative, the parabola opens down.

But what if the functions are not expressed in that form? We're more used to seeing them written as $y = ax^2 + bx + c$. For such a function, you graph it by first putting it into the form we used above, and then graphing it. And the way you get it into the right form is...completing the square! This process is almost identical to the way we used completing the square to **solve** quadratic equations, but some of the details are different.

Example 4.10

Graphing a Quadratic Function	
Graph $2x^2 - 20x + 58$	The problem.
<i>continued on next page</i>	

instead, look over every **individual step** carefully and make sure you understand why it works—that is, why it keeps the function fundamentally unchanged, while moving us toward our goal of a form that we can graph.

The good news is, this process is basically the same every time. A different example is worked through in the worksheet “Graphing Quadratic Functions II”—that example differs only because the x^2 term does not have a coefficient, which changes a few of the steps in a minor way. You will have plenty of opportunity to practice this process, which will help you get the “big picture” if you understand all the individual steps.

And don’t forget that what we’re really creating here is an algebraic generalization!

$$2x^2 - 20x + 58 = 2(x - 5)^2 + 8 \quad (4.29)$$

This is exactly the sort of generalization we discussed in the first unit—the assertion that these two very different functions will **always give the same answer** for any x -value you plug into them. For this very reason, we can also assert that the two graphs will look the same. So we can graph the first function by graphing the second.

4.8 Solving Problems by Graphing Quadratic Functions⁸

Surprisingly, there is a fairly substantial class of real world problems that can be solved by graphing quadratic functions.

These problems are commonly known as “optimization problems” because they involve the question: “When does this important function reach its **maximum**?” (Or sometimes, its **minimum**?) In real life, of course, there are many things we want to maximize—a company wants to maximize its revenue, a baseball player his batting average, a car designer the leg room in front of the driver. And there are many things we want to minimize—a company wants to minimize its costs, a baseball player his errors, a car designer the amount of gas used. Mathematically, this is done by writing a function for that quantity and finding where that function reaches its highest or lowest point.

Example 4.11

If a company manufactures x items, its total cost to produce these items is $x^3 - 10x^2 + 43x$. How many items should the company make in order to **minimize its average cost per item**?

Solution

“Average cost per item” is the total cost, divided by the number of items. For instance, if it costs \$600 to manufacture 50 items, then the average cost per item was \$12. It is important for companies to minimize average cost because this enables them to sell at a low price.

In this case, the total cost is $x^3 - 10x^2 + 43x$ and the number of items is x . So the **average cost per item** is

$$A(x) = \frac{x^3 - 10x^2 + 43x}{x} = x^2 - 10x + 43 \quad (4.30)$$

What the question is asking, mathematically, is: what value of x makes this function the **lowest**?

Well, suppose we were to graph this function. We would complete the square by rewriting it as:

$$A(x) = x^2 - 10x + 25 + 18 = (x - 5)^2 + 18 \quad (4.31)$$

⁸This content is available online at <<http://cnx.org/content/m18220/1.2/>>.

The graph opens up (since the $(x - 5)^2$ term is positive), and has its vertex at $(5, 18)$ (since it is moved 5 to the right and 18 up). So it would look something like this:

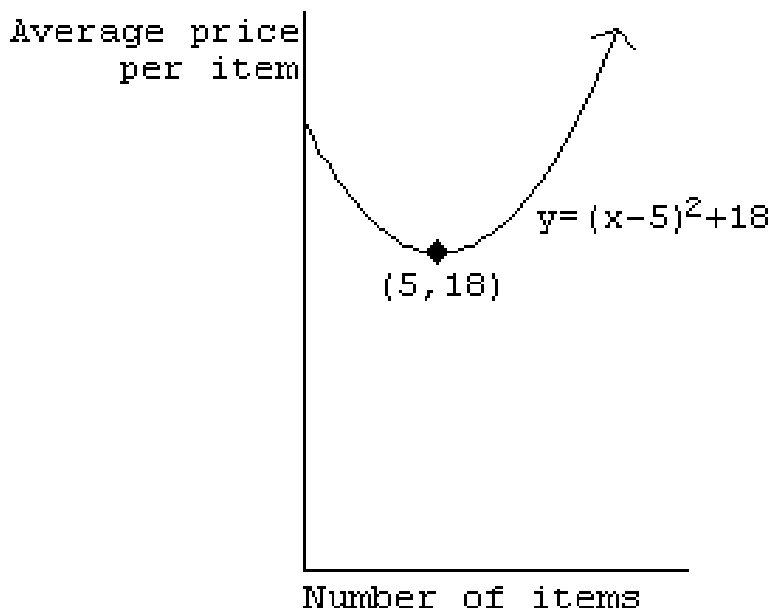


Figure 4.5

I have graphed only the first quadrant, because negative values are not relevant for this problem (why?).

The real question here is, what can we learn from that graph? Every point on that graph represents one possibility for our company: if they manufacture x items, the graph shows what $A(x)$, the average cost per item, will be.

The point $(5, 18)$ is the **lowest point on the graph**. It is possible for x to be higher or lower than 5, but it is never possible for A to be lower than 18. So if its goal is to minimize average cost, their best strategy is to manufacture 5 items, which will bring their average cost to \$18/item.

4.9 Quadratic Inequalities⁹

Consider the following inequality:

$$-x^2 + 6x - 8 \leq 0 \quad (4.32)$$

There are a number of different ways to approach such a problem. The one that is stressed in the text is by **graphing**.

We begin by graphing the function $y = -x^2 + 6x - 8$. We know how to graph this function by completing the square, but we're going to take a shortcut—one that will not actually tell us what the vertex is, but

⁹This content is available online at <<http://cnx.org/content/m18230/1.2/>>.

that will give us what we need to know. The shortcut relies on finding only two facts about the quadratic function: what are its **roots** (the places it crosses the x -axis), and which way does it open?

Of course we know three ways to find the roots of a quadratic equation: the easiest is always factoring when it works, as it does in this case.

$$-x^2 + 6x - 8 = 0 \quad (4.33)$$

$$(x - 4)(-x + 2) = 0 \quad (4.34)$$

$$x = 0, x = 2 \quad (4.35)$$

Second, which way does the parabola open? Since it is a vertical parabola with a **negative coefficient of the x^2 term**, it opens down.

So the graph looks like this:

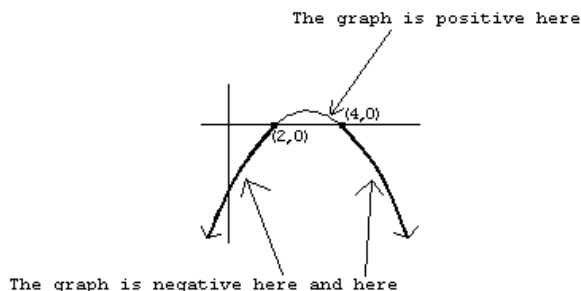


Figure 4.6

(Hey, if it's that easy to graph a quadratic function, why did we spend all that time completing the square? Well, this method of graphing does not tell you the vertex. It tells us all we need to solve the quadratic inequality, but not everything about the graph. **Oh. Darn.** Anyway, back to our original problem.)

If that is the graph of $y = -x^2 + 6x - 8$, then let us return to the original question: when is $-x^2 + 6x - 8$ **less than or equal to 0**? What this question is asking is: when does this graph dip **below** the x -axis? Looking at the graph, the answer is clear: the graph is below the x -axis, and therefore the function is negative, whenever $x \leq 2$ or $x \geq 4$.

When students graph these functions, they tend to get the right answer. Where students go wrong is by trying to take "shortcuts." They find where the function equals zero, and then attempt to quickly find a solution based on that. To see where this goes wrong, consider the following:

$$x^2 - 2x + 2 \geq 0 \quad (4.36)$$

If you attempt to find where this function **equals** zero—using, for instance, the quadratic formula—you will find that it never does. (Try it!) Many students will therefore quickly answer "no solution." Quick, easy—and wrong. To see why, let's try graphing it.

$$y = x^2 - 2x + 2 = x^2 - 2x + 1 + 1 = (x - 1)^2 + 1 \quad (4.37)$$

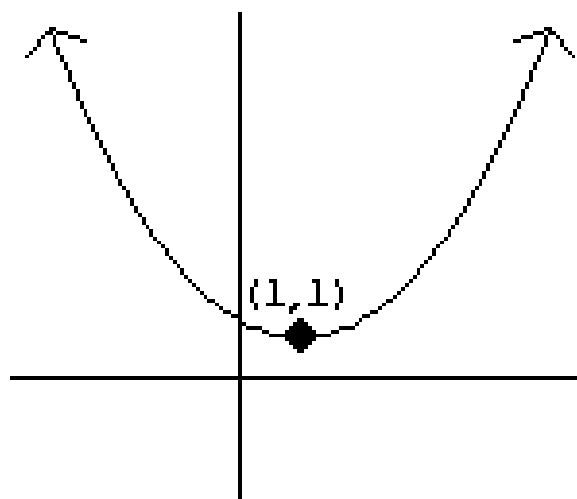


Figure 4.7

The original question asked: when is this function greater than or equal to zero? The graph makes it clear: **everywhere**. Any x value you plug into this function will yield a positive answer. So the solution is: "All real numbers."

4.9.1 "Extra for Experts": Quadratic Inequalities by Factoring

This technique is **not** stressed in this book. But it is possible to solve quadratic inequalities without graphing, if you can factor them.

Let's return to the previous example:

$$-x^2 + 6x - 8 \leq 0 \quad (4.38)$$

$$(x - 4)(-x + 2) \leq 0 \quad (4.39)$$

When we solved quadratic **equations** by factoring, we asked the question: "How can you multiply two numbers and get 0?" (Answer: when one of them is 0.) Now we ask the question: "How can you multiply two numbers and get a **negative answer**?" Answer: when the two numbers have **different signs**. That is, the product will be negative if...

The first is positive and the second negative	OR	The first is negative and the second positive
$x - 4 \geq 0$ AND $-x + 2 \leq 0$	OR	$x - 4 \leq 0$ AND $-x + 2 \geq 0$
$x \geq 4$ AND $-x \leq -2$	OR	$x \leq 4$ AND $-x \geq -2$
$x \geq 4$ AND $x \geq 2$	OR	$x \leq 4$ AND $x \leq 2$
$x \geq 4$	OR	$x \leq 2$

Table 4.7

This is, of course, the same answer we got by graphing.

Solutions to Exercises in Chapter 4

Solution to Exercise 4.1 (p. 69)

What **multiplies** to 18? $1 \cdot 18$, or $2 \cdot 9$, or $3 \cdot 6$.

Which of those **adds** to 11? $2 + 9$.

$$(x + 2)(x + 9)$$

Solution to Exercise 4.2 (p. 69)

What **multiplies** to 12? $1 \cdot 12$, or $2 \cdot 6$, or $3 \cdot 4$

Which of those **adds** to 13? $1 + 12$

$$(x - 1)(x - 12)$$

Solution to Exercise 4.3 (p. 69)

What **multiplies** to 24? $1 \cdot 24$, or $2 \cdot 12$, or $3 \cdot 8$, or $4 \cdot 6$

Which of those **adds** to 12? None of them.

It can't be factored. It is "prime."

Solution to Exercise 4.4 (p. 70)

What **multiplies** to 15? $1 \cdot 15$, or $3 \cdot 5$

Which of those **subtracts** to 2? $5 - 3$

$$(x + 5)(x - 3)$$

Solution to Exercise 4.5 (p. 70)

$$2(x^2 + 12x + 36)$$

$$2(x + 6)^2$$

Solution to Exercise 4.6 (p. 70)

What **multiplies** to 16? $1 \cdot 16$, or $2 \cdot 8$, or $4 \cdot 4$

Which of those adds to 14 **after tripling one number**? $8 + 3 \cdot 2$

$$(3x + 8)(x + 2)$$

Chapter 5

Exponents

5.1 Exponent Concepts¹

An exponent means repeated multiplication. For instance, 10^6 means $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$, or 1,000,000.

You've probably noticed that there is a logical progression of operations. When you **add the same number repeatedly**, that's multiplication. When you **multiply the same number repeatedly**, that's an exponent.

However, there is one vital difference: addition and multiplication **commute**, but exponentiation does **not commute**. This is a fancy way of saying that order matters. $2 + 3 = 3 + 2$; and $2 \cdot 3 = 3 \cdot 2$, but 2^3 is **not** the same as 3^2 .

5.2 Laws of Exponents²

The following are generally referred to as the “laws” or “rules” of exponents.

$$x^a x^b = x^{a+b} \quad (5.1)$$

$$\frac{x^a}{x^b} = x^{a-b} \quad \text{or} \quad \frac{1}{x^{b-a}} \quad (5.2)$$

$$(x^a)^b = x^{ab} \quad (5.3)$$

As with any formula, the most important thing is to be able to **use** them—that is, to understand what they mean. But it is also important to know **where these formulae come from**. And finally, in this case, the three should be memorized.

So...what do they mean? They are, of course, algebraic generalizations—statements that are true for any x , a , and b values. For instance, the first rule tells us that:

$$7^{12} \cdot 7^4 = 7^{16} \quad (5.4)$$

which you can confirm on your calculator. Similarly, the third rule promises us that

$$(7^{12})^4 = 7^{48} \quad (5.5)$$

¹This content is available online at <<http://cnx.org/content/m18232/1.2/>>.

²This content is available online at <<http://cnx.org/content/m18235/1.2/>>.

These rules can be used to combine and simplify expressions.

Example 5.1

Simplifying with the Rules of Exponents	
$\frac{(x^3)^4 \cdot x^5}{x^9 \cdot x^{11}}$	
$= \frac{x^{12} \cdot x^5}{x^9 \cdot x^{11}}$	Third rule: $(x^a)^b = x^{ab}$, where $a = 3$ and $b = 4$
$= \frac{x^{17}}{x^{20}}$	First rule: $x^a x^b = x^{a+b}$, done on both the top and bottom
$= \frac{1}{x^3}$	Second rule: $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$, where we choose this form to avoid a negative exponent

Table 5.1

Why do these rules work? It's very easy to see, based on what an exponent is.

Why does the first rule work?	
19^3	19^4
$= (19 \cdot 19 \cdot 19)$	$(19 \cdot 19 \cdot 19)$
$= 19^7$	

Table 5.2

You see what happened? 19^3 means three 19s multiplied; 19^4 means four 19s multiplied. Multiply them together, and you get **seven** 19s multiplied.

Why does the second rule work?	
First form	Second form
$\frac{19^8}{19^5}$	$\frac{19^5}{19^8}$
$= \frac{19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19}{19 \cdot 19 \cdot 19 \cdot 19 \cdot 19}$	$= \frac{19 \cdot 19 \cdot 19 \cdot 19 \cdot 19}{19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19 \cdot 19}$
$= \frac{\cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot 19 \cdot 19 \cdot 19}{\cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19}}$	$= \frac{\cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19}}{\cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot \cancel{19} \cdot 19 \cdot 19 \cdot 19}$
continued on next page	

$= 19^3$	$= \frac{1}{19^3}$
----------	--------------------

Table 5.3

In this case, the key is **fraction cancellations**. When the top is multiplied by 19 and the bottom is multiplied by 19, canceling these 19s has the effect of dividing **the top and bottom** by 19. When you divide the top and bottom of a fraction by the same number, the fraction is unchanged.

You can also think of this rule as the inevitable consequence of the **first** rule. If $19^3 \cdot 19^5 = 19^8$, then $\frac{19^8}{19^5}$ (which asks the question “ 19^5 times what equals 19^8 ?”) must be 19^3 .

Why does the third rule work?
$(19^3)^4$
$= (19 \cdot 19 \cdot 19) \cdot (19 \cdot 19 \cdot 19) \cdot (19 \cdot 19 \cdot 19) \cdot (19 \cdot 19 \cdot 19)$
$= 19^{12}$

Table 5.4

What does it mean to raise something to the fourth power? It means to multiply it by itself, four times. In this case, what we are multiplying by itself four times is 19^3 , or $(19 \cdot 19 \cdot 19)$. Three 19s multiplied **four times** makes twelve 19s multiplied.

5.3 Zero, Negative Numbers, and Fractions as Exponents³

The definition of exponents given at the beginning of this section— 10^6 means $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ —does **not** enable us to answer questions such as:

$$4^0 = ? \quad (5.6)$$

$$5^{-4} = ? \quad (5.7)$$

$$9^{\frac{1}{2}} = ? \quad (5.8)$$

You can’t “multiply 9 by itself half a time” or “multiply 5 by itself -4 times.” In general, the original definition only applies if the exponent is a **positive integer**.

It’s very important to understand this point. The question is **not** “What answer does our original definition give in these cases?” The original definition does **not** give any answer in these cases. If we want to include these numbers, we need a whole new definition of what an exponent is.

In principle, many such definitions are possible. We **could** define 5^{-4} as $5/5/5/5$: in other words, divide four times instead of multiplying four times. Or we **could** define 5^{-4} as 5^{-4} : take 5 to the fourth power, and then multiply it by -1 . Or we **could** define 5^{-4} as $-(5)^4$: take -5 to the fourth power (which gives a different answer from the previous definition). It seems at first that we are at liberty to choose any definition we want.

Given that degree of freedom, you may be very surprised at the definitions that are actually used: they seem far more arbitrary and complicated than some others you could come up with.

³This content is available online at <<http://cnx.org/content/m18234/1.3/>>.

Definitions: When the exponent is not a positive integer

Zero exponents	Negative exponents	Fractional exponents (numerator = 1)	Fractional exponents (numerator $\neq$ 1)
Always 1	Go in the denominator	Act as roots	The numerator is an exponent The denominator is a root
$7^0 = 1$ $9^0 = 1$ $x^0 = 1$	$7^{-3} = \frac{1}{7^3} = \frac{1}{343}$ $x^{-5} = \frac{1}{x^5}$ $\frac{1}{5^{-3}} = 5^3$	$9^{\frac{1}{2}} = \sqrt{9} = 3$ $2^{\frac{1}{2}} = \sqrt{2}$ $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ $x^{\frac{1}{4}} = \sqrt[4]{x}$	$8^{\frac{2}{3}} = \sqrt[3]{8^2}$ or $(\sqrt[3]{8})^2$ Order doesn't matter! $\sqrt[3]{8^2} = \sqrt[3]{64} = 4$ or $(\sqrt[3]{8})^2 = 2^2 = 4$ $8 = \sqrt{8^3}$ or $(\sqrt{8})^3$

Table 5.5

Note that you can **combine** these definitions. For instance, $8^{-\frac{2}{3}}$ is a negative, fractional exponent. The negative exponent means, as always, “put me in the denominator.” So we can write:

$$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{4} \quad (5.9)$$

5.3.1 OK, so why define exponents that way?

These are obviously not chosen to be the **simplest** possible definitions. But they are chosen to be **consistent** with the behavior of positive-integer exponents.

One way to see that consistency is to consider the following progression:

$$19^4 = 19 \cdot 19 \cdot 19 \cdot 19 \quad (5.10)$$

$$19^3 = 19 \cdot 19 \cdot 19 \quad (5.11)$$

$$19^2 = 19 \cdot 19 \quad (5.12)$$

$$19^1 = 19 \quad (5.13)$$

What happens each time we decrease the exponent by 1? Your first response might be “we have one less 19.” But what is really happening, mathematically, to the numbers on the right? The answer is that, with each step, they are **dividing by 19**. If you take $19 \cdot 19 \cdot 19 \cdot 19$, and **divide it by 19**, you get $19 \cdot 19 \cdot 19$. Divide that by 19 again, and you get $19 \cdot 19$...and so on. From this we can formulate the following principle for the powers of 19:

Whenever you **subtract 1** from the exponent, you **divide the answer by 19**.

As I said earlier, we want the behavior of our new exponents to be consistent with the behavior of the **old** (positive-integer) exponents. So we can continue this progression as follows:

$$19^0 = \frac{19}{19} = 1 \quad (5.14)$$

$$19^{-1} = \frac{1}{19} \quad (5.15)$$

$$19^{-2} = \frac{1}{19} = \frac{1}{19^2} \quad (5.16)$$

...and so on. We can arrive at our definitions anything⁰ = 1 and **negative exponents go in the denominator** by simply requiring this progression to be consistent.

More rigorously, we can find all our exponent definitions by using the **laws of exponents**. For instance, what is 4^0 ? We can approach this question indirectly by asking: what is $\frac{4^2}{4^2}$?

- The second law of exponents tells us that $\frac{4^2}{4^2} = 4^{2-2}$, which is of course 4^0 .
- But of course, $\frac{4^2}{4^2}$ is just $\frac{16}{16}$, or 1.
- Since $\frac{4^2}{4^2}$ is both 4^0 and 1, 4^0 and 1 must be the same thing!

The proofs given below all follow this pattern. They use the laws of exponents to rewrite expressions such as $\frac{4^2}{4^2}$, and go on to show how zero, negative, and fractional exponents must be defined. We started with the definition of an exponent for a positive integer, $10^6 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$. From there, we developed the laws of exponents. Now we find that, if we want those same laws to apply to other kinds of exponents, there is only one correct way to define those other kinds of exponents.

Proofs: When the exponent is not a positive integer

Zero exponents	Negative exponents	Fractional exponents (numerator = 1)	Fractional exponents (numerator $\neq$ 1)
Always 1	Go in the denominator	Act as roots	The numerator is an exponent The denominator is a root
$\frac{4^2}{4^2} = 4^{2-2} = 4^0$ but $\frac{4^2}{4^2} = \frac{16}{16} = 1$ so 4^0 must be 1!	$\frac{10^1}{10^3} = 10^{1-3} = 10^{-2}$ but $\frac{10^1}{10^3} = \frac{10}{10 \cdot 10 \cdot 10} = \frac{1}{10 \cdot 10}$ so 10^{-2} must be $\frac{1}{10^2}$	$(9^{\frac{1}{2}})^2 = 9^{\frac{1}{2} \cdot 2} = 9^1 = 9$ So what is $9^{\frac{1}{2}}$? Well, when you square it, you get 9. So it must be $\sqrt{9}$, or 3!	$8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2$ or $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = \sqrt[3]{8^2}$

Table 5.6

You may want to experiment with making these proofs more general and more rigorous by using letters instead of numbers. For instance, in the third case, we could write:

$$\left(x^{\frac{1}{a}}\right)^a = x^{\left(\frac{1}{a}\right)(a)} = x^1 \quad (5.17)$$

$$\left(x^{\frac{1}{a}}\right)^a = x \quad (5.18)$$

$$\sqrt[a]{(x)^a} = \sqrt[a]{x} \quad (5.19)$$

$$x^{\frac{1}{a}} = \sqrt[a]{x} \quad (5.20)$$

5.4 Exponential Curves⁴

By plotting points, you can discover that the graph of $y = 2^x$ looks like this:

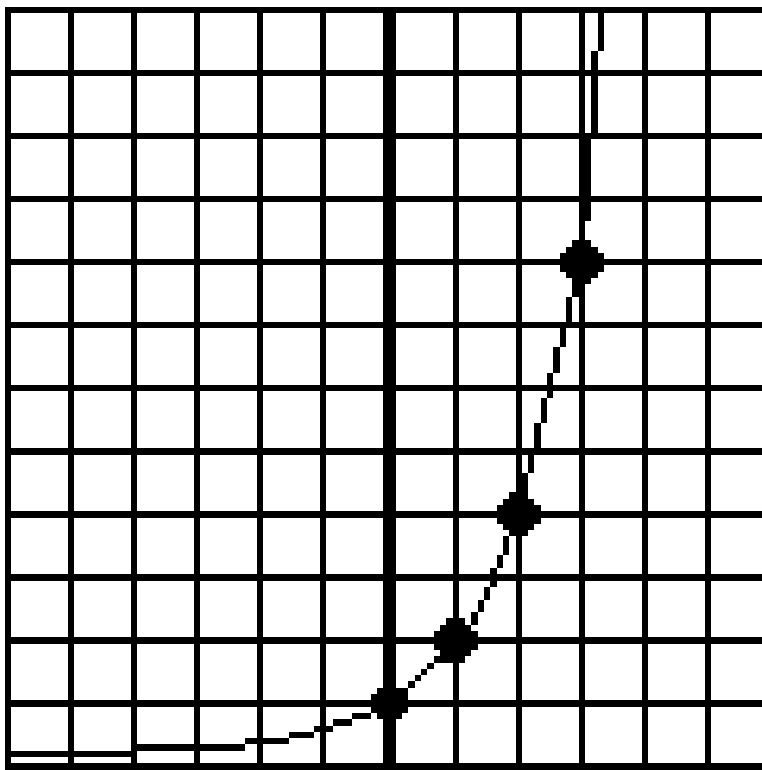


Figure 5.1: $y = 2^x$

A few points to notice about this graph.

- It goes through the point $(0, 1)$ because $2^0 = 1$.
- It never dips below the x -axis. The **domain** is unlimited, but the **range** is $y > 0$. (*Think about our definitions of exponents: whether x is positive or negative, integer or fraction, 2^x is **always** positive.)
- Every time you move one unit to the right, the graph height doubles. For instance, 2^5 is twice 2^4 , because it multiplies by **one more** 2. So as you move to the right, the y -values start looking like 8, 16, 32, 64, 128, and so on, going up more and more sharply.
- Conversely, every time you move one unit to the left, the graph height drops in half. So as you move to the left, the y -values start looking like $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and so on, falling closer and closer to 0.

What would the graph of $y = 3^x$ look like? Of course, it would also go through $(0, 1)$ because $3^0 = 1$. With each step to the right, it would **triple**; with each step to the left, it would drop in a **third**. So the overall shape would look similar, but the rise (on the right) and the drop (on the left) would be faster.

⁴This content is available online at <<http://cnx.org/content/m18233/1.2/>>.

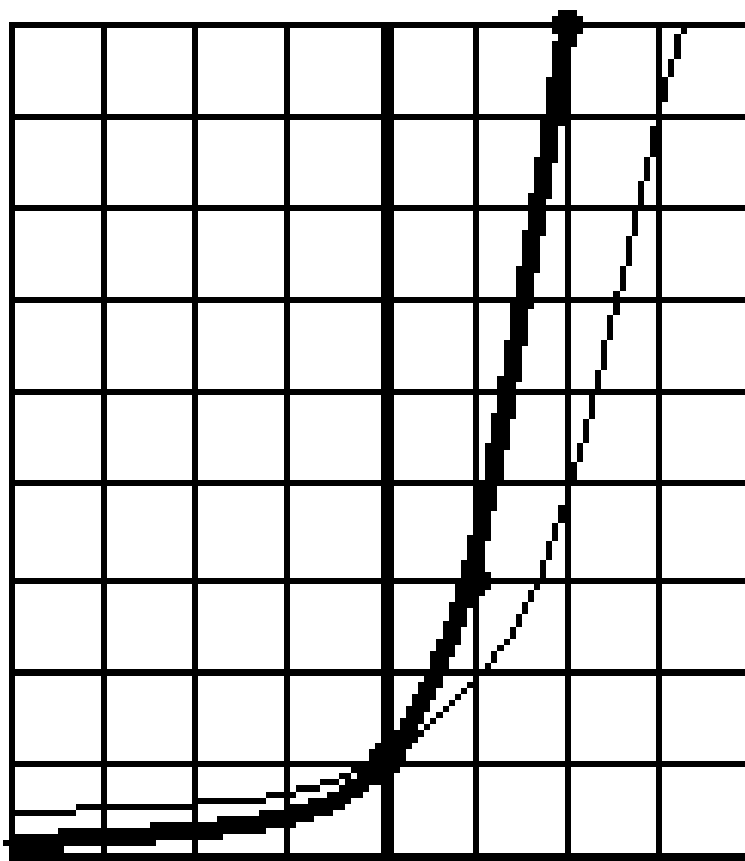


Figure 5.2: $y = 2^x$ in thin line; $y = 5^x$ in thick line; They cross at $(0, 1)$

As you might guess, graphs such as 5^x and 10^x all have this same characteristic shape. In fact, any graph a^x where $a > 1$ will look basically the same: starting at $(0, 1)$ it will rise more and more sharply on the right, and drop toward zero on the left. This type of graph models **exponential growth**—functions that keep multiplying by the same number. A common example, which you work through in the text, is compound interest from a bank.

The opposite graph is $\left(\frac{1}{2}\right)^x$.

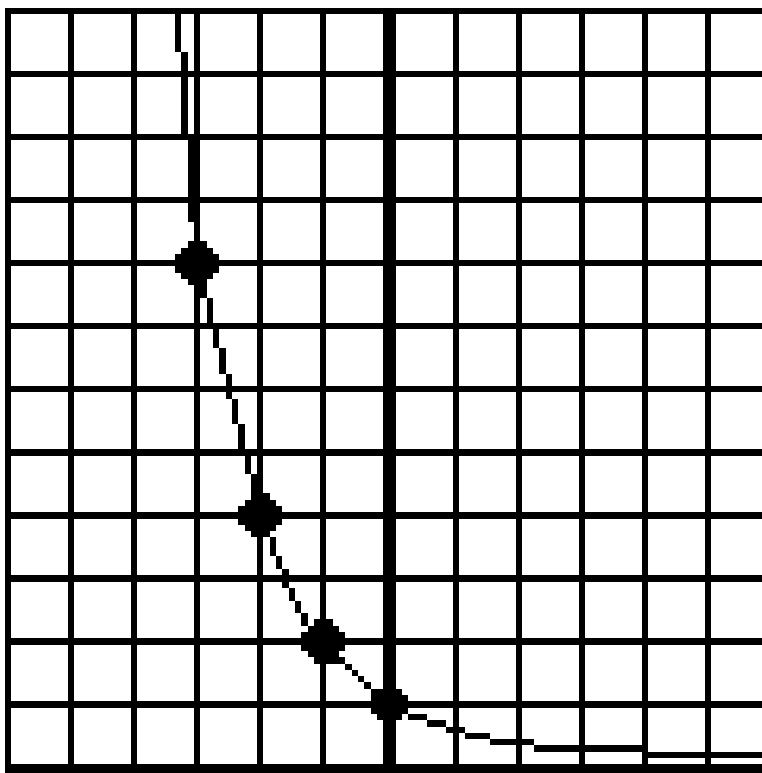


Figure 5.3: $y = \left(\frac{1}{2}\right)^x$

Each time you move to the right on this graph, it multiplies by $\frac{1}{2}$: in other words, it **divides** by 2, heading closer to zero the further you go. This kind of equation is used to model functions that keep **dividing** by the same number; for instance, radioactive decay. You will also be working through examples like this one.

Of course, all the **permutations** from the first chapter on “functions” apply to these graphs just as they apply to any graph. A particularly interesting example is 2^{-x} . Remember that when you replace x with $-x$, $f(3)$ becomes the old $f(-3)$ and vice-versa; in other words, the graph flips around the y -axis. If you take the graph of 2^x and permute it in this way, you get a familiar shape:

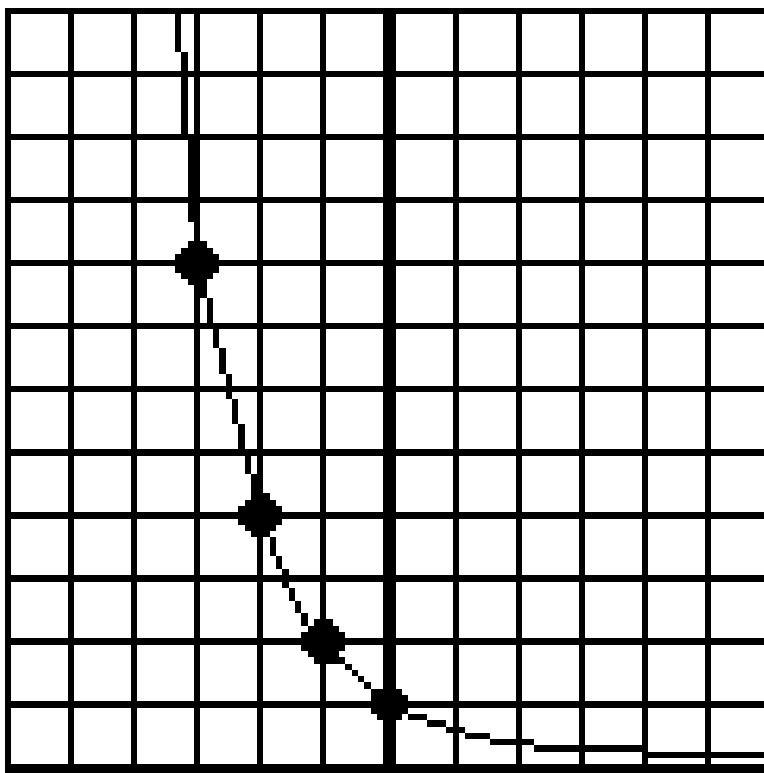


Figure 5.4: $y = 2^{-x}$

Yes, it's $\left(\frac{1}{2}\right)^x$ in a new disguise!

Why did it happen that way? Consider that $\left(\frac{1}{2}\right)^x = \frac{1^x}{2^x}$. But 1^x is just 1 (in other words, 1 to the **anything** is 1), so $\left(\frac{1}{2}\right)^x = \frac{1}{2^x}$. But negative exponents go in the denominator: $\frac{1}{2^x}$ is the same thing as 2^{-x} ! So we arrive at: $\left(\frac{1}{2}\right)^x = 2^{-x}$. The two functions are the same, so their graphs are of course the same.

Another fun pair of permutations is:

$y = 2 \cdot 2^x$ Looks just like $y = 2^x$ but vertically stretched: all y-values double

$y = 2^{x+1}$ Looks just like $y = 2^x$ but horizontally shifted: moves 1 to the left

If you permute 2^x in these two ways, you will find that they create the same graph.

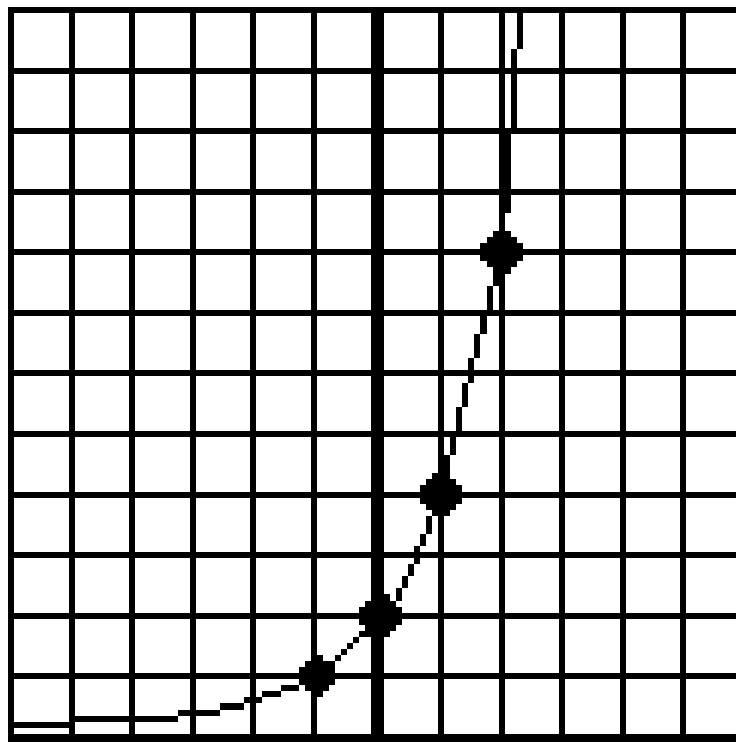


Figure 5.5: $y = 2 \cdot 2^x$ aka $y = 2^{x+1}$

Once again, this is predictable from the rules of exponents: $2 \cdot 2^x = 2^1 \cdot 2^x = 2^{x+1}$

5.4.1 Using exponential functions to model behavior

In the first chapter, we talked about **linear functions** as functions that **add the same amount every time**. For instance, $y = 3x + 4$ models a function that starts at 4; every time you increase x by 1, you add 3 to y .

Exponential functions are conceptually very analogous: they **multiply by the same amount every time**. For instance, $y = 4 \times 3^x$ models a function that starts at 4; every time you increase x by 1, you **multiply** y by 3.

Linear functions can go down, as well as up, by having **negative slopes**: $y = -3x + 4$ starts at 4 and **subtracts** 3 every time. Exponential functions can go down, as well as up, by having **fractional bases**: $y = 4 \times \left(\frac{1}{3}\right)^x$ starts at 4 and **divides** by 3 every time.

Exponential functions often defy intuition, because they grow much faster than people expect.

Modeling exponential functions

Your father's house was worth \$100,000 when he bought it in 1981. Assuming that it increases in value by 8% every year, what was the house worth in the year 2001? (*Before you work through the math, you may want to make an intuitive guess as to what you think the house is worth. Then, after we crunch the numbers, you can check to see how close you got.)

Often, the best way to approach this kind of problem is to begin by making a chart, to get a sense of the growth pattern.

Year	Increase in Value	Value
1981	N/A	100,000
1982	8% of 100,000 = 8,000	108,000
1983	8% of 108,000 = 8,640	116,640
1984	8% of 116,640 = 9,331	125,971

Table 5.7

Before you go farther, make sure you understand where the numbers on that chart come from. It's OK to use a calculator. But if you blindly follow the numbers without understanding the calculations, the whole rest of this section will be lost on you.

In order to find the pattern, look at the "Value" column and ask: what is happening to these numbers every time? Of course, we are adding 8% each time, but what does that really mean? With a little thought—or by looking at the numbers—you should be able to convince yourself that the numbers are **multiplying by 1.08 each time**. That's why this is an exponential function: the value of the house multiplies by 1.08 every year.

So let's make that chart again, in light of this new insight. Note that I can now skip the middle column and go straight to the answer we want. More importantly, note that I am not going to use my calculator this time—I don't want to multiply all those 1.08s, I just want to note each time that the answer is 1.08 **times the previous answer**.

Year	House Value
1981	100,000
1982	$100,000 \times 1.08$
1983	$100,000 \times 1.08^2$
1984	$100,000 \times 1.08^3$
1985	$100,000 \times 1.08^4$
y	$100,000 \times 1.08^{\text{something}}$

Table 5.8

If you are not clear where those numbers came from, think again about the conclusion we reached earlier: each year, the value **multiplies by 1.08**. So if the house is worth $100,000 \times 1.08^2$ in 1983, then its value in 1984 is $(100,000 \times 1.08^2) \times 1.08$, which is $100,000 \times 1.08^3$.

Once we write it this way, the pattern is clear. I have expressed that pattern by adding the last row, the value of the house in any year y . And what is the mystery exponent? We see that the exponent is 1 in 1982, 2 in 1983, 3 in 1984, and so on. In the year y , the exponent is $y - 1981$.

So we have our house value function:

$$v(y) = 100,000 \times 1.08^{y-1981} \quad (5.21)$$

That is the pattern we needed in order to answer the question. So in the year 2001, the value of the house is $100,000 \times 1.08^{20}$. Bringing the calculator back, we find that the value of the house is now \$466,095 and change.

Wow! The house is over four times its original value! That's what I mean about exponential functions growing faster than you expect: they start out slow, but given time, they explode. This is also a practical life lesson about the importance of saving money early in life—a lesson that many people don't realize until it's too late.

Chapter 6

Logarithms

6.1 Logarithm Concepts¹

Suppose you are a biologist investigating a population that doubles every year. So if you start with 1 specimen, the population can be expressed as an exponential function: $p(t) = 2^t$ where t is the number of years you have been watching, and p is the population.

Question: How long will it take for the population to exceed 1,000 specimens?

We can rephrase this question as: “2 to **what power** is 1,000?” This kind of question, where you know the base and are looking for the exponent, is called a **logarithm**.

$\log_2 1000$ (read, “the logarithm, base two, of a thousand”) means “2, raised to what power, is 1000?”

In other words, the logarithm always asks “**What exponent should we use?**” This unit will be an exploration of logarithms.

6.1.1 A few quick examples to start things off

Problem	Means	The answer is	because
$\log_2 8$	2 to what power is 8?	3	2^3 is 8
$\log_2 16$	2 to what power is 16?	4	2^4 is 16
$\log_2 10$	2 to what power is 10?	somewhere between 3 and 4	$2^3 = 8$ and $2^4 = 16$
$\log_8 2$	8 to what power is 2?	$\frac{1}{3}$	$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$
$\log_{10} 10,000$	10 to what power is 10,000?	4	$10^4 = 10,000$
$\log_{10} \left(\frac{1}{100}\right)$	10 to what power is $\frac{1}{100}$?	-2	$10^{-2} = \frac{1}{10^2} = \frac{1}{100}$
$\log_5 0$	5 to what power is 0?	There is no answer	$5^{\text{something}}$ will never be 0

Table 6.1

As you can see, one of the most important parts of finding logarithms is being very familiar with how exponents work!

¹This content is available online at <<http://cnx.org/content/m18242/1.2/>>.

6.2 The Logarithm Explained by Analogy to Roots²

The logarithm may be the first really new concept you've encountered in Algebra II. So one of the easiest ways to understand it is by comparison with a familiar concept: roots.

Suppose someone asked you: "Exactly what does root mean?" You do understand roots, but they are difficult to define. After a few moments, you might come up with a definition very similar to the "question" definition of logarithms given above. $\sqrt[3]{8}$ means "what number cubed is 8?"

Now the person asks: "How do you find roots?" Well...you just play around with numbers until you find one that works. If someone asks for $\sqrt{25}$, you just have to know that $5^2 = 25$. If someone asks for $\sqrt{30}$, you know that has to be bigger than 5 and smaller than 6; if you need more accuracy, it's time for a calculator.

All that information about roots applies in a very analogous way to logarithms.

	Roots	Logs
The question	$\sqrt[n]{x}$ means "what number, raised to the a power, is x?" As an equation, $?^a = x$	$\log_a x$ means "a, raised to what power, is x?" As an equation, $a^? = x$
Example that comes out even	$\sqrt[3]{8} = 2$	$\log_2 8 = 3$
Example that doesn't	$\sqrt[3]{10}$ is a bit more than 2	$\log_2 10$ is a bit more than 3
Out of domain example	$\sqrt{-4}$ does not exist (x^2 will never give -4)	$\log_2(0)$ and $\log_2(-1)$ do not exist (2^x will never give 0 or a negative answer)

Table 6.2

6.3 Rewriting Logarithm Equations as Exponent Equations³

Both root equations and logarithm equations can be rewritten as exponent equations.

$\sqrt{9} = 3$ can be rewritten as $3^2 = 9$. These two equations are the same statement about numbers, written in two different ways. $\sqrt{9}$ asks the question "What number squared is 9?" So the equation $\sqrt{9} = 3$ asks this question, and then answers it: "3 squared is 9."

We can rewrite logarithm equations in a similar way. Consider this equation:

$$\log_3 \left(\frac{1}{3} \right) = -1 \quad (6.1)$$

If you are asked to rewrite that logarithm equation as an exponent equation, think about it this way. The left side asks: "3 to what power is $\left(\frac{1}{3}\right)$?" And the right side answers: "3 to the -1 power is $\left(\frac{1}{3}\right)$." $3^{-1} = \left(\frac{1}{3}\right)$.

²This content is available online at <<http://cnx.org/content/m18236/1.2/>>.

³This content is available online at <<http://cnx.org/content/m18241/1.2/>>.

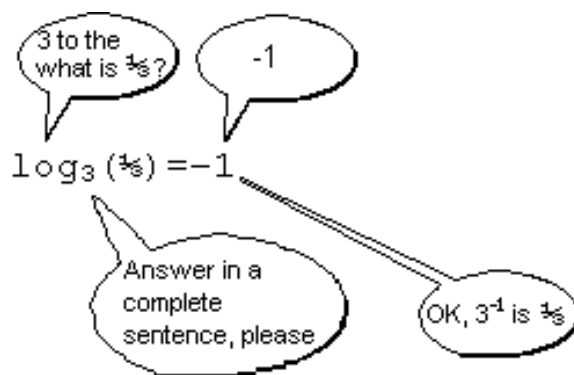


Figure 6.1

These two equations, $\log_3\left(\frac{1}{3}\right) = -1$ and $3^{-1} = \left(\frac{1}{3}\right)$, are two different ways of expressing the same numerical relationship.

6.4 The Logarithm Defined as an Inverse Function⁴

$\sqrt{x}$ can be defined as the **inverse function** of x^2 . Recall the definition of an inverse function— $f^{-1}(x)$ is defined as the inverse of $f^1(x)$ if it reverses the inputs and outputs. So we can demonstrate this inverse relationship as follows:

$\sqrt{x}$ is the inverse function of x^2
$3 \rightarrow x^2 \rightarrow 9$
$9 \rightarrow \sqrt{x} \rightarrow 3$

Table 6.3

Similarly, $\log_2 x$ is the **inverse function** of the exponential function 2^x .

$\log_2 x$ is the inverse function of 2^x
$3 \rightarrow 2^x \rightarrow 8$
$8 \rightarrow \log_2 x \rightarrow 3$

Table 6.4

(You may recall that during the discussion of inverse functions, 2^x was the only function you were given that you **could not find** the inverse of. Now you know!)

In fact, as we noted in the first chapter, $\sqrt{x}$ is **not** a perfect inverse of x^2 , since it does not work for negative numbers. $(-3)^2 = 9$, but $\sqrt{9}$ is not -3 . Logarithms have no such limitation: $\log_2 x$ is a perfect inverse for 2^x .

⁴This content is available online at <<http://cnx.org/content/m18240/1.2/>>.

The inverse of addition is subtraction. The inverse of multiplication is division. Why do exponents have two completely different kinds of inverses, roots and logarithms? **Because exponents do not commute.** 3^2 and 2^3 are not the same number. So the question “what number squared equals 10?” and the question “2 to what power equals 10?” are different questions, which we express as $\sqrt{10}$ and $\log_2 10$, respectively, and they have different answers. x^2 and 2^x are not the same function, and they therefore have different inverse functions $\sqrt{x}$ and $\log_2 10$.

6.5 Properties of Logarithms⁵

Just as there are three fundamental laws of exponents, there are three fundamental laws of logarithms.

$$\log_x (ab) = \log_x a + \log_x b \quad (6.2)$$

$$\log_x \frac{a}{b} = \log_x a - \log_x b \quad (6.3)$$

$$\log_x (a^b) = b \log_x a \quad (6.4)$$

As always, these **algebraic generalizations** hold for any a , b , and x .

Example 6.1: Properties of Logarithms

1. Suppose you are given these two facts:

- $\log_4 5 = 1.16$
- $\log_4 10 = 1.66$

2. Then we can use the laws of logarithms to conclude that:

- $\log_4 (50) = \log_4 5 + \log_4 10 = 2.82$
- $\log_4 (2) = \log_4 10 - \log_4 5 = 0.5$
- $\log_4 (100,000) = 5 \log_4 10 = 8.3$

NOTE: All three of these results can be found quickly, and without a calculator. Note that the second result could also be figured out directly, since $4^{\frac{1}{2}} = 2$.

These properties of logarithms were very important historically, because they enabled pre-calculator mathematicians to perform **multiplication** (which is very time-consuming and error prone) by doing **addition** (which is faster and easier). These rules are still useful in simplifying complicated expressions and solving equations.

Example 6.2: Solving an equation with the properties of logarithms

$\log_2 x - \log_x (x - 1) = 5$	The problem	
$\log_2 \left(\frac{x}{x-1} \right) = 5$	Second property of logarithms	
$\frac{x}{x-1} = 2^5 = 32$	Rewrite the log as an exponent. (2-to-what is? $\frac{x}{x-1}$ 2-to-the-5!)	
$x = 32(x - 1)$	Multiply. We now have an easy equation to solve.	
$x = 32x - 32$		
$-31x = -32$		
$x = \frac{32}{31}$		

⁵This content is available online at <<http://cnx.org/content/m18239/1.2/>>.

Table 6.5

6.5.1 Proving the Properties of Logarithms

If you understand what an exponent is, you can very quickly see why the three rules of exponents work. But why do logarithms have these three properties?

As you work through the text, you will demonstrate these rules intuitively, by viewing the logarithm as a **counter**. ($\log_2 8$ asks “**how many** 2s do I need to multiply, in order to get 8?”) However, these rules can also be rigorously proven, using the laws of exponents as our starting place.

Proving the First Law of Logarithms, $\log_x(ab) = \log_x a + \log_x b$	
$m = \log_x a$	I’m just inventing m to represent this log
$x^m = a$	Rewriting the above expression as an exponent. ($\log_x a$ asks “ x to what power is a ?” And the equation answers: “ x to the m is a .”)
$n = \log_x b$	Similarly, n will represent the other log.
$x^n = b$	
$\log_x(ab) = \log_x(x^m x^n)$	Replacing a and b based on the previous equations
$= \log_x(x^{m+n})$	This is the key step! It uses the first law of exponents. Thus you can see that the properties of logarithms come directly from the laws of exponents.
$= m + n$	$= \log_x(x^{m+n})$ asks the question: “ x to what power is x^{m+n} ?” Looked at this way, the answer is obviously $(m+n)$. Hence, you can see how the logarithm and exponential functions cancel each other out, as inverse functions must.
$= \log_x a + \log_x b$	Replacing m and n with what they were originally defined as. Hence, we have proven what we set out to prove.

Table 6.6

To test your understanding, try proving the second law of logarithms: the proof is very similar to the first. For the third law, you need invent only one variable, $m = \log_x a$. In each case, you will rely on a different one of the three rules of exponents, showing how each exponent law corresponds to one of the logarithms laws.

6.6 Common Logarithms⁶

When you see a root without a number in it, it is assumed to be a **square** root. That is, $\sqrt{25}$ is a shorthand way of writing $\sqrt[2]{25}$. This rule is employed because **square** roots are more common than other types.

⁶This content is available online at <<http://cnx.org/content/m18237/1.2/>>.

When you see a logarithm without a number in it, it is assumed to be a **base 10** logarithm. That is, $\log(1000)$ is a shorthand way of writing $\log_{10}(1000)$. A base 10 logarithm is also known as a “common” log.

Why are common logs particularly useful? Well, what is $\log_{10}(1000)$? By now you know that this asks the question “10 to what power is 1000?” The answer is 3. Similarly, you can confirm that:

$$\log(10) = 1 \quad (6.5)$$

$$\log(100) = 2 \quad (6.6)$$

$$\log(1,000,000) = 6 \quad (6.7)$$

We can also follow this pattern backward:

$$\log(1) = 0 \quad (6.8)$$

$$\log\left(\frac{1}{10}\right) = -1 \quad (6.9)$$

$$\log\left(\frac{1}{100}\right) = -2 \quad (6.10)$$

and so on. In other words, the common log tells you the **order of magnitude** of a number: how many zeros it has. Of course, $\log_{10}(500)$ is difficult to determine exactly without a calculator, but we can say immediately that it must be somewhere between 2 and 3, since 500 is between 100 and 1000.

6.7 Graphing Logarithmic Functions⁷

Suppose you want to graph the function $y = \log_2(x)$. You might start by making a table that looks something like this:

x	$y = \log_2(x)$
1	0
2	1
3	um....I’m not sure
4	2
5	can I use a calculator?

Table 6.7

This doesn’t seem to be the right strategy. Many of those numbers are just too hard to work with.

So, you start looking for numbers that **are** easy to work with. And you remember that it’s important to look at numbers that are less than 1, as well as greater. And eventually, you end up with something more like this.

⁷This content is available online at <<http://cnx.org/content/m18238/1.2/>>.

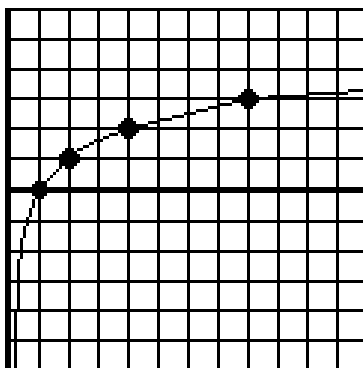
x	$y = \log_2(x)$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Table 6.8

As long as you keep putting powers of 2 in the x column, the y column is very easy to figure.

In fact, the easiest way to generate this table is to recognize that it is the table of $y = 2^x$ values, only with the x and y coordinates switched! In other words, we have re-discovered what we already knew: that $y = 2^x$ and $y = \log_2(x)$ are inverse functions.

When you graph it, you end up with something like this:

Figure 6.2: $y = \log_2(x)$

As always, you can learn a great deal about the log function by reading the graph.

- The domain is $x > 0$. (You can't take the log of 0 or a negative number—do you remember why?).
- The range, on the other hand, is all numbers. Of course, all this inverts the function 2^x , which has a **domain** of all numbers and a **range** of $y > 0$.
- As x gets closer and closer to 0, the function dives down to smaller and smaller negative numbers. So the y -axis serves as an “asymptote” for the graph, meaning a line that the graph **approaches closer and closer to** without ever touching.
- As x moves to the right, the graph grows—but more and more slowly. As x goes from 4 to 8, the graph goes up by 1. As x goes from 8 to 16, the graph goes up by another 1. It doesn't make it up another 1 until x reaches 32...and so on.

This pattern of **slower and slower growth** is one of the most important characteristics of the log. It can be used to “slow down” functions that have too wide a range to be practical to work with.

Example 6.3: Using the log to model a real world problem

Lewis Fry Richardson (1881–1953) was a British meteorologist and mathematician. He was also an active Quaker and committed pacifist, and was one of the first men to apply statistics to the study of human conflict. Richardson catalogued 315 wars between 1820 and 1950, and categorized them by **how many deaths** they caused. At one end of the scale is a deadly quarrel, which might result in 1 or 2 deaths. At the other extreme are World War I and World War II, which are responsible for roughly 10 million deaths each.

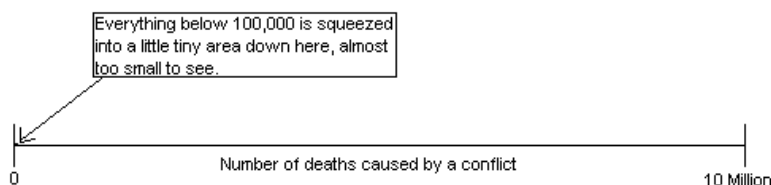


Figure 6.3

As you can see from the chart above, working with these numbers is extremely difficult: on a scale from 0 to 10 Million, there is no visible difference between (say) 1 and 100,000. Richardson solved this problem by taking the **common log of the number of deaths**. So a conflict with 1,000 deaths is given a magnitude of $\log(1000) = 3$. On this scale, which is now the standard for conflict measurement, the magnitudes of all wars can be easily represented.

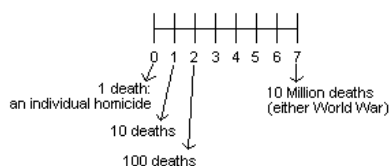


Figure 6.4

Richardson’s scale makes it practical to chart, discuss, and compare wars and battles from the smallest to the biggest. For instance, he discovered that each time you move up by one on the scale—that is, each time the number of deaths multiplies by 10—the number of conflicts drops in a third. (So there are roughly three times as many “magnitude 5” wars as “magnitude 6,” and so on.)

The log is useful here because the logarithm function itself **grows so slowly** that it compresses the entire 1-to-10,000,000 range into a 0-to-7 scale. As you will see in the text, the same trick is used—for the same reason—in fields ranging from earthquakes to sound waves.

Chapter 7

Rational Expressions

7.1 Rational Expression Concepts¹

The term “rational” in math is not used in the sense of “sane” or “sensible.” It is instead used to imply a **ratio**, or fraction. A **rational expression** is the ratio of two polynomials: for instance, $\frac{x^2+1}{x^2-1}$ is a rational expression.

There are two rules for working with rational expressions.

1. Begin every problem by factoring everything you can.
2. Remember that, despite all the complicated looking functions, a rational expression is just a fraction: you manipulate them using all the rules of fractions that you are familiar with.

7.2 Simplifying Rational Expressions²

How do you **simplify** a fraction? The answer is, you divide the top and bottom by the same thing.

$$\frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3} \quad (7.1)$$

So $\frac{4}{6}$ and $\frac{2}{3}$ are two different ways of writing the same number.

¹This content is available online at <<http://cnx.org/content/m18304/1.1/>>.

²This content is available online at <<http://cnx.org/content/m18296/1.1/>>.

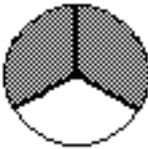
	On the left, a pizza divided into six equal slices: the four shaded-in regions represent $\frac{4}{6}$ of a pizza. On the right, a pizza divided into three equal slices: the two shaded-in regions represent $\frac{2}{3}$ of a pizza. The two areas are identical: $\frac{4}{6}$ and $\frac{2}{3}$ are two different ways of expressing the same amount of pizza.	
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Table 7.1

In some cases, you have to repeat this process more than once before the fraction is fully simplified.

$$\frac{40}{48} = \frac{40 \div 4}{48 \div 4} = \frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6} \quad (7.2)$$

It is vital to remember that **we have not divided this fraction by 4, or by 2, or by 8**. We have rewritten the fraction in another form: $\frac{40}{48}$ is the **same number** as $\frac{5}{6}$. In strictly practical terms, if you are given the choice between $\frac{40}{48}$ of a pizza or $\frac{5}{6}$ of a pizza, it does not matter which one you choose, because they are the same amount of pizza.

You can divide the top and bottom of a fraction by the same number, but you cannot subtract the same number from the top and bottom of a fraction!

$$\frac{40}{48} = \frac{40-39}{48-39} = \frac{1}{9} \times \text{Wrong!}$$

Given the choice, a hungry person would be wise to choose $\frac{40}{48}$ of a pizza instead of $\frac{1}{9}$.

Dividing the top and bottom of a fraction by the same number leaves the fraction unchanged, and that is how you simplify fractions. **Subtracting** the same number from the top and bottom changes the value of the fraction, and is therefore an illegal simplification.

All this is review. But if you understand these basic fraction concepts, you are ahead of many Algebra II students! And if you can **apply these same concepts when variables are involved**, then you are ready to simplify rational expressions, because there are no new concepts involved.

As an example, consider the following:

$$\frac{x^2 - 9}{x^2 + 6x + 9} \quad (7.3)$$

You might at first be tempted to cancel the common x^2 terms on the top and bottom. But this would be, mathematically, **subtracting** x^2 from both the top and the bottom; which, as we have seen, is an illegal fraction operation.

$\frac{x^2 - 9}{x^2 + 6x + 9}$	$= \frac{\cancel{x^2} - 9}{\cancel{x^2} + 6x + 9}$	$= \frac{-9}{6x + 9}$	$\times \text{Wrong!}$
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Table 7.2

To properly simplify this expression, begin by factoring both the top and the bottom, and **then** see if anything cancels.

Example 7.1: Simplifying Rational Expressions

$\frac{x^2-9}{x^2+6x+9}$	The problem
$= \frac{(x+3)(x-3)}{(x+3)^2}$	Always begin rational expression problems by factoring! This factors easily, thanks to $(x+a)(x-a) = x^2 - a^2$ and $(x+a)^2 = x^2 + 2ax + a^2$
$= \frac{x-3}{x+3}$	Cancel a common $(x+3)$ term on both the top and the bottom. This is legal because this term was multiplied on both top and bottom; so we are effectively dividing the top and bottom by $(x+3)$, which leaves the fraction unchanged.

Table 7.3

What we have created, of course, is an algebraic generalization:

$$\frac{x^2-9}{x^2+6x+9} = \frac{x-3}{x+3} \quad (7.4)$$

For any x value, the complicated expression on the left will give the same answer as the much simpler expression on the right. You may want to try one or two values, just to confirm that it works.

As you can see, the skills of **factoring** and **simplifying fractions** come together in this exercise. No new skills are required.

7.3 Multiplying Rational Expressions³

Multiplying fractions is easy: you just multiply the tops, and multiply the bottoms. For instance,

$$\frac{6}{7} \times \frac{7}{11} = \frac{6 \times 7}{7 \times 11} = \frac{42}{77} \quad (7.5)$$

Now, you may notice that $\frac{42}{77}$ can be simplified, since 7 goes into the top and bottom. $\frac{42}{77} = \frac{42 \div 7}{77 \div 7} = \frac{6}{11}$. So $\frac{42}{77}$ is the correct answer, but $\frac{6}{11}$ is **also** the correct answer (since they are the same number), and it's a good bit simpler.

In fact, we could have jumped straight to the simplest answer first, and avoided dealing with all those big numbers, if we had noticed that we have a 7 in the numerator and a 7 in the denominator, and **cancelled** them before we even multiplied!

³This content is available online at <<http://cnx.org/content/m18301/1.1/>>.

$$\frac{\cancel{6}}{\cancel{7}} \times \frac{\cancel{7}}{11} = \frac{6}{11}$$

Figure 7.1

This is a great time-saver, and you're also a lot less likely to make mistakes.

When multiplying fractions...

If the same number appears on the top and the bottom, you can cancel it before you multiply. This works regardless of whether the numbers appear in the **same fraction or different fractions**.

But it's critical to remember that this rule **only applies when you are multiplying fractions**: not when you are adding, subtracting, or dividing.

As you might guess, all this review of basic fractions is useful because, once again, rational expressions work the same way.

Example 7.2: Multiplying Rational Expressions

$\frac{3x^2-21x-24}{x^2-16} \cdot \frac{x^2-6x+8}{3x+3}$	The problem
$= \frac{3(x-8)(x+1)}{(x+4)(x-4)} \cdot \frac{(x-2)(x-4)}{3(x+1)}$	Always begin rational expression problems by factoring! Note that for the first element you begin by factoring out the common 3, and then factoring the remaining expression.
$= \frac{\cancel{3}(x-8)\cancel{(x+1)}}{(x+4)\cancel{(x-4)}} \cdot \frac{(x-2)\cancel{(x-4)}}{\cancel{3}(x+1)}$	When multiplying fractions, you can cancel anything on top with anything on the bottom, even across different fractions
$= \frac{(x-8)(x-2)}{x+4}$	Now, just see what you're left with. Note that you could rewrite the top as $x^2 - 10x + 16$ but it's generally easier to work with in factored form.

Table 7.4

7.3.1 Dividing Rational Expressions

To divide fractions, you flip the bottom one, and then multiply.

$$\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \cdot 3 = \frac{3}{2} \quad (7.6)$$

After the "flipping" stage, all the considerations are exactly the same as multiplying.

Example 7.3: Dividing Rational Expressions

$\frac{\frac{x^2-3x}{2x^2-13x+6}}{\frac{x^3+4x}{x^2-12x+36}}$	This problem could also be written as: $\frac{x^2-3x}{2x^2-13x+6} \div \frac{x^3+4x}{x^2-12x+36}$. However, the symbol is rarely seen at this level of math. $12 \div 4$ is written as $\frac{12}{4}$.
$\frac{x^2-3x}{2x^2-13x+6} \times \frac{x^2-12x+36}{x^3+4x}$	Flip the bottom and multiply. From here, it's a straight multiplication problem.
$= \frac{x(x-3)}{(2x-1)(x-6)} \times \frac{(x-6)^2}{x(x^2+4)}$	Always begin rational expression problems by factoring! Now, cancel a factor of x and an $(x-6)$ and you get...
$= \frac{(x-3)(x-6)}{(2x-1)(x^2+4)}$	That's as simple as it gets, I'm afraid. But it's better than what we started with!

Table 7.5

7.4 Adding and Subtracting Rational Expressions⁴

Adding and subtracting fractions is harder—but once again, it is a familiar process.

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6} \quad (7.7)$$

The key is finding the **least common denominator**: the smallest multiple of both denominators. Then you rewrite the two fractions with this denominator. Finally, you add the fractions by **adding the numerators and leaving the denominator alone**.

But how do you find the least common denominator? Consider this problem:

$$\frac{5}{12} + \frac{7}{30} \quad (7.8)$$

You could probably find the least common denominator if you played around with the numbers long enough. But what I want to show you is a **systematic method** for finding least common denominators—a method that works with rational expressions just as well as it does with numbers. We start, as usual, by factoring. For each of the denominators, we find all the **prime factors**, the prime numbers that multiply to give that number.

$$\frac{5}{2 \cdot 2 \cdot 3} + \frac{7}{2 \cdot 3 \cdot 5} \quad (7.9)$$

If you are not familiar with the concept of prime factors, it may take a few minutes to get used to. $2 \times 2 \times 3$ is 12, broken into its **prime factors**: that is, it is the list of prime numbers that multiply to give 12. Similarly, the prime factors of 30 are $2 \times 3 \times 5$.

Why does that help? Because $12 = 2 \times 2 \times 3$, any number whose prime factors include two 2s and one 3 will be a multiple of 12. Similarly, any number whose prime factors include a 2, a 3, and a 5 will be a multiple of 30.

⁴This content is available online at <<http://cnx.org/content/m18303/1.1/>>.

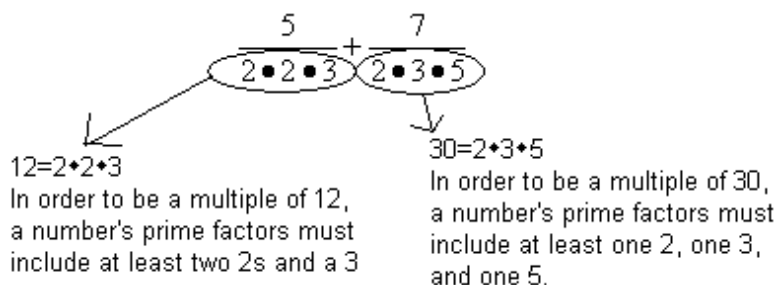


Figure 7.2

The least common denominator is the smallest number that meets both these criteria: it must have two 2s, one 3, and one 5. Hence, the least common denominator must be $2 \times 2 \times 3 \times 5$, and we can finish the problem like this.

$$\frac{5}{2 \cdot 2 \cdot 3} + \frac{7}{2 \cdot 3 \cdot 5} = \frac{55}{(2 \cdot 2 \cdot 3) \underline{5}} + \frac{72}{(2 \cdot 3 \cdot 5) \underline{2}} = \frac{25}{60} + \frac{14}{60} = \frac{39}{60} \quad (7.10)$$

This may look like a very strange way of solving problems that you've known how to solve since the third grade. However, I would urge you to spend a few minutes carefully following that solution, focusing on the question: **why is $2 \times 2 \times 3 \times 5$ guaranteed to be the least common denominator?** Because once you understand that, you have the key concept required to add and subtract rational expressions.

Example 7.4: Subtracting Rational Expressions

$\frac{3}{x^2+12x+36} - \frac{4x}{x^3+4x^2-12x}$ $= \frac{3}{(x+6)^2} - \frac{4x}{x(x+6)(x-2)}$	The problem
$= \frac{3(x)(x-2)^2}{(x+6)^2} (x)(x-2) - \frac{4x(x+6)^2}{x(x+6)} (x-2)$	Always begin rational expression problems by factoring! The least common denominator must have two $(x+6)$ s, one x , and one $(x-2)$.
$= \frac{3(x)(x-2)-4x(x+6)^2}{x(x-2)(x+6)}$	Rewrite both fractions with the common denominator.
	Subtracting fractions is easy when you have a common denominator! It's best to leave the bottom alone, since it is factored. The top, however, consists of two separate factored pieces, and will be simpler if we multiply them out so we can combine them.
<i>continued on next page</i>	

$= \frac{3x^2 - 6x - (4x^2 + 24x)^2}{x(x-2)(x+6)}$	A common student mistake here is forgetting the parentheses. The entire second term is subtracted; without the parentheses, the $24x$ ends up being added.
$= \frac{-x^2 - 30x}{x(x-2)(x+6)}$	Almost done! But finally, we note that we can factor the top again. If we factor out an x it will cancel with the x in the denominator.
$= \frac{-x - 30}{(x-2)(x+6)}$	A lot simpler than where we started, isn't it?

Table 7.6

The problem is long, and the math is complicated. So after following all the steps, it's worth stepping back to realize that even this problem results simply from the two rules we started with.

First, always factor rational expressions before doing anything else.

Second, follow the regular processes for fractions: in this case, the procedure for subtracting fractions, which involves finding a common denominator. After that, you subtract the numerators while leaving the denominator alone, and then simplify.

7.5 Rational Equations⁵

7.5.1 Rational Equations

A **rational equation** means that you are setting two rational **expressions** equal to each other. The goal is to solve for x ; that is, find the x value(s) that make the equation true.

Suppose I told you that:

$$\frac{x}{8} = \frac{3}{8} \quad (7.11)$$

If you think about it, the x in this equation has to be a 3. That is to say, if $x=3$ then this equation is true; for any other x value, this equation is false.

This leads us to a very general rule.

A very general rule about rational equations

If you have a rational equation where the **denominators** are the same, then the **numerators** must be the same.

This in turn suggests a strategy: find a common denominator, and then set the numerators equal.

Example: Rational Equation
<i>continued on next page</i>

⁵This content is available online at <<http://cnx.org/content/m18302/1.1/>>.

$\frac{3}{x^2+12x+36} = \frac{4x}{x^3+4x^2-12x}$	Same problem we worked before, but now we are equating these two fractions, instead of subtracting them.
$\frac{3(x)(x-2)^2}{(x+6)^2} (x)(x-2) = \frac{4x(x+6)^2}{x(x+6)} (x-2)$	Rewrite both fractions with the common denominator.
$3x(x-2) = 4x(x+6)$	Based on the rule above—since the denominators are equal, we can now assume the numerators are equal.
$3x^2 - 6x = 4x^2 + 24x$	Multiply it out
$x^2 + 30x = 0$	What we're dealing with, in this case, is a quadratic equation. As always, move everything to one side...
$x(x+30) = 0$	...and then factor. A common mistake in this kind of problem is to divide both sides by x ; this loses one of the two solutions.
$x=0$ or $x = -30$	Two solutions to the quadratic equation. However, in this case, $x = 0$ is not valid, since it was not in the domain of the original right-hand fraction. (Why?) So this problem actually has only one solution, $x = -30$.

Table 7.7

As always, it is vital to remember what we have found here. We started with the equation $\frac{3}{x^2+12x+36} = \frac{4x}{x^3+4x^2-12x}$. We have concluded now that if you plug $x = -30$ into that equation, you will get a true equation (you can verify this on your calculator). For any other value, this equation will evaluate false.

To put it another way: if you graphed the functions $\frac{3}{x^2+12x+36}$ and $\frac{4x}{x^3+4x^2-12x}$, the two graphs would intersect at one point only: the point when $x = -30$.

7.6 Dividing Polynomials⁶

Simplifying, multiplying, dividing, adding, and subtracting rational expressions are all based on the basic skills of working with fractions. Dividing polynomials is based on an even earlier skill, one that pretty much everyone remembers with horror: long division.

To refresh your memory, try dividing $\frac{745}{3}$ by hand. You should end up with something that looks something like this:

⁶This content is available online at <<http://cnx.org/content/m18299/1.1/>>.

$$\begin{array}{r}
 248 \text{ r}1 \\
 3 \overline{) 745} \\
 \underline{6} \\
 14 \\
 \underline{12} \\
 25 \\
 \underline{24} \\
 1
 \end{array}$$

So we conclude that $\frac{745}{3}$ is 248 with a remainder of 1; or, to put it another way, $\frac{745}{3} = 248\frac{1}{3}$.

You may have decided years ago that you could forget this skill, since calculators will do it for you. But now it comes roaring back, because here is a problem that your calculator will not solve for you: $\frac{6x^3-8x^2+4x-2}{2x-4}$. You can solve this problem in much the same way as the previous problem.

Example 7.5

Polynomial Division

$\frac{6x^3-8x^2+4x-2}{2x-4}$	The problem
$2x-4 \overline{) 6x^3-8x^2+4x-2}$	The problem, written in standard long division form.
$2x-4 \overline{) 6x^3-8x^2+4x-2}$ $ \underline{3x^2}$	Why $3x^2$? This comes from the question: "How many times does $2x$ go into $6x^3$?" Or, to put the same question another way: "What would I multiply $2x$ by, in order to get $6x^3$?" This is comparable to the first step in our long division problem: "What do I multiply 3 by, to get 7?"
$2x-4 \overline{) 6x^3-8x^2+4x-2}$ $ \underline{6x^3-12x^2}$ $ \underline{4x^2}$	Now, multiply the $3x^2$ times the $(2x - -4)$ and you get $6x^3 - -12x^2$. Then subtract this from the line above it. The $6x^3$ terms cancel—that shows we picked the right term above! Note that you have to be careful with signs here. $- - 8x^2 - - (- - 12x^2)$ gives us positive $4x^2$.
continued on next page	

$ \begin{array}{r} 3x^2 \\ 2x-4 \overline{) 6x^3-8x^2+4x-2} \\ \underline{6x^3-12x^2} \\ 4x^2+4x \end{array} $	Bring down the $4x$. We have now gone through all four steps of long division—divide, multiply, subtract, and bring down. At this point, the process begins again, with the question “How many times does $2x$ go into $4x^2$?”
$ \begin{array}{r} 3x^2+2x+6 \text{ r } 22 \\ 2x-4 \overline{) 6x^3-8x^2+4x-2} \\ \underline{6x^3-12x^2} \\ 4x^2+4x \\ \underline{4x^2-8x} \\ 12x-2 \\ \underline{12x-24} \\ 22 \end{array} $	This is not the next step...this is what the process looks like after you’ve finished all the steps. You should try going through it yourself to make sure it ends up like this.

Table 7.8

So we conclude that $\frac{6x^3-8x^2+4x-2}{2x-4}$ is $3x^2 + 2x + 6$ with a remainder of 22, or, to put it another way, $3x^2 + 2x + 6 + \frac{22}{2x-4}$.

7.6.1 Checking your answers

As always, checking your answers is not just a matter of catching careless errors: it is a way of making sure that you **know what you have come up with**. There are two different ways to check the answer to a division problem, and both provide valuable insight

The first is by plugging in numbers. We have created an algebraic generalization:

$$\frac{6x^3 - 8x^2 + 4x - 2}{2x - 4} = 3x^2 + 2x + 6 + \frac{22}{2x - 4} \quad (7.12)$$

In order to be valid, this generalization must hold for $x = 3$, $x = -4$, $x = 0$, $x = \omega$, or any other value except $x = 2$ (which is outside the domain). Let’s try $x = 3$.

Checking the answer by plugging in $x = 3$

$$\frac{6(3)^3 - 8(3)^2 + 4(3) - 2}{2(3) - 4} \stackrel{?}{=} 3(3)^2 + 2(3) + 6 + \frac{22}{2(3) - 4} \quad (7.13)$$

$$\frac{162 - 72 + 12 - 2}{6 - 4} \stackrel{?}{=} 27 + 6 + 6 + \frac{22}{6 - 4} \quad (7.14)$$

$$\frac{100}{2} \stackrel{?}{=} 39 + \frac{22}{2} \quad (7.15)$$

$$50 \stackrel{?}{=} 39 + 11 \text{ [U+2713]} \quad (7.16)$$

The second method is by multiplying back. Remember what division is: it is the opposite of multiplication! If $\frac{745}{3}$ is 248 with a remainder of 1, that means that $248 \cdot 3$ will be 745, with 1 left over. Similarly, if our long division was correct, then $(3x^2 + 2x + 6)(2x - 4) + 22$ should be $6x^3 - 8x^2 + 4x - 2$.

Checking the answer by multiplying back

$$(3x^2 + 2x + 6)(2x - 4) + 22 \quad (7.17)$$

$$= (6x^3 - 12x^2 + 4x^2 - 8x - 24) + 22 \quad (7.18)$$

$$= 6x^3 - 8x^2 + 4x - 2 \text{ [U+2713]} \quad (7.19)$$

Chapter 8

Radicals

8.1 Radical Concepts¹

The concept of a radical (or root) is a familiar one, and was reviewed in the conceptual explanation of logarithms in the previous chapter. In this chapter, we are going to explore some possibly unfamiliar properties of radicals, and solve equations involving radicals.

8.2 Properties of Radicals²

What is $\sqrt{x^2 + 9}$? Many students will answer quickly that the answer is $(x + 3)$ and have a very difficult time believing this answer is wrong. But it is wrong.

$\sqrt{x^2}$ is $x^* 3$ and $\sqrt{9}$ is 3, but $\sqrt{x^2 + 9}$ is **not** $(x + 3)$.

Why not? Remember that $\sqrt{x^2 + 9}$ is asking a question: “what **squared** gives the answer $x^2 + 9$?” So $(x + 3)$ is not an answer, because $(x + 3)^2 = x^2 + 6x + 9$, **not** $x^2 + 9$.

As an example, suppose $x = 4$. So $\sqrt{x^2 + 9} = \sqrt{4^2 + 9} = \sqrt{25} = 5$. But $(x + 3) = 7$.

NOTE: If two numbers are **added** or **subtracted** under a square root, you cannot split them up. In symbols: $\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$ or, to put it another way, $\sqrt{x^2 + y^2} \neq x + y$

$\sqrt{x^2 + 9}$ cannot, in fact, be simplified at all. It is a perfectly valid function, but cannot be rewritten in a simpler form.

How about $\sqrt{9x^2}$? By analogy to the previous discussion, you might expect that this cannot be simplified either. But in fact, it can be simplified:

$$\sqrt{9x^2} = 3x$$

Why? Again, $\sqrt{9x^2}$ is asking “what **squared** gives the answer $9x^2$?” The answer is $3x$ because $(3x)^2 = 9x^2$.

Similarly, $\sqrt{\frac{9}{x^2}} = \frac{3}{x}$, because $(\frac{3}{x})^2 = \frac{9}{x^2}$.

¹This content is available online at <<http://cnx.org/content/m18244/1.2/>>.

²This content is available online at <<http://cnx.org/content/m18271/1.1/>>.

³I’m fudging a bit here: $\sqrt{x^2}$ is x only if you ignore negative numbers. For instance, if $x = -3$, then $x^2 = 9$, and $\sqrt{x^2}$ is 3; so in that case, $\sqrt{x^2}$ is not x . In general, $\sqrt{x^2} = |x|$. However, this subtlety is not relevant to the overall point, which is that you cannot break up two terms that are added under a radical.

NOTE: If two numbers are multiplied or divided under a square root, you **can** split them up. In symbols: $\sqrt{ab} = \sqrt{a}\sqrt{b}$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

8.3 Simplifying Radicals⁴

8.3.1 Simplifying Radicals

The property $\sqrt{ab} = \sqrt{a}\sqrt{b}$ can be used to simplify radicals. The key is to break the number inside the root into two factors, **one of which is a perfect square**.

Example 8.1: Simplifying a Radical

$\sqrt{75}$	
$= \sqrt{25 \bullet 3}$	because $25 \bullet 3$ is 75, and 25 is a perfect square
$= \sqrt{25}\sqrt{3}$	because $\sqrt{ab} = \sqrt{a}\sqrt{b}$
$= 5\sqrt{3}$	because $\sqrt{25} = 5$

Table 8.1

So we conclude that $\sqrt{75} = 5\sqrt{3}$. You can confirm this on your calculator (both are approximately 8.66).

We rewrote 75 as $25 \bullet 3$ because 25 is a perfect square. We could, of course, also rewrite 75 as $5 \bullet 15$, but—although correct—that would not help us simplify, because neither number is a perfect square.

Example 8.2: Simplifying a Radical in Two Steps

$\sqrt{180}$	
$= \sqrt{9 \bullet 20}$	because $9 \bullet 20$ is 180, and 9 is a perfect square
$= \sqrt{9}\sqrt{20}$	because $\sqrt{ab} = \sqrt{a}\sqrt{b}$
$= 3\sqrt{20}$	So far, so good. But wait! We're not done!
$= 3\sqrt{4 \bullet 5}$	There's another perfect square to pull out!
$= 3\sqrt{4}\sqrt{5}$	
$= 3(2)\sqrt{5}$	
$= 6\sqrt{5}$	Now we're done.

Table 8.2

The moral of this second example is that **after** you simplify, you should always look to see if you can simplify **again**.

A secondary moral is, try to pull out the biggest perfect square you can. We could have jumped straight to the answer if we had begun by rewriting 180 as $36 \bullet 5$.

⁴This content is available online at <<http://cnx.org/content/m18274/1.1/>>.

This sort of simplification can sometimes allow you to **combine** radical terms, as in this example:

Example 8.3: Combining Radicals

$\sqrt{75} - - \sqrt{12}$	
$= 5 \sqrt{3} - - 2 \sqrt{3}$	We found earlier that $\sqrt{75} = 5 \sqrt{3}$. Use the same method to confirm that $\sqrt{12} = 2 \sqrt{3}$.
$= 3 \sqrt{3}$	5 of anything minus 2 of that same thing is 3 of it, right?

Table 8.3

That last step may take a bit of thought. It can only be used when the radical is the same. Hence, $\sqrt{2} + \sqrt{3}$ cannot be simplified at all. We were able to simplify $\sqrt{75} - \sqrt{12}$ only by making the radical in both cases the same .

So why does $5 \sqrt{3} - - 2 \sqrt{3} = 3 \sqrt{3}$? It may be simplest to think about verbally: 5 of these things, minus 2 of the same things, is 3 of them. But you can look at it more formally as a factoring problem, if you see a common factor of $\sqrt{3}$.

$$5 \sqrt{3} - - 2 \sqrt{3} = \sqrt{3}(5 - -2) = \sqrt{3}(3).$$

Of course, the process is exactly the same if variable are involved instead of just numbers!

Example 8.4: Combining Radicals with Variables

$x^{\frac{3}{2}} + x^{\frac{5}{2}}$	
$= x^3 + x^5$	Remember the definition of fractional exponents!
$= \sqrt{x^2 * x} + \sqrt{x^4 * x}$	As always, we simplify radicals by factoring them inside the root...
$\sqrt{x^2} + \sqrt{x} + \sqrt{x^4} + \sqrt{x}$	and then breaking them up...
$= x\sqrt{x} + x^2\sqrt{x}$	and then taking square roots outside!
$= (x^2 + x) \sqrt{x}$	Now that the radical is the same, we can combine.

Table 8.4

8.3.1.1 Rationalizing the Denominator

It is always possible to express a fraction with no square roots in the denominator.

Is it always desirable? Some texts are religious about this point: "You should never have a square root in the denominator." I have absolutely no idea why. To me, $\frac{1}{\sqrt{2}}$ looks simpler than $\frac{\sqrt{2}}{2}$; I see no overwhelming reason for forbidding the first or preferring the second.

However, there are times when it is useful to remove the radicals from the denominator: for instance, when adding fractions. The trick for doing this is based on the basic rule of fractions: **if you multiply the top**

and bottom of a fraction by the same number, the fraction is unchanged. This rule enables us to say, for instance, that $\frac{2}{3}$ is exactly the same number as $\frac{2 \cdot 3}{3 \cdot 3} = \frac{6}{9}$.

In a case like $\frac{1}{\sqrt{2}}$, therefore, you can multiply the top and bottom by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} = \frac{1 \cdot 2}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}}{2}$$

What about a more complicated case, such as $\frac{\sqrt{12}}{1+\sqrt{3}}$? You might think we could simplify this by multiplying the top and bottom by $(1 + \sqrt{3})$, but that doesn't work: the bottom turns into $(1 + 3)^2 = 1 + 2\sqrt{3} + 3$, which is at least as ugly as what we had before.

The correct trick for getting rid of $(1 + \sqrt{3})$ is to multiply it by $(1 - \sqrt{3})$. These two expressions, identical except for the replacement of $a +$ by $a -$, are known as **conjugates**. What happens when we multiply them? We don't need to use FOIL if we remember that

$$(x + y)(x - y) = x^2 - y^2$$

Using this formula, we see that

$$(1 + \sqrt{3})(1 - \sqrt{3}) = 1^2 - (\sqrt{3})^2 = 1 - 3 = -2$$

So the square root does indeed go away. We can use this to simplify the original expression as follows.

Example 8.5: Rationalizing Using the Conjugate of the Denominator

$$\frac{\sqrt{12}}{1+\sqrt{3}} = \frac{\sqrt{12}(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} = \frac{\sqrt{12}-\sqrt{36}}{1-3} = \frac{2\sqrt{3}-6}{-2} = -\sqrt{3} + 3$$

As always, you may want to check this on your calculator. Both the original and the simplified expression are approximately 1.268.

Of course, the process is the same when variables are involved.

Example 8.6: Rationalizing with Variables

$$\frac{1}{x-\sqrt{x}} = \frac{1(x+\sqrt{x})}{(x-\sqrt{x})(x+\sqrt{x})} = \frac{x+\sqrt{x}}{x^2-x}$$

Once again, we multiplied the top and the bottom by the **conjugate of the denominator**: that is, we replaced $a -$ with $a +$. The formula $(x + a)(x - a) = x^2 - a^2$ enabled us to quickly multiply the terms on the bottom, and eliminated the square roots in the denominator.

8.4 Radical Equations⁵

When solving **equations that involve radicals**, begin by asking yourself: **is there an x under the square root?** The answer to this question will determine the way you approach the problem.

If there is **not** an x under the square root—if only numbers are under the radicals—you can solve much the same way you would solve with no radicals at all.

Example 8.7: Radical Equation with No Variables Under Square Roots

⁵This content is available online at <<http://cnx.org/content/m18273/1.1/>>.

$\sqrt{2}x + 5 = 7 - \sqrt{3}x$	Sample problem: no variables under radicals
$\sqrt{2} + \sqrt{3}x = 7 - 5$	Get everything with an x on one side, everything else on the other
$x(\sqrt{2} + \sqrt{3}) = 2$	Factor out the x
$x = \frac{2}{\sqrt{2} + \sqrt{3}}$	Divide, to solve for x

Table 8.5

The key thing to note about such problems is that **you do not have to square both sides of the equation**. $\sqrt{2}$ may look ugly, but it is just a number—you could find it on your calculator if you wanted to—it functions in the equation just the way that the number 10, or $\frac{1}{3}$, or π would.

If there is an x under the square root, the problem is completely different. You will have to square both sides to get rid of the radical. However, there are two important notes about this kind of problem.

1. Always get the radical **alone, on one side of the equation**, before squaring.
2. Squaring both sides can introduce **false answers**—so it is important to check your answers after solving!

Both of these principles are demonstrated in the following example.

Example 8.8: Radical Equation with Variables under Square Roots

$\sqrt{x+2} + 3x = 5x + 1$	Sample problem with variables under radicals
$\sqrt{x+2} = 2x + 1$	Isolate the radical before squaring!
$x + 2 = (2x + 1)^2$	Now, square both sides
$x + 2 = 4x^2 + 4x + 1$	Multiply out. Hey, it looks like a quadratic equation now!
$x + 2 = 4x^2 + 4x + 1$	As always with quadratics, get everything on one side.
$(4x - 1)(x + 1) = 0$	Factoring: the easiest way to solve quadratic equations.
$x = \frac{1}{4}$ or $x = -1$	Two solutions. Do they work? Check in the original equation!

Table 8.6

Check $x = \frac{1}{4}$	Check $x = -1$
$\sqrt{\frac{1}{4} + 2} + 3\left(\frac{1}{4}\right) \stackrel{?}{=} 5\left(\frac{1}{4}\right) + 1$	$\sqrt{-1 + 2} + 3(-1) \stackrel{?}{=} 5(-1) + 1$
$\sqrt{\frac{1}{4} + \frac{8}{4}} + \frac{3}{4} \stackrel{?}{=} \frac{5}{4} + 1$	$\sqrt{1} - 3 \stackrel{?}{=} -5 + 1$
$\sqrt{\frac{9}{4} + \frac{3}{4}} \stackrel{?}{=} \frac{5}{4} + \frac{4}{4}$	$1 - 3 \stackrel{?}{=} -5 + 1$
$\frac{3}{2} + \frac{3}{4} \stackrel{?}{=} \frac{5}{4} + \frac{4}{4}$	$-2 = -4$ Not equal!
$\frac{9}{4} = \frac{9}{4}$	

Table 8.7

So the algebra yielded two solutions: $\frac{1}{4}$ and -1 . Checking, however, we discover that only the first solution is valid. This problem demonstrates how important it is to check solutions whenever squaring both sides of an equation.

If variables under the radical occur more than once, you will have to go through this procedure multiple times. Each time, you isolate a radical and then square both sides.

Example 8.9: Radical Equation with Variables under Square Roots Multiple Times

$\sqrt{x+7} - x = 1$	Sample problem with variables under radicals multiple times
$\sqrt{x+7} = \sqrt{x} + 1$	Isolate one radical. (I usually prefer to start with the bigger one.)
$x + 7 = x + 2\sqrt{x} + 1$	Square both sides. The two -radical equation is now a one -radical equation.
$6 = 2\sqrt{x}$	
$3 = \sqrt{x}$	Isolate the remaining radical, then square both sides again..
$9 = x$	In this case, we end up with only one solution. But we still need to check it.

Table 8.8

Check $x=9$

$\sqrt{9+7} - \sqrt{9} \stackrel{?}{=} 1$
$\sqrt{16} - \sqrt{9} \stackrel{?}{=} 1$
$4 - 3 = 1$

Table 8.9

Remember, the key to this problem was recognizing that **variables under the radical** occurred in the original problem **two times**. That cued us that we would have to go through the process—isolate a radical, then square both sides—twice, before we could solve for x . And **whenever** you square both sides of the equation, it's vital to check your answer(s)!

8.4.1 When good math leads to bad answers

Why is it that—when squaring both sides of an equation—perfectly good algebra can lead to invalid solutions? The answer is in the redundancy of squaring. Consider the following equation:

$-5 = 5$ False. But square both sides, and we get...

$25 = 25$ True. So squaring both sides of a **false** equation can produce a **true** equation.

To see how this affects our equations, try plugging $x = -1$ into the various steps of the first example.

Example 8.10: Why did we get a false answer of $x=-1$ in Example 1?

$\sqrt{x+2} + 3x = 5x + 1$	Does $x = -1$ work here? No, it does not.
$\sqrt{x+2} = 2x + 1$	How about here? No, $x = -1$ produces the false equation $1 = -1$.
$x + 2 = (2x + 1)^2$	Suddenly, $x = -1$ works. (Try it!)

Table 8.10

When we squared both sides, we “lost” the difference between 1 and -1 , and they “became equal.” From here on, when we solved, we ended up with $x = -1$ as a valid solution.

Test your memory: When you square both sides of an equation, you can introduce false answers. We have encountered one other situation where **good algebra** can lead to a **bad answer**. When was it?

Answer: It was during the study of absolute value equations, such as $|2x + 3| = -11x + 42$. In those equations, we also found the hard-and-fast rule that you **must check your answers** as the last step.

What do these two types of problem have in common? The function $|x|$ actually has a lot in common with x^2 . Both of them have the peculiar property that they always turn $-a$ and a into the same response. (For instance, if you plug -3 and 3 into the function, you get the same thing back.) This property is known as being an **even function**. Dealing with such “redundant” functions leads, in both cases, to the possibility of false answers.

The similarity between these two functions can also be seen in the graphs: although certainly not identical, they bear a striking resemblance to each other. In particular, both graphs are symmetric about the y-axis, which is the fingerprint of an “even function”.

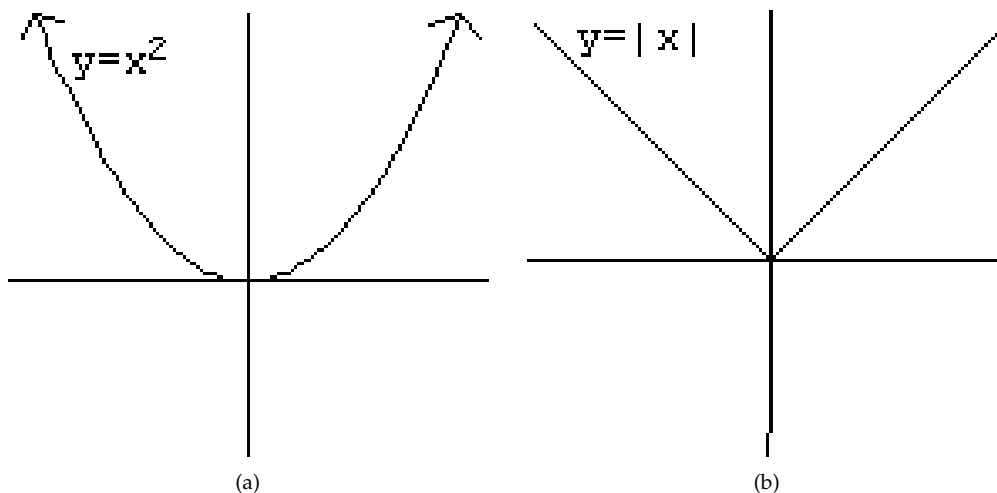


Figure 8.1

Chapter 9

Imaginary Numbers

9.1 Imaginary Numbers Concepts¹

$$-1^2 = 1$$

$$1^2 = 1$$

Whether you square a positive or a negative number, the answer is positive. It is impossible to square **any** number and get a negative answer.

So what is $\sqrt{-1}$? Since it asks the question “What number **squared** is -1 ?”, and since nothing squared ever gives the answer -1 , we say that the question has no answer. More generally, we say that the domain of $\sqrt{x}$ is all numbers x such that $x \geq 0$. -1 is not in the domain.

However, it turns out that for a certain class of problems, it is useful to define a new kind of number that has the peculiar property that when you square them, you **do** get negative answers.

Definition of i

The definition of the imaginary number i is that it is the square root of -1 :

$$i = \sqrt{-1} \text{ or, equivalently, } i^2 = -1$$

i is referred to as an “imaginary number” because it cannot represent real quantities such as “the number of rocks” or “the length of a stick.” However, surprisingly, imaginary numbers can be useful in solving many real world problems!

I often like to think of x as being like a science fiction story. Many science fiction stories are created by starting with **one false premise**, such as “time travel is possible” or “there are men on Mars,” and then following that premise logically to see where it would lead. With imaginary numbers, we start with the premise that “a number exists whose square is -1 .” We then follow that premise logically, using all the established rules of math, to see where it leads.

“The imaginary number is a fine and wonderful resource of the human spirit, almost an amphibian between being and not being.”

-Gottfried Wilhelm Leibniz

¹This content is available online at <<http://cnx.org/content/m18285/1.2/>>.

9.2 Playing with i^2

Let's begin with a few very simple exercises designed to show how we apply the normal rules of algebra to this new, abnormal number.

A few very simple examples of expressions involving i	
Simplify:	$i \bullet 5$
Answer:	$5i$
Simplify:	$i + 5i$
Answer:	$6i$ (Add anything to 5 of itself, and you get 6 of it. Or, you can think of this as "pulling out" an i as follows: $i + 5i = i(1 + 5) = 6i$)
Simplify:	$2i + 3$
Answer:	You can't simplify it.

Table 9.1

Now let's try something a little more involved.

Example: Simplify the expression $(3+2i)^2$	
$(3 + 2i)^2 = 3^2 + 2(3)(2i) + (2i)^2$	because $(x + a)^2 = x^2 + 2ax + a^2$ as always
$= 9 + 12i - 4$	$(2i)^2 = (2i)(2i) = (2)(2)(i)(i) = 4i^2 = -4$
$= 5 + 12i$	we can combine the 9 and -4, but not the 12i.

Table 9.2

It is vital to remember that i is not a variable, and this is not an algebraic generalization. You cannot plug $i = 3$ into that equation and expect anything valid to come out. The equation $(3+2i)^2 = 5 + 12i$ has been shown to be true for only one number: that number is i , the square root of -1 .

²This content is available online at <<http://cnx.org/content/m18286/1.1/>>.

In the next example, we simplify a radical using **exactly the same technique that we used in the unit on radicals**, except that $a = -1$ is thrown into the picture.

Example: Simplify $\sqrt{-20}$	
$\sqrt{-20} = \sqrt{(4)(5)(-1)}$	as always, factor out the perfect squares
$= \sqrt{4}\sqrt{5}\sqrt{-1}$	then split it, because $\sqrt{ab} = \sqrt{a}\sqrt{b}$
$= 2i\sqrt{5}$	$\sqrt{4}=2$, $\sqrt{-1}=i$, and $\sqrt{5}$ is just $\sqrt{5}$
Check	
Is $2i\sqrt{5}$ really the square root of -20 ? If it is, then when we square it, we should get -20 .	
$(2i\sqrt{5})^2 = 2^2 i^2 5^2 = 4 * -1 * 5 = -20$  It works!	

Table 9.3

The problem above has a very important consequence. We began by saying “You can’t take the square root of any negative number.” Then we defined i as the square root of -1 . But we see that, using i , we can now take the square root of **any** negative number.

9.3 Complex Numbers³

A “complex number” is the sum of two parts: a real number by itself, and a real number multiplied by i . It can therefore be written as $a + bi$, where a and bi are real numbers.

The first part, a , is referred to as the **real part**. The second part, bi , is referred to as the **imaginary part**.

Examples of complex numbers $a + bi$ (a is the “real part”; bi is the “imaginary part”)	
$3 + 2i$	$a = 3, b = 2$
π	$a = \pi, b = 0$ (no imaginary part: a “pure real number”)
$-i$	$a = 0, b = -1$ (no real part: a “pure imaginary number”)

Table 9.4

Some numbers are not obviously in the form $a + bi$. However, any number can be **put** in this form.

Example 1: Putting a fraction into $a + bi$ form (i in the numerator)
$\frac{3-4i}{5}$ is a valid complex number. But it is not in the form $a + bi$, and we cannot immediately see what the real and imaginary parts are.
<i>continued on next page</i>

³This content is available online at <<http://cnx.org/content/m18282/1.1/>>.

To see the parts, we rewrite it like this:
$\frac{3-4i}{5} = \frac{3}{5} - \frac{4}{5}i$
Why does that work? It's just the ordinary rules of fractions, applied backward. (Try multiplying and then subtracting on the right to confirm this.) But now we have a form we can use:
$\frac{3-4i}{5} a = \frac{3}{5}, b = - - \frac{4}{5}$
So we see that fractions are very easy to break up, if the i is in the numerator. An i in the denominator is a bit trickier to deal with.

Table 9.5

Example 2: Putting a fraction into $a + bi$ form (i in the denominator)	
$\frac{1}{i} = \frac{1 \cdot i}{i \cdot i}$	Multiplying the top and bottom of a fraction by the same number never changes the value of the fraction: it just rewrites it in a different form.
$= \frac{i}{-1}$	Because $i \bullet i$ is i^2 , or -1 .
$= -i$	This is not a property of i , but of -1 . Similarly, $\frac{5}{-1} = - - 5$.
$\frac{1}{i}: a = 0, b = -1$	since we rewrote it as $-i$, or $0 - 1i$

Table 9.6

Finally, what if the denominator is a more complicated complex number? The trick in this case is similar to the trick we used for rationalizing the denominator: we multiply by a quantity known as the **complex conjugate of the denominator**.

Definition of Complex Conjugate

The complex conjugate of the number $a + bi$ is $a - bi$. In words, you leave the real part alone, and change the sign of the imaginary part.

Here is how we can use the “complex conjugate” to simplify a fraction.

Example: Using the Complex Conjugate to put a fraction into $a + bi$ form	
$\frac{5}{3-4i}$	The fraction: a complex number not currently in the form $a + bi$
$= \frac{5(3+4i)}{(3-4i)(3+4i)}$	Multiply the top and bottom by the complex conjugate of the denominator
$= \frac{15+20i}{3^2-(4i)^2}$	Remember, $(x+a)(x-a) = x^2 - a^2$
$= \frac{15+20i}{9+16}$	$(4i)^2 = 4^2 i^2 = 16(-1) = - - 16$, which we are subtracting from 9
$= \frac{15+20i}{25}$	Success! The top has i , but the bottom doesn't. This is easy to deal with.
$= \frac{15}{25} + \frac{20i}{25}$	Break the fraction up, just as we did in a previous example.
$= \frac{3}{5} + \frac{4}{5}i$	So we're there! $a = \frac{3}{5}$ and $b = \frac{4}{5}$

Table 9.7

Any number of any kind can be written as $a + bi$. The above examples show how to rewrite fractions in this form. In the text, you go through a worksheet designed to rewrite $\sqrt[3]{-1}$ as three different complex numbers. Once you understand this exercise, you can rewrite other radicals, such as $\sqrt{i}$, in $a + bi$ form.

9.4 Equality and Inequality in Complex Numbers⁴

What does it mean for two complex numbers to be **equal**? As always, equality asserts that two things are exactly the same. $7 + 3i$ is **not** equal to 7, or to $3i$, or to $7 - 3i$, or to $3 + 7i$. It is not equal to anything except $7 + 3i$.

Definition of Equality

Two complex numbers are **equal** to each other only if **their real parts are equal, and their imaginary parts are equal**.

So if we say that two complex numbers equal each other, we are actually making two separate, independent statements. We can use this, for instance, to solve for two separate variables.

Example: Complex Equality	
If $3x + 4yi + 7 = 4x + 8i$, what are x and y ?	
Normally, it is impossible to solve one equation for two unknowns. But this is really two separate equations!	
Real part on the left = real part on the right:	$3x + 7 = 4x$
Imaginary part on the left = imaginary part on the right:	$4y = 8$
We can now solve both of these equations trivially.	$x = 7, y = 2$

Table 9.8

And what about **inequalities**? The answer may surprise you: **there are no inequalities with complex numbers**, at least not in the form we're seeing.

The real numbers have the property that for any two real numbers a and b , exactly one of the following three statements must be true: $a = b$, $a > b$, or $a < b$. This is one of those properties that seems almost too obvious to bother with. But it becomes more interesting when you realize that the complex numbers do **not** have that property. Consider two simple numbers, i and 1. Which of the following is true?

- $i = 1$ ☒
- $i > 1$ ☒
- $i < 1$ ☒

None of them is true. It is not generally possible to describe two complex numbers as being “greater than” or “less than” each other.

Visually, this corresponds to the fact that all the real numbers can be laid out on a number line: “greater than” means “to the right of” and so on. The complex numbers cannot be laid out on a number line. They are sometimes pictured on a 2-dimensional graph, where the real part is the x coordinate and the imaginary part is the y coordinate. But one point on a graph is neither greater than, nor less than, another point!

9.5 Quadratic Equations and Complex Numbers⁵

In the unit on quadratic equations and complex numbers, we saw that a quadratic equation can have **two answers, one answer, or no answers**.

⁴This content is available online at <<http://cnx.org/content/m18283/1.1/>>.

⁵This content is available online at <<http://cnx.org/content/m18288/1.1/>>.

We can now modify this third case. In cases where we described “no answers” there are actually two answers, but both are complex! This is easy to see if you remember that we found “no answers” when the **discriminant was negative**—that is, when the quadratic formula gave us a negative answer in the square root.

As an example, consider the equation:

$$2x^2 + 3x + 5 = 0$$

The quadratic equation gives us:

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(5)}}{4} = \frac{-3 \pm \sqrt{-31}}{4}$$

This is the point where, in the “old days,” we would have given up and declared “no answer.” Now we can find two answers—both complex.

$$= \frac{-3}{4} \pm \frac{\sqrt{31}\sqrt{-1}}{4} = \frac{-3}{4} \pm \frac{\sqrt{31}}{4}i$$

So we have two answers. Note that the two answers are **complex conjugates** of each other—this relationship comes directly from the quadratic formula.

9.6 A Few “Extra for Experts” Thoughts on Imaginary Numbers⁶

9.6.1 Illegal Operations

So far, we have seen three different **illegal operations** in math.

1. You cannot take the square root of a negative number. (Hence, the domain of $\sqrt{x}$ is $x \geq 0$.)
2. You cannot divide by zero. (Hence, the domain of $\frac{1}{x}$ is $x \neq 0$.)
3. You cannot take the log of 0 or a negative number. (Hence, the domain of $\log(x)$ is $x \geq 0$.)

Imaginary numbers give us a way of violating the first restriction. Less obviously, they also give us a way of violating the third restriction: with imaginary numbers, you can take the log of a negative number.

So, how about that second restriction? Do you ever reach a point in math where the teacher admits “OK, we **really can** divide by 0 now”? Can we define a new imaginary number $j = \frac{1}{0}$?

The answer is emphatically no: **you really can’t divide by 0**. If you attempt to define an imaginary way around this problem, all of math breaks down. Consider the following simple example:

$5 \bullet 0 = 3 \bullet 0$	That’s true
$5 = 3$	Divide both sides by 0

Table 9.9

You see? Dividing by 0 takes us from true conclusions to false ones.

The astonishing thing about the definition $i = \sqrt{-1}$ is that, although it is imaginary and nonsensical, it is **consistent**: it does not lead to any logical contradictions. You can find many ways to simplify $\frac{1}{i}$ and it will always reduce to $-i$ in the end. Division by zero can never be consistent in this way, so it is always forbidden.

⁶This content is available online at <<http://cnx.org/content/m18284/1.1/>>.

A great deal of Calculus is concerned with getting around this problem, by dividing by numbers that are **very close** to zero.

9.6.2 The World of Numbers

When you first learn about numbers, you learn the **counting numbers**:

1,2,3,4...	counting numbers
------------	-------------------------

Table 9.10

These numbers are perfect for answering questions such as “How many sticks do I have?” “How many days until Christmas?” “How many years old are you?”

For other questions, however, you run into limitations. In measuring temperature, for instance, we find that we need lower numbers than 1. Hence, we arrive at a broader list:

...-4,-3,-2,-1,0,1,2,3,4...	integers
-----------------------------	-----------------

Table 9.11

The addition of 0 and the negative numbers gives us a new, broader set. The original idea of numbers is now seen as a special case of this more general idea; the original set is a subset of this one.

Still, if we are measuring lengths of sticks, we will find that often they fall **between** our numbers. Now we have to add fractions, or decimals, to create the set of **rational numbers**. I can no longer list the set, but I can give examples.

$\frac{1}{2}, -3, \frac{22}{7}, 0, 2.718, 0.14141414...$	rational numbers
--	-------------------------

Table 9.12

The word “rational” implies a **ratio**, or fraction: the ratio of **two integers**. Hence, we define our new, broader set (rational numbers) in terms of our older, more limited set (integers). Rational numbers can be expressed as either fractions, or as decimals (which either end after a certain number of digits, or repeat the same loop of digits forever).

This set seems to be all-inclusive, but it isn’t: certain numbers cannot be expressed in this form.

$\sqrt{2}, \pi$	irrational numbers
-----------------	---------------------------

Table 9.13

The square root of any non-perfect square is “irrational” and so is π . They can be **approximated** as fractions, but not expressed exactly. As decimals, they go on forever but do not endlessly repeat the same loop.

If you take the rationals and irrationals together, you get the **real numbers**. The real numbers are all the numbers represented on a number line.

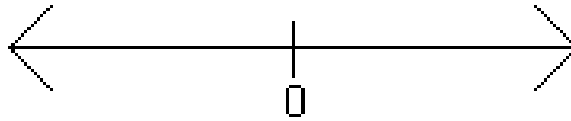


Figure 9.1: All the numbers on a number line are the real numbers

Now, with this unit, we have added the final piece of the puzzle, the **complex numbers**. A complex number is any number $a + bi$ where a and b are real numbers. Hence, just as our definition of rational numbers was **based on** our definition of integers, so our definition of complex numbers is **based on** our definition of real numbers. And of course, if $b = 0$ then we have a real number: the old set is a subset of the new.

All of this can be represented in the following diagram.

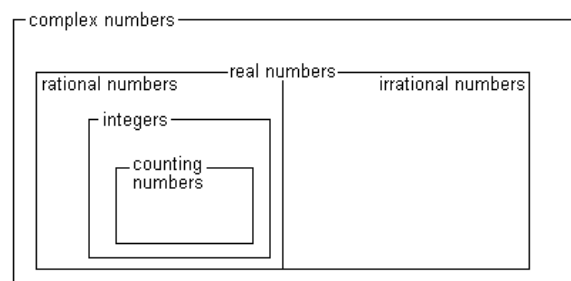


Figure 9.2

The diagram captures the vital idea of subsets: all real numbers are complex numbers, but not all complex numbers are real.

Similarly, the diagram shows that if you take all the rational numbers, and all the irrational numbers, together they make up the set of real numbers.

Chapter 10

Matrices

10.1 Matrices¹

10.1.1 Conceptual Explanations: Matrices

A “matrix” is a grid, or table, of numbers. For instance, the following matrix represents the prices at the store “Nuthin’ But Bed Stuff.”

	King-sized	Queen-sized	Twin
Mattress	\$649	\$579	\$500
Box spring	\$350	\$250	\$200
Fitted sheet	\$15	\$12	\$10
Top sheet	\$15	\$12	\$10
Blanket	\$20	\$20	\$15

Table 10.1

(The matrix is the numbers, not the words that label them.)

Of course, these prices could be displayed in a simple list: “King-sized mattress,” “Queen-sized mattress,” and so on. However, this two-dimensional display makes it much easier to compare the prices of mattresses to box springs, or the prices of king-sized items to queen-sized items, for instance.

Each horizontal list of numbers is referred to as a **row**; each vertical list is a **column**. Hence, the list of all mattresses is a row; the list of all king-sized prices is a column. (It’s easy to remember which is which if you think of Greek columns, which are big posts that hold up buildings and are very tall and...well, you know...vertical.) This particular matrix has 5 rows and 3 columns. It is therefore referred to as a 5×3 (read, “5 by 3”) matrix.

If a matrix has the same number of columns as rows, it is referred to as a **square matrix**.

10.1.2 Adding and Subtracting Matrices

Adding matrices is very simple. You just add each number in the first matrix, to the corresponding number in the second matrix.

¹This content is available online at <<http://cnx.org/content/m18311/1.1/>>.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 60 & 50 & 40 \\ 30 & 20 & 10 \end{bmatrix} = \begin{bmatrix} 61 & 52 & 43 \\ 34 & 25 & 16 \end{bmatrix}$$

For instance, for the upper-right-hand corner, the calculation was $3 + 40 = 43$. Note that both matrices being added are 2×3 , and the resulting matrix is also 2×3 . **You cannot add two matrices that have different dimensions.**

As you might guess, subtracting works much the same way, except that you subtract instead of adding.

$$\begin{bmatrix} 60 & 50 & 40 \\ 30 & 20 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 59 & 48 & 37 \\ 26 & 15 & 4 \end{bmatrix}$$

Once again, note that the resulting matrix has the same dimensions as the originals, and that **you cannot subtract two matrices that have different dimensions.**

10.1.3 Multiplying a Matrix by a Constant

What does it mean to multiply a number by 3? It means you add the number to itself 3 times.

Multiplying a **matrix** by 3 means the same thing...you add the matrix to itself 3 times.

$$3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

Note what has happened: each element in the original matrix has been multiplied by 3. Hence, we arrive at the method for multiplying a matrix by a constant: you multiply each element by that constant. The resulting matrix has the same dimensions as the original.

$$\frac{1}{2} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & \frac{5}{2} & 3 \end{bmatrix}$$

10.1.4 Matrix Equality

For two matrices to be “equal” they must be exactly the same. That is, they must have the same dimensions, and each element in the first matrix must be equal to the corresponding element in the second matrix.

For instance, consider the following matrix equation.

$$\begin{bmatrix} 1 & x+y \\ 12 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 18 \\ x-y & 10 \end{bmatrix}$$

Both matrices have the same dimensions. And the upper-left and lower-right elements are definitely the same.

But for the matrix to be equal, we also need the other two elements to be the same. So

$$x + y = 18$$

$$x - y = 12$$

Solving these two equations (for instance, by elimination) we find that $x = 15$, $y = 3$.

You may notice an analogy here to complex numbers. When we assert that two complex numbers equal each other, we are actually making two statements: the **real** parts are equal, and the **imaginary** parts are equal. In such a case, we can use one equation to solve for two unknowns. A very similar situation exists with matrices, except that one equation actually represents **many more** statements. For 2×2 matrices, setting them equal makes four separate statements; for 2×3 matrices, six separate statements; and so on.

OK, take a deep breath. Even if you've never seen a matrix before, the concept is not too difficult, and everything we've seen so far should be pretty simple, if not downright obvious.

Let that breath out now. This is where it starts to get weird.

10.2 Multiplying Matrices²

10.2.1 Multiplying a Row Matrix by a Column Matrix

A “row matrix” means a matrix with only one row. A “column matrix” means a matrix with only one column. When a row matrix has the same number of elements as a column matrix, they can be multiplied. So the following is a perfectly **legal** matrix multiplication problem:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix}$$

These two matrices could not be added, of course, since their dimensions are different, but they **can** be multiplied. Here's how you do it. You multiply the first (left-most) item in the row, by the first (top) item in the column. Then you do the same for the second items, and the third items, and so on. Finally, you **add** all these products to produce the final number.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \cdot 10 \\ + \\ 2 \cdot 20 \\ + \\ 3 \cdot 30 \\ + \\ 4 \cdot 40 \end{bmatrix} = [300]$$

Figure 10.1

A couple of my students (Nakisa Asefnia and Laura Parks) came up with an ingenious trick for visualizing this process. Think of the row as a dump truck, backing up to the column dumpster. When the row dumps its load, the numbers line up with the corresponding numbers in the column, like so:

²This content is available online at <<http://cnx.org/content/m18291/1.1/>>.

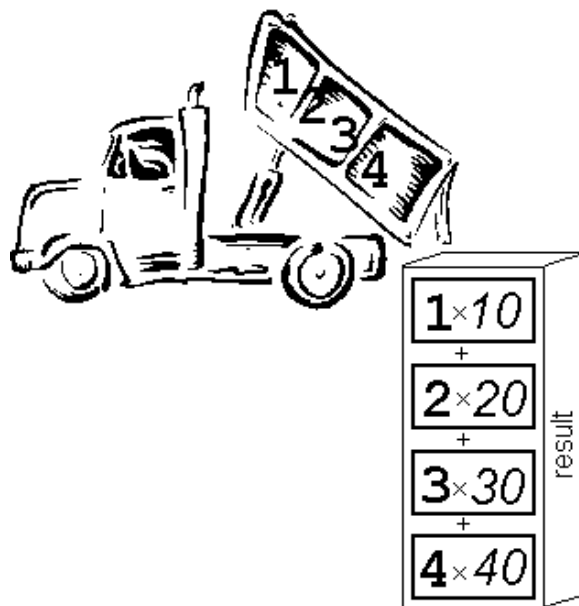


Figure 10.2

So, without the trucks and dumpsters, we express the result—a row matrix, times a column matrix—like this:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 300 \end{bmatrix}$$

There are several subtleties to note about this operation.

- The picture is a bit deceptive, because it might appear that you are multiplying two columns. In fact, **you cannot multiply a column matrix by a column matrix**. We are multiplying a row matrix by a column matrix. The picture of the **row** matrix “dumping down” only demonstrates which numbers to multiply.
- The answer to this problem is not a number: it is a 1-by-1 matrix.
- The multiplication can only be performed if the **number of elements** in each matrix is the same. (In this example, each matrix has 4 elements.)
- Order matters! We are multiplying a **row matrix times a column matrix**, not the other way around.

It’s important to practice a few of these, and get the hang of it, before you move on.

10.2.2 Multiplying Matrices in General

The general algorithm for multiplying matrices is built on the row-times-column operation discussed above. Consider the following example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix}$$

The key to such a problem is to think of the first matrix as a list of **rows** (in this case, 4 rows), and the second matrix as a list of **columns** (in this case, 2 columns). You are going to multiply each **row** in the first matrix, by each **column** in the second matrix. In each case, you will use the “dump truck” method illustrated above.

Start at the beginning: first row, times first column.

Multiply the first row of the first matrix, times the first column of the second matrix. The result becomes the upper-left-hand corner of the answer matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 & \\ & \end{bmatrix}$$

$1 \cdot 10 + 2 \cdot 20 + 3 \cdot 30 = 140$

Now, move **down** to the next row. As you do so, move **down** in the answer matrix as well.

Now continue working in the same column (first column, second matrix). But begin moving down the rows of the first matrix, and moving down the rows of the answer matrix you are building.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 & \\ 320 & \end{bmatrix}$$

$4 \cdot 10 + 5 \cdot 20 + 6 \cdot 30 = 320$

Now, move down the rows in the first matrix, multiplying each one by that same column on the right. List the numbers below each other.

And so on, down the rows of the first matrix. Each one multiplies by the first column of the second matrix, to produce the whole first column of the answer.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 \\ 320 \\ 500 \\ 680 \end{bmatrix}$$

The **first column** of the second matrix has become the **first column** of the answer. We now move on to the **second column** and repeat the entire process, starting with the first row.

Same steps, new column. Begin with the top row of the first matrix, and it becomes the top row in the answer.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 & 320 \\ & \\ & \\ & \end{bmatrix}$$

$1 \cdot 40 + 2 \cdot 50 + 3 \cdot 60 = 320$

And so on, working our way once again through all the rows in the first matrix.

And so the second column in the second matrix creates the second column in the answer. And so on...if the second matrix had 17 columns, then our answer would have 17 columns. In this case, since it has only two columns, we're done.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 & 320 \\ 320 & 770 \\ 500 & 1220 \\ 680 & 1670 \end{bmatrix}$$

We're done. We can summarize the results of this entire operation as follows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 & 320 \\ 320 & 770 \\ 500 & 1220 \\ 680 & 1670 \end{bmatrix}$$

It's a strange and ugly process—but everything we're going to do in the rest of this unit builds on this, so it's vital to be comfortable with this process. **The only way to become comfortable with this process is to do it. A lot.** Multiply a lot of matrices until you are confident in the steps.

Note that we could add more rows to the first matrix, and that would add more rows to the answer. We could add more **columns** to the second matrix, and that would add more columns to the answer. However—if we added a column to the first matrix, or added a row to the second matrix, we would have an illegal multiplication. As an example, consider what happens if we try to do this multiplication in reverse:

$$\begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \quad \text{Illegal multiplication}$$

If we attempt to multiply these two matrices, we start (as always) with the first row of the first matrix, times

the first column of the second matrix: $\begin{bmatrix} 10 & 40 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}$. But this is an illegal multiplication; the items

don't line up, since there are two elements in the row and four in the column. So you cannot multiply these two matrices.

This example illustrates two vital properties of matrix multiplication.

- The number of columns in the first matrix, and the number of rows in the second matrix, must be equal. Otherwise, you cannot perform the multiplication.
- Matrix multiplication is not **commutative**—which is a fancy way of saying, order matters. If you reverse the order of a matrix multiplication, you may get a different answer, or you may (as in this case) get no answer at all.

10.3 The Identity Matrix³

When multiplying numbers, the number 1 has a special property: when you multiply 1 by any number, you get that same number back. We can express this property as an algebraic generalization:

$$1x = x \quad (10.1)$$

The matrix that has this property is referred to as the **identity matrix**.

Definition of Identity Matrix

The **identity matrix**, designated as $[I]$, is defined by the property: $[A][I] = [I][A] = [A]$

Note that the definition of $[I]$ stipulates that the multiplication must **commute**—that is, it must yield the same answer no matter which order you multiply in. This is important because, for most matrices, multiplication does not commute.

What matrix has this property? Your first guess might be a matrix full of 1s, but that doesn't work:

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$=$	$\begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix}$	so	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	is not an identity matrix
--	--	-----	--	----	--	---------------------------

Table 10.2

The matrix that **does** work is a diagonal stretch of 1s, with all other elements being 0.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$=$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	so	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	is the identity for 2x2 matrices
$\begin{bmatrix} 2 & 5 & 9 \\ \pi & -2 & 8 \\ -3 & 1/2 & 8.3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$=$	$\begin{bmatrix} 2 & 5 & 9 \\ \pi & -2 & 8 \\ -3 & 1/2 & 8.3 \end{bmatrix}$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	is the identity for 3x3 matrices

Table 10.3

You should confirm those multiplications for yourself, and also confirm that they work in reverse order (as the definition requires).

Hence, we are led from the definition to:

The Identity Matrix

For any square matrix, its identity matrix is a diagonal stretch of 1s going from the upper-left-hand corner to the lower-right, with all other elements being 0. **Non-square** matrices do not have an identity. That is, for a non-square matrix $[A]$, there is no matrix such that $[A][I] = [I][A] = [A]$.

Why no identity for a non-square matrix? Because of the requirement of commutativity. For a non-square matrix $[A]$ you might be able to find a matrix $[I]$ such that $[A][I] = [A]$; however, if you reverse the order, you will be left with an illegal multiplication.

³This content is available online at <<http://cnx.org/content/m18293/1.1/>>.

10.4 The Inverse Matrix⁴

We have seen that the number 1 plays a special role in multiplication, because $1x = x$.

The **inverse of a number** is defined as the number that multiplies by that number to give 1: b is the inverse of a if $ab = 1$. Hence, the inverse of 3 is $\frac{1}{3}$; the inverse of $\frac{-5}{8} = \frac{-8}{5}$. Every number except 0 has an inverse.

By analogy, the **inverse of a matrix** multiplies by that matrix to give the identity matrix.

Definition of Inverse Matrix

The **inverse of matrix** $[A]$, designated as $[A]^{-1}$, is defined by the property: $[A][A]^{-1} = [A]^{-1}[A] = [I]$

The superscript -1 is being used here in a similar way to its use in functions. Recall that $f^{-1}(x)$ does not designate an exponent of any kind, but instead, an inverse function. In the same way, $[A]^{-1}$ does not denote an exponent, but an inverse **matrix**.

Note that, just as in the definition of the identity matrix, this definition requires commutativity—the multiplication must work the same in either order.

Note also that only square matrices can have an inverse. Why? The definition of an inverse matrix is **based on** the identity matrix $[I]$, and we already said that only square matrices even **have** an identity!

How do you **find** an inverse matrix? The method comes directly from the definition, with a little algebra.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as the inverse that we are looking for, by asserting that it fills the definition of an inverse matrix: when you multiply this mystery matrix by our original matrix, you get $[I]$. When we solve for the four variables a , b , c , and d , we will have found our inverse matrix.

$\begin{bmatrix} 3a+4c & 3b+4d \\ 5a+6c & 5b+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Do the multiplication. (You should check this step for yourself, it's great practice. For instance, you start by multiplying first row x first column, and you get $3a+4c$.)

$3a+4c = 1 \quad 3b+4d = 0 \quad 5a+6c = 0 \quad 5b+6d = 1$ Remember what it means for two matrices to be equal: every element in the left must equal its corresponding element on the right. So, for these two matrices to equal each other, all four of these equations must hold.

$a = - - 3 \quad b = 2 \quad c = 2\frac{1}{2} \quad d = - - 1\frac{1}{2}$ Solve the first two equations for a and c by using either elimination or substitution. Solve the second two equations for b and d by using either elimination or substitution. (The steps are not shown here.)

So the inverse is: $\begin{bmatrix} -3 & 2 \\ 2\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$ Having found the four variables, we have found the inverse.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as the inverse that we are looking for, by asserting that it fills the definition of an inverse matrix: when you multiply this mystery matrix by our original matrix, you get $[I]$. When we solve for the four variables a , b , c , and d , we will have found our inverse matrix.

⁴This content is available online at <<http://cnx.org/content/m18294/1.1/>>.

$$\begin{bmatrix} 3a+4c & 3b+4d \\ 5a+6c & 5b+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
 Do the multiplication. (You should check this step for yourself, it's great practice. For instance, you start by multiplying first row x first column, and you get $3a+4c$.)

$3a+4c=1$ $3b+4d=0$ $5a+6c=0$ $5b+6d=1$ Remember what it means for two matrices to be equal: every element in the left must equal its corresponding element on the right. So, for these two matrices to equal each other, all four of these equations must hold.

$a = - - 3$ $b = 2$ $c = 2\frac{1}{2}$ $d = - - 1\frac{1}{2}$ Solve the first two equations for a and c by using either elimination or substitution. Solve the second two equations for b and d by using either elimination or substitution. (The steps are not shown here.)

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$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as the inverse that we are looking for, by asserting that it fills the definition of an inverse matrix: when you multiply this mystery matrix by our original matrix, you get $[I]$. When we solve for the four variables a , b , c , and d , we will have found our inverse matrix.

$$\begin{bmatrix} 3a+4c & 3b+4d \\ 5a+6c & 5b+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
 Do the multiplication. (You should check this step for yourself, it's great practice. For instance, you start by multiplying first row x first column, and you get $3a+4c$.)

$3a+4c=1$ $3b+4d=0$ $5a+6c=0$ $5b+6d=1$ Remember what it means for two matrices to be equal: every element in the left must equal its corresponding element on the right. So, for these two matrices to equal each other, all four of these equations must hold.

$a = - - 3$ $b = 2$ $c = 2\frac{1}{2}$ $d = - - 1\frac{1}{2}$ Solve the first two equations for a and c by using either elimination or substitution. Solve the second two equations for b and d by using either elimination or substitution. (The steps are not shown here.)

So the inverse is: $\begin{bmatrix} -3 & 2 \\ 2\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$ Having found the four variables, we have found the inverse.

Example: Finding an Inverse Matrix				$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as the inverse that
Find the inverse of	$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$		The problem	
$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$=$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	This is the key step. It establishes	
b				
c			d	
we are looking for, by asserting that it fills the definition of an inverse matrix: when you multiply this				

mystery matrix by our original matrix, you get [I]. When we solve for the four variables a , b , c , and d , we will have found our inverse matrix.

$$\begin{bmatrix} 3a + 4c & 3b + 4d \\ 5a + 6c & 5b + 6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
 Do the multiplication. (You should check this step for yourself, it's great practice. For instance, you start by multiplying first row $\times$ first column, and you get $3a+4c$.)

$3a + 4c = 1$ $3b + 4d = 0$ $5a + 6c = 0$ $5b + 6d = 1$ Remember what it means for two matrices to be equal: every element in the left must equal its corresponding element on the right. So, for these two matrices to equal each other, all four of these equations must hold.

$a = - - 3$ $b = 2$ $c = 2\frac{1}{2}$ $d = - - 1\frac{1}{2}$ Solve the first two equations for a and c by using either elimination or substitution. Solve the second two equations for b and d by using either elimination or substitution. (The steps are not shown here.)

So the inverse is:
$$\begin{bmatrix} -3 & 2 \\ 2\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$$
 Having found the four variables, we have found the inverse.

Table 10.4

Did it work? Let's find out.

Testing our Inverse Matrix	
$\begin{bmatrix} -3 & 2 \\ 2\frac{1}{2} & -1\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	The definition of an inverse matrix: if we have indeed found an inverse, then when we multiply it by the original matrix, we should get [I].
$\stackrel{?}{=} \begin{bmatrix} (-3)(3) + (2)(5) & (-3)(4) + (2)(6) \\ (2\frac{1}{2})(3) + (-1\frac{1}{2})(5) & (2\frac{1}{2})(4) + (-1\frac{1}{2})(6) \end{bmatrix}$	Do the multiplication.
continued on next page	

$\begin{bmatrix} -9+10 & -12+12 \\ 7\frac{1}{2}-7\frac{1}{2} & 10-9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		It works!
---	---	-----------

Table 10.5

Note that, to fully test it, we would have to try the multiplication in both orders. Why? Because, in general, changing the order of a matrix multiplication changes the answer; but the definition of an inverse matrix specifies that it must work both ways! Only one order was shown above, so technically, we have only half-tested this inverse.

This process does not have to be memorized: it should make logical sense. Everything we have learned about matrices should make logical sense, except for the very arbitrary-looking definition of matrix multiplication.

10.5 Matrices on a TI-83 or TI-84 Calculator⁵

Many modern graphing calculators have all the basic matrix operations built into them. The following is a brief overview of how to work with matrices on a TI-83, TI-83 Plus, TI-84, or TI-84 Plus.

The calculator has room to store up to ten matrices at once. It refers to these matrices as [A], [B], and so on, through [J]. Note that these are not the same as the 26 lettered memories used for numbers.

The following steps will walk you through the process of entering and manipulating matrices.

1. Hit the **MATRIX** button. On a TI-83, this is a standalone button; on a TI-83 Plus, you first hit **2nd** and then **MATRIX** (above the x^{-1} button). The resulting display is a list of all the available matrices. (You have to scroll down if you want to see the ones below [G].)

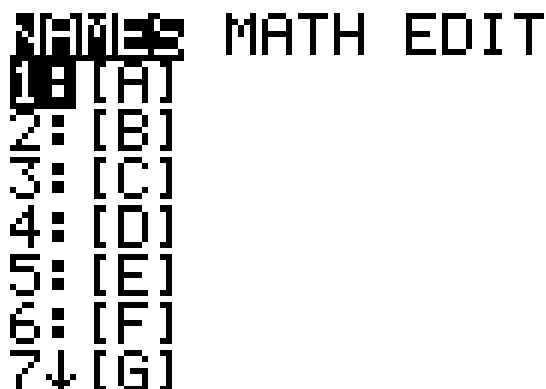


Figure 10.3

2. Hit the right arrow key **[U+25BA]** twice, to move the focus from **NAMES** to **EDIT**. This signals that you want to create, or change, a matrix.

⁵This content is available online at <http://cnx.org/content/m18290/1.1/>.

```

NAMES  MATH  [EQU]
[1] [A]
2: [B]
3: [C]
4: [D]
5: [E]
6: [F]
7↓ [G]

```

Figure 10.4

3. Hit the number 1 to indicate that you want to edit the first matrix, [A].
4. Hit 4 ENTER 3 ENTER to indicate that you want to create a 4x3 matrix. (4 rows, 3 columns.)

```

MATRIX[A]  4  x3
[ 0  0  0 ]
[ 0  0  0 ]
[ 0  0  0 ]
[ 0  0  0 ]

```

```

1 , 1 = 0

```

Figure 10.5

5. Hit

```

1 ENTER 2 ENTER 3 ENTER
4 ENTER 5 ENTER 6 ENTER
7 ENTER 8 ENTER 9 ENTER
10 ENTER 11 ENTER 12 ENTER

```

This fills in the matrix with those numbers (you can watch it fill as you go). If you make a mistake, you can use the arrow keys to move around in the matrix until the screen looks like the picture below.

```

MATRIX[A]  4  x3
[ 1      2      3      ]
[ 4      5      6      ]
[ 7      8      9      ]
[ 10     11     12     ]

4, 3=12

```

Figure 10.6

6. Hit 2nd Quit to return to the main screen.
7. Return to the main matrix menu, as before. However, this time, do not hit the right arrow to go to the EDIT menu. Instead, from the NAMES menu, hit the number 1. This puts [A] on the main screen. Then hit ENTER to display matrix [A].

```

[A]
[[1  2  3 ]
 [4  5  6 ]
 [7  8  9 ]
 [10 11 12]]

```

Figure 10.7

8. Go through the process (steps 1-7) again, with a few changes. This time, define matrix [B] instead of matrix [A]. (This will change step 3: once you are in the EDIT menu, you will hit a 2 instead of a 1.) Define [B] as a 3x2 matrix in step 4. Then, in step 5, enter the following numbers:

$$\begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} \quad (10.2)$$

When you are done, and have returned to the main screen and punched 2 in the NAMES menu (step 7), your main screen should look like this:

```

      [4  5  6 ]
      [7  8  9 ]
      [10 11 12]]
[B]
      [ [10 40]
        [20 50]
        [30 60] ]

```

Figure 10.8

9. Now, type the following keys, watching the calculator as you do so. TI-83 Plus users should always remember to hit 2nd MATRIX instead of just MATRIX. MATRX 1 + MATRX 2

```

      [4  5  6 ]
      [7  8  9 ]
      [10 11 12]]
[B]
      [ [10 40]
        [20 50]
        [30 60] ]
[A] + [B] ■

```

Figure 10.9

This instructs the computer to add the two matrices. Now hit ENTER

```

ERR: DIM MISMATCH
1:Quit
2:Goto

```

Figure 10.10

Hey, what happened? You asked the computer to add two matrices. But these matrices have different dimensions. Remember that you can only add two matrices if they have the same dimensions—that is, the same number of rows as columns. So you got an “Error: Dimension Mismatch.” Hit ENTER to get out of this error and return to the main screen.

10. Now try the same sequence without the + key: MATRX 1 MATRX 2 ENTER

```

[20 50]
[30 60] ]
[A] [B]
[ [140 320 ]
  [320 770 ]
  [500 1220]
  [680 1670] ]

```

Figure 10.11

This instructs the calculator to multiply the two matrices. This is a legal multiplication—in fact, you may recognize it as the multiplication that we did earlier. The calculator displays the result

that we found by hand:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 & 320 \\ 320 & 770 \\ 500 & 1220 \\ 680 & 1670 \end{bmatrix}$$

11. Enter a third matrix, matrix [C]= $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$. When you confirm that it is entered correctly, the screen should look like this:


```
[C]
      [[3 4]
      [5 6]]
```

Figure 10.12

Now type `MATRIX 3 x-1 ENTER`

```
[C]
      [[3 4]
      [5 6]]
[C]-1
      [[-3 2]
      [2.5 -1.5]]
```

Figure 10.13

This takes the inverse of matrix [C]. Note that the answer matches the inverse matrix that we found before.

12. Type `MATRIX 3 x-1 MATRIX 3 ENTER`

```
[C]-1
      [[5 6]
      [-3 2]
      [2.5 -1.5]]
[C]-1[C]
      [[1 0]
      [0 1]]
```

Figure 10.14

This instructs the calculator to multiply matrix [C]-1 times matrix [C]. The answer, of course, is the

2×2 identity matrix $[I]$.

10.6 Determinants⁶

10.6.1 The Determinant of a 2×2 Matrix

In the exercise “Inverse of the Generic 2×2 Matrix,” you found that the inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. This formula can be used to very quickly find the inverse of any 2×2 matrix.

Note that if $ad - bc = 0$, the formula does not work, since it puts a 0 in the denominator. This tells us that, for any 2×2 matrix, if $ad - bc = 0$ the matrix has no inverse.

The quantity $ad - bc$ is therefore seen to have a special importance for 2×2 matrices, and it is accorded a special name: the “determinant.” Determinants are represented mathematically with absolute value signs: the determinant of matrix $[A]$ is $|A|$.

Definition of the Determinant of a 2×2 Matrix

If matrix $[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is the number $|A| = ad - bc$.

For instance, for the matrix $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, the determinant is $(3)(6) - (4)(5) = -2$.

Note that the determinant is a number, not a matrix. It is a special number that is associated with a matrix.

We said earlier that “if $ad - bc = 0$ the matrix has no inverse.” We can now restate this result.

Any square matrix whose determinant is not 0, has an inverse matrix. Any square matrix with determinant 0 has no inverse.

This very important result is analogous to the result stated earlier for numbers: every number except 0 has an inverse.

10.6.2 The Determinant of a 3×3 Matrix (or larger)

Any square matrix has a determinant—an important number associated with that matrix. Non-square matrices do not have a determinant.

How do you find the determinant of a 3×3 matrix? The method presented here is referred to as “expansion by minors.” There are other methods, but they turn out to be mathematically equivalent to this one: that is, they end up doing the same arithmetic and arriving at the same answer.

⁶This content is available online at <<http://cnx.org/content/m18289/1.1/>>.

Example: Finding the Determinant of a 3x3 Matrix		
Find the determinant of	$\begin{bmatrix} 2 & 4 & 5 \\ 10 & 8 & 3 \\ 1 & 1 & 1 \end{bmatrix}$	The problem.
SORRY, THIS MEDIA TYPE IS NOT SUPPORTED.		We're going to walk through the top row, one element at a time, starting with the first element (the 2). In each case, begin by crossing out the row and column that contain that number.
$\begin{vmatrix} 8 & 3 \\ 1 & 1 \end{vmatrix} = (8)(1) - (3)(1) = 5$		Once you cross out one row and column, you are left with a 2x2 matrix (a "minor"). Take the determinant of that matrix.
$2(5) = 10$		Now, that "minor" is what we got by crossing out a 2 in the top row. Multiply that number in the top row (2) by the determinant of the minor (5).
SORRY, THIS MEDIA TYPE IS NOT SUPPORTED. $(10)(1) - (3)(1) = 7(4) = 28$		Same operation for the second element in the row (the 4 in this case)...
SORRY, THIS MEDIA TYPE IS NOT SUPPORTED. $(10)(1) - (8)(1) = 2(5) = 10$		...and the third (the 5 in this case).
$+10 - 28 + 10 = -8$		Take these numbers, and alternately add and subtract them; add the first, subtract the second, add the third. The result of all that is the determinant.

Table 10.6

This entire process can be written more concisely as:

$$\begin{vmatrix} 2 & 4 & 5 \\ 10 & 8 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 8 & 3 \\ 1 & 1 \end{vmatrix} - 4 \begin{vmatrix} 10 & 3 \\ 1 & 1 \end{vmatrix} + 5 \begin{vmatrix} 10 & 8 \\ 1 & 1 \end{vmatrix} = 2(5) - 4(7) + 5(2) = -8$$

This method of "expansion of minors" can be extended upward to any higher-order square matrix. For instance, for a 4x4 matrix, each "minor" that is left when you cross out a row and column is a 3x3 matrix. To find the determinant of the 4x4, you have to find the determinants of all four 3x3 minors!

Fortunately, your calculator can also find determinants. Enter the matrix given above as matrix [D]. Then type:

MATRIX [U+25BA] 1MATRIX 4) ENTER

The screen should now look like this:

```

[0]
      [ [2  4  5]
        [10 8  3]
        [1  1  1] ]
det([0])
                                     -8
■

```

Figure 10.15

If you watched the calculator during that sequence, you saw that the right-arrow key took you to the **MATH** submenu within the **MATRIX** menus. The first item in that submenu is **DET** (which means “determinant of.”

What does the determinant mean? It turns out that this particular odd set of operations has a surprising number of applications. We have already seen one—in the case of a 2x2 matrix, the determinant is part of the inverse. And for any square matrix, the determinant tells you whether the matrix has an inverse at all.

Another application is for finding the area of triangles. To find the area of a triangle whose vertices are

(a,b), (c,d), and (e,f), you can use the formula: $\text{Area} = \frac{1}{2} \begin{vmatrix} a & c & e \\ b & d & f \\ 1 & 1 & 1 \end{vmatrix}$. Hence, if you draw a triangle with

vertices (2,10), (4,8), and (5,3), the above calculation shows that the area of this triangle will be 4.

10.7 Solving Linear Equations⁷

At this point, you may be left with a pretty negative feeling about matrices. The initial few ideas—adding matrices, subtracting them, multiplying a matrix by a constant, and matrix equality—seem almost too obvious to be worth talking about. On the other hand, multiplying matrices and taking determinants seem to be strange, arbitrary sequences of steps with little or no purpose.

A great deal of it comes together in solving linear equations. We have seen, in the chapter on simultaneous equations, how to solve two equations with two unknowns. But suppose we have three equations with three unknowns? Or four, or five? Such situations are more common than you might suppose in the real world. And even if you are allowed to use a calculator, it is not at all obvious how to solve such a problem in a reasonable amount of time.

Surprisingly, the things we have learned about matrix multiplication, about the identity matrix, about inverse matrices, and about matrix equality, give us a very fast way to solve such problems on a calculator!

⁷This content is available online at <<http://cnx.org/content/m18292/1.1/>>.

Consider the following example, three equations with three unknowns:

$$x + 2y - z = 11 \quad (10.3)$$

$$2x - y + 3z = 7 \quad (10.4)$$

$$7x - 3y - 2z = 2 \quad (10.5)$$

Define a 3×3 matrix $[A]$ which is the coefficients of all the variables on the left side of the equal signs:

$$[A] = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{bmatrix}$$

Define a 3×1 matrix $[B]$ which is the numbers on the right side of the equal signs:

$$[B] = \begin{bmatrix} 11 \\ 7 \\ 2 \end{bmatrix}$$

Punch these matrices into your calculator, and then ask the calculator for $[A^{-1}][B]$: that is, the inverse of matrix $[A]$, multiplied by matrix $[B]$.

$$[A]^{-1}[B] = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

Figure 10.16

The calculator responds with a 3×1 matrix which is all three answers. In this case, $x = 3$, $y = 5$, and $z = 2$.

The whole process takes no longer than it takes to punch a few matrices into the calculator. And it works just as quickly for 4 equations with 4 unknowns, or 5, etc.

10.7.1 Huh? Why the heck did that work?

Solving linear equations in this way is fast and easy. But with just a little work—and with the formalisms that we have developed so far about matrices—we can also show why this method works.

10.7.1.1 Step 1: In Which We Replace Three Linear Equations With One Matrix Equation

First of all, consider the following matrix equation:

$$\begin{bmatrix} x + 2y - z \\ 2x - y + 3z \\ 7x - 3y - 2z \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \\ 2 \end{bmatrix}$$

The matrix on the left may look like a 3×3 matrix, but it is actually a 3×1 matrix. The top element is $x + 2y - z$ (all one big number), and so on.

Remember what it means for two matrices to be equal to each other. They have to have the same dimensions (✓). And all the elements have to be equal to each other. So for this matrix equation to be true, all three of the following equations must be satisfied:

$$x + 2y - z = 11 \quad (10.6)$$

$$2x - y + 3z = 7 \quad (10.7)$$

$$7x - 3y - 2z = 2 \quad (10.8)$$

Look familiar? Hey, this is the three equations we started with! The point is that this one matrix equation is equivalent to those three linear equations. We can replace the original three equations with one matrix equation, and then set out to solve that.

10.7.1.2 Step 2: In Which We Replace a Simple Matrix Equation with a More Complicated One

Do the following matrix multiplication. (You will need to do this by hand—since it has variables, your calculator can't do it for you.)

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

If you did it correctly, you should have wound up with the following 3×1 matrix:

$$\begin{bmatrix} x + 2y - z \\ 2x - y + 3z \\ 7x - 3y - 2z \end{bmatrix} \quad (10.9)$$

Once again, we pause to say...hey, that looks familiar! Yes, it's the matrix that we used in Step 1. So we can now rewrite the matrix equation from Step 1 in this way:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \\ 2 \end{bmatrix}$$

Stop for a moment and make sure you're following all this. I have shown, in two separate steps, that this matrix equation is equivalent to the three linear equations that we started with.

But this matrix equation has a nice property that the previous one did not. The first matrix (which we called $[A]$ a long time ago) and the third one ($[B]$) contain only numbers. If we refer to the middle matrix as $[X]$ then we can write our equation more concisely:

$$[A] [X] = [B], \text{ where } [A] = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{bmatrix}, [X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } [B] = \begin{bmatrix} 11 \\ 7 \\ 2 \end{bmatrix}$$

Most importantly, $[X]$ contains the three variables we want to solve for! If we can solve this equation for $[X]$ we will have found our three variables x , y , and z .

10.7.1.3 Step 3: In Which We Solve a Matrix Equation

We have rewritten our original equations as $[A] [X] = [B]$, and redefined our original goal as “solve this matrix equation for $[X]$.” If these were numbers, we would divide both sides by $[A]$. But these are matrices, and we have never defined a division operation for matrices. Fortunately, we can do something just as good, which is multiplying both sides by $[A]^{-1}$. (Just as, with numbers, you can replace “dividing by 3” with “multiplying by $\frac{1}{3}$.”)

Solving a Matrix Equation	
$[A] [X] = [B]$	The problem.
$[A]^{-1} [A] [X] = [A]^{-1} [B]$	Multiply both sides by $[A]^{-1}$, on the left. (Remember order matters! If we multiplied by $[A]^{-1}$ on the right, that would be doing something different.)
$[I] [X] = [A]^{-1} [B]$	$[A]^{-1} [A] = [I]$ by the definition of an inverse matrix.
$[X] = [A]^{-1} [B]$	$[I]$ times anything is itself, by definition of the identity matrix.

Table 10.7

So we’re done! $[X]$, which contains exactly the variables we are looking for, has been shown to be $[A]^{-1} [B]$. This is why we can punch that formula into our calculator and find the answers instantly.

10.7.2 Let’s try one more example

$$5x - 3y - 2z = 4 \quad (10.10)$$

$$x + y - 7z = 7 \quad (10.11)$$

$$10x - 6y - 4z = 10 \quad (10.12)$$

We don’t have to derive the formula again—we can just use it. Enter the following into your calculator:

$$[A] = \begin{bmatrix} 5 & -3 & -2 \\ 1 & 1 & -7 \\ 10 & -6 & -4 \end{bmatrix} \quad [B] = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}$$

Then ask the calculator for $[A]^{-1}[B]$.

```

      [ [5  -3  -2]
      [1   1  -7]
      [10 -6  -4]]
[B]
      [ [4 ]
      [7 ]
      [10]]
[A]^-1[B]

```

Figure 10.17

The result?

```

ERR: SINGULAR MAT
1: Quit
2: Goto

```

Figure 10.18

What happened? To understand this error, try the following:

Hit ENTER to get out of the error, and then hit <MATRX> [U+25BA] 1 <MATRX> 1) ENTER

$$\begin{array}{rcl}
 [B] & & \\
 & & \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} \\
 [A]^{-1}[B] & & \\
 \det([A]) & & 0 \\
 \blacksquare & &
 \end{array}$$

Figure 10.19

Aha! Matrix $[A]$ has a determinant of 0. A matrix with 0 determinant has no inverse. So the operation you asked the calculator for, $[A]^{-1}[B]$, is impossible.

What does this tell us about our original equations? They have no solution. To see why this is so, double the first equation and compare it with the third—it should become apparent that both equations cannot be true at the same time.

Chapter 11

Modeling Data with Functions

11.1 Data Modeling Concepts¹

11.1.1 Conceptual Explanations: Modeling Data with Functions

In school, you generally start with a function and work from there to numbers. “Newton’s Law tells us that $F = ma$. So if you push on a 3kg object with a 12N force, what will the acceleration be?”

In real life, the work often goes the other way. Newton didn’t start out knowing that $F = ma$; he observed the world around him, and concluded that $F = ma$. In science, you begin with data—that is, numbers—and attempt to find a mathematical function that will model the data. Then you use that function to make predictions for new data. If the predictions come true, you gain confidence in your model.

So, this unit is about a few processes that can be used to look at a set of numbers and find a function that relates them.

11.2 Direct and Inverse Variations²

11.2.1 Direct Variation

As a simple example, consider the variable c which is the number of cars in a parking lot, and the variable t which is the number of tires in the parking lot. Assuming each car has four tires, we might see numbers like this.

c (number of cars)	t (number of tires)
0	0
1	4
2	8
3	12
4	16

Table 11.1

¹This content is available online at <<http://cnx.org/content/m18277/1.1/>>.

²This content is available online at <<http://cnx.org/content/m18281/1.1/>>.

These two columns stand in a very particular relationship to each other which is referred to as direct variation.

Definition of “Direct Variation”

Two variables are in “direct variation” with each other if the following relationship holds: whenever one variable doubles, the other variable doubles. Whenever one variable triples, the other variable triples. And so on.

When the left-hand column goes up, the right-hand column goes up. This is characteristic of direct variation, but it does not prove a direct variation. The function $y = x + 1$ has the characteristic that whenever x goes up, y also goes up; however, it does not fulfill the definition of direct variation.

The equation for this particular function is, of course, $t(c) = 4c$. In general, direct variation always takes the form $y = kx$, where k is some constant—a number, not a function of x . This number is referred to as the constant of variation.

Note that, in real life, these relationships are not always exact! For instance, suppose m is the number of men in the room, and w is the weight of all the men in the room. The data might appear something like this:

m (number of men)	w (total weight of men, in pounds)
0	0
1	160
2	330
3	475
4	655

Table 11.2

Not all men weigh the same. So this is not exactly a direct variation. However, looking at these numbers, you would have a very good reason to suspect that the relationship is more or less direct variation.

How can you confirm this? Recall that if this is direct variation, then it follows the equation $w = km$, or $w/m = k$. So for direct variation, we would expect the ratio w/m to be approximately the same in every case. If you compute this ratio for every pair of numbers in the above table, you will see that it does indeed come out approximately the same in each case. (Try it!) So this is a good candidate for direct variation.

11.2.2 Inverse Variation

Suppose 5 cars all travel 120 miles. These cars get different mileage. How much gas does each one use? Let m be the miles per gallon that a car gets, and g be the number of gallons of gas it uses. Then the table might look something like this.

m (miles/gallon)	g (gallons of gas used to travel 120 miles)
10	12
20	6
30	4
40	3
60	2

Table 11.3

These variables display an inverse relationship.

Definition of “Inverse Variation”

Two variables are in “inverse variation” with each other if the following relationship holds: whenever one variable doubles, the other variable halves. Whenever one variable triples, the other variable drops in a third. And so on.

Note that as the first column gets bigger, the second column gets smaller. This is suggestive of an inverse relationship, but it is not a guarantee. $y = 10 - x$ would also have that property, and it is not inverse variation.

The equation for this particular function is $g = 120/m$. In general, inverse variation can always be expressed as $y = k/x$, where k is once again the constant of variation.

If $y = k/x$, then of course $xy = k$. So inverse variation has the characteristic that when you multiply the two variables, you get a constant. In this example, you will always get 120. With real life data, you may not always get exactly the same answer; but if you always get approximately the same answer, that is a good indication of an inverse relationship.

11.2.3 More Complex Examples

In the year 1600, Johannes Kepler sat down with the data that his teacher, Tycho Brahe, had collected after decades of carefully observing the planets. Among Brahe’s data was the period of each planet’s orbit (how many years it takes to go around the sun), and the semimajor axis of the orbit (which is sort of like a radius, but not quite—more on this in “Ellipses”). Today, these figures look something like this.

Planet	Semimajor Axis a (1010 meters)	Period T (years)
Mercury	5.79	0.241
Venus	10.8	0.615
Earth	15.0	1 (*duh)
Mars	22.8	1.88
Jupiter	77.8	11.9
Saturn	143	29.5
Uranus	287	84
Neptune	450	165

Table 11.4

What can we make of this data? As a goes up, T clearly also goes up. But they are not directly proportional. For instance, looking at the numbers for Uranus and Neptune, we see that 165 is almost exactly twice 84; but 450 is much less than twice 287. Is there a consistent pattern? Kepler went down in history for figuring out that the square of the period is directly proportional to the cube of the semimajor axis: in numbers, $T^2 = ka^3$. You can confirm this for yourself, using the numbers above. (What is k ?)

So we see that the concepts of “directly proportional” and “inversely proportional” can be applied to situations more complex than $y = kx$ or $y = k/x$.

The situation becomes more interesting still when multiple independent variables are involved. For instance, Isaac Newton was able to explain Kepler’s results by proposing that every body in the world exerts a gravitational field that obeys the following two laws.

- When the mass of the body doubles, the strength of the gravitational field doubles
- When the distance from the body doubles, the strength of the field drops in a fourth

Science texts express these laws more concisely: the field is directly proportional to the mass, and inversely proportional to the square of the radius. It may seem as if these two statements require two different equations. But instead, they are two different clues to finding the one equation that allows you to find the gravitational field F at a distance r from a given mass m . That one equation is $F = \frac{Gm}{r^2}$ where G , the constant of proportionality, is one of the universal constants of nature. This does not come from combining the two equations $F = km$ and $F = \frac{k}{r^2}$ as a composite function or anything else. Rather, it is one equation that expresses both relationships properly: doubling the mass doubles the field, and doubling the radius drops the field in a fourth.

11.3 Finding a Linear Function for any Two Points³

11.3.1 Finding a Linear Function for any Two Points

In an earlier unit, we did a great deal of work with the equation for the height of a ball thrown straight up into the air. Now, suppose you want an equation for the speed of such a ball. Not knowing the correct formula, you run an experiment, and you measure the two data points.

t (time)	v (velocity, or speed)
1 second	50 ft/sec
3 seconds	18 ft/sec

Table 11.5

Obviously, the ball is slowing down as it travels upward. Based on these two data points, what function $v(t)$ might model the speed of the ball?

Given any two points, the simplest equation is always a line. We have two points, (1,50) and (3,18). How do we find the equation for that line? Recall that every line can be written in the form:

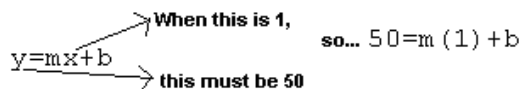
$$y = mx + b \tag{11.1}$$

If we can find the m and b for our particular line, we will have the formula.

³This content is available online at <<http://cnx.org/content/m18278/1.1/>>.

Here is the key: if our line contains the point (1,50) that means that when we plug in the x-value 1, we must get the y-value 50.

The equation contains the point (1, 50). That tells us that...



Similarly, we can use the point (3,18) to generate the equation $18 = m(3) + b$. So now, in order to find m and b , we simply have to solve two equations and two unknowns! We can solve them either by substitution or elimination: the example below uses substitution.

$$m + b = 50 \rightarrow b = 50 - m \quad (11.2)$$

$$3m + b = 18 \rightarrow 3m + (50 - m) = 18 \quad (11.3)$$

$$2m + 50 = 18 \quad (11.4)$$

$$2m = - - 32 \quad (11.5)$$

$$m = -16 \quad (11.6)$$

$$b = 50 - (-16) = 66 \quad (11.7)$$

So we have found m and b . Since these are the unknowns in the equation $y = mx + b$, the equation we are looking for is:

$$y = - - 16x + 66 \quad (11.8)$$

Based on this equation, we would expect, for instance, that after 4 seconds, the speed would be 2 ft/sec. If we measured the speed after 4 seconds and found this result, we would gain confidence that our formula is correct.

11.4 Finding a Parabolic Function for any Three Points⁴

11.4.1 Finding a Parabolic Function for any Three Points

Any two points are joined by a line. Any three points are joined by a vertical parabola.

Let's start once again with the exceptions. Once again, if any two of the points are vertically aligned, then no function can join them. However, there is no additional exception—if all three points lie on a line, then no parabola joins them. For instance, no parabola contains the three points (1,3), (2,5), and (5,11). In real life, of course, if we wanted to model those three points, we would be perfectly happy to use the line $y = 2x + 1$ instead of a parabola.

However, if three points are not vertically aligned and do not lie on a line, it is always possible to find a vertical parabola that joins them. The process is very similar to the process we used for a line, except that the starting equation is different.

⁴This content is available online at <<http://cnx.org/content/m18279/1.3/>>.

Example 11.1: Finding a Vertical Parabola to Fit Three Points

Find a vertical parabola containing the points $(-2,5)$, $(-1,6)$, and $(3,-10)$.

The problem. As with our example earlier, this problem could easily come from an attempt to find a function to model real-world data.

$$y = ax^2 + bx + c$$

This is the equation for any vertical parabola. Our job is to find a , b , and c . Note that this starting point is the same for any problem with three points, just as any problem with two points starts out $y = mx + b$.

$$\begin{aligned} 5 &= a(-2)^2 + b(-2) + c \\ 6 &= a(-1)^2 + b(-1) + c \\ -10 &= a(3)^2 + b(3) + c \end{aligned}$$

Each point represents an (x, y) pair that must create a true equation in our function. Hence, we can plug each point in for x and y to find three equations that must be true. We can now solve for our 3 unknowns.

Rewrite the above three equations in a more standard form:

$$\begin{aligned} 4a - 2b + c &= 5 \\ a - b + c &= 6 \\ 9a + 3b + c &= -10 \end{aligned}$$

Uh-oh. Now what? In the linear example, we used elimination or substitution to solve for the two variables. How do we solve three? Oh, yeah. Matrices! Rewrite the above three equations

as $[A][X] = [B]$, where $[X] = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is what we want.

$$[A] = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 9 & 3 & 1 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$$[A]^{-1} [B] = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

From the calculator, of course. Remember what this means! It means that $a = -1$, $b = -2$, and $c = 5$. We can now plug these into our original equation, $y = ax^2 + bx + c$.

$$y = -x^2 - 2x + 5$$

So this is the equation we were looking for.

Did it work? Remember that we were looking for a parabola that contained the three points $(-2,5)$, $(-1,6)$, and $(3,-10)$. If this parabola contains those three points, then our job is done. Let's try the first point.

$$5 \stackrel{?}{=} -(-2)^2 - 2(-2) + 5 \quad (11.9)$$

$$5 = -4 + 4 + 5$$

So the parabola does contain the point $(-2,5)$. You can confirm for yourself that it also contains the other two points.

Finally, remember what this means! If we had measured some real-world phenomenon and found the three points $(-2,5)$, $(-1,6)$, and $(3,-10)$, we would now suspect that the function $y = -x^2 - 2x + 5$ might serve as a model for this phenomenon.

This model predicts that if we make a measurement at $x = -3$ we will find that $y = 2$. If we made such a measurement and it matched the prediction, we would gain greater confidence in our model. On the other hand, if the measurement was far off the prediction, we would have to rethink our model.

11.4.1.1 A surprising application: "secret sharing"

Bank vaults are commonly secured by a method called "secret sharing." The goal of a secret sharing system runs something like this: if any three employees enter their secret codes at the same time, the vault will open. But any two employees together cannot open the vault.

Secret sharing is implemented as follows.

- Choose a parabolic function—that is, choose the numbers a , b , and c in the equation $y = ax^2 + bx + c$. This function is chosen at random, and is not programmed into the vault or given to any employee.
- The actual number that will open the vault is the y-intercept of the parabola: that is, the y-value of the parabola when $x = 0$. This number is not given to any employee.
- Each employee's secret code is one point on the parabola.

When three employees enter their secret codes at the same time, the vault computer uses the three points to compute a , b , and c for the parabola. As we have seen, this computation can be done quickly and easily using inverse matrices and matrix multiplication, both of which are easy algorithms to program into a computer. Once the computer has those three numbers, it computes the y-value when $x = 0$, and uses this number to open the vault.

Any three employees—that is, any three points—are enough to uniquely specify the parabola. But if you only have two points, you are no closer to the answer than when you started: the secret y value could still be, literally, any number at all.

Note also that the system is easily extendable. That is, if you want to say that four employees are required to open the vault, you just move up to a third-order polynomial, $y = ax^3 + bx^2 + c + d$. The resulting equations—four equations with four unknowns—are just as easy, with matrices, as three were.

11.5 Regression on the Calculator⁵

11.5.1 Regression on the Calculator

What Kepler did is an example of “regression”: finding an equation that models a particular set of data.

Kepler became famous because regression is hard. Who would have thought to look for $T^2 = ka^3$? Especially when you consider all the other equations that would still have a “this goes up, that goes up” relationship, such as $T = 2^a + 7$, or $T = \frac{1}{10-a}$, or maybe $T = 3\log 5a$?

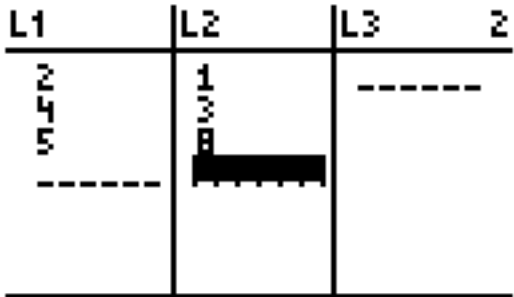
Fortunately, we have a tool that Kepler did not have: the modern computer. Mathematical programs and graphing calculators can take a set of points, and find the line or curve of “best fit” to model the data.

As an example of this process, suppose that you have run an experiment and generated three data points: (2,1), (4,3), and (5,8). What function might model those data?

NOTE: Note: the directions below are given for a TI-83 or compatible calculator. Many other calculators can perform the same functions, but the implementation details may look quite different.

11.5.1.1 Entering the data

1. Hit STAT to go into the Statistics menu.
2. Choose Edit... This brings you to a screen where you enter a bunch of L1 and L2 values.

FIXME: A LIST CAN NOT BE A TABLE ENTRY. Enter the L1 and L2 values as follows: for each data point, the x-coordinate is in the L1 list, and the y-coordinate is in the L2 list. The screen to the right shows the points (2,1), (4,3), and (5,8).	
	L2(4) =

⁵This content is available online at <<http://cnx.org/content/m18280/1.1/>>.

Table 11.6

1. Hit 2nd QUIT return to the main screen.

11.5.1.2 Viewing the data, and guessing at the shape

Once you have entered your points into the L1 and L2 lists, your calculator can show you a “scatterplot”—which is a pointlessly fancy word for “a graph of a bunch of points,” like you used to make when you were first learning what graphing was.

1. Hit WINDOW (near the upper-left-hand corner of the calculator).
2. Set the variables Xmin, Xmax, Ymin, and Ymax appropriately. For instance, for the three points shown above, the x-values are 2, 4, and 5, and the y-values are 1, 3, and 8. So it might make sense to set Xmin=0, XMax=10, Ymin=0, and Ymax=10—and that is how I did it in the drawing below. Of course, there are many other settings you could use. But if you go through the whole process and don’t see any points, it’s a reasonable guess that your window is not set properly.
3. Hit Y= (upper-left-hand corner of the calculator).
4. Then, hit the up-arrow key, so the focus moves to Plot1 (which will start blinking).
5. Hit ENTER. It’s actually impossible, at this point, to see that anything has happened. But if you down-arrow away from Plot1, you should see that it remains darkened (white letters on a black background, instead of the other way around). This indicates that it has been selected.

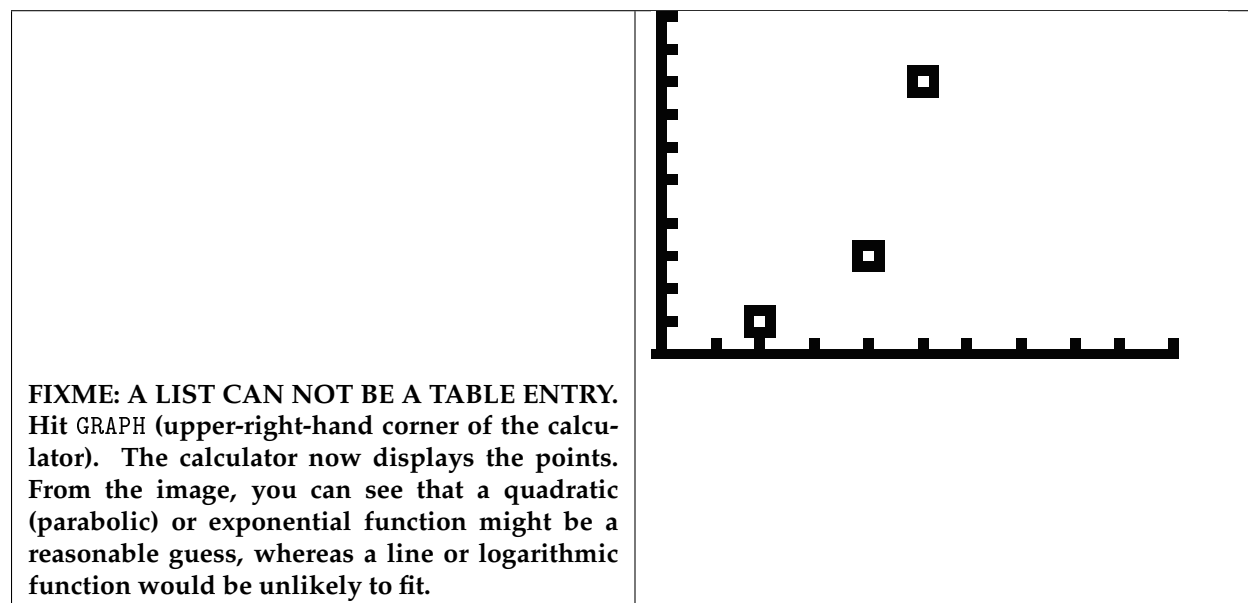


Table 11.7

At this point, looking at the data, it is often useful to categorize it in two ways.

First: is it increasing or decreasing? In our example, of course, the points are increasing. (Some data, of course, may be doing both at different times: consider, for instance, a parabola.)

Now, in the case of an increasing function, you can categorize it as one of the following.

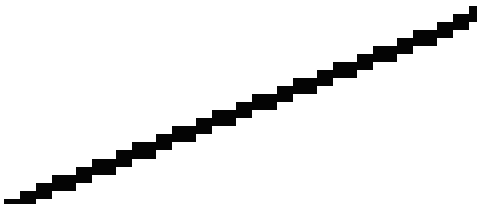
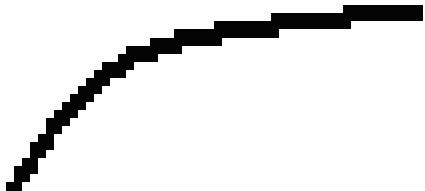
	If it is increasing steadily, that suggests a line. (Remember that what makes a linear function linear is that it always goes up at the same rate, or slope!)
	If it is increasing more and more slowly, that suggests a logarithmic function. (A square root would also have this basic shape, but you cannot do a square-root-regression.)
	If it is increasing more and more quickly, that suggests an exponential function, or possibly the right side of a quadratic function (a parabola).

Table 11.8

Decreasing functions can be categorized similarly, of course. If a function decreases and then increases, a parabola is probably the best fit. Functions that go up, then down, then up again, are most likely to be higher-order polynomials.

11.5.1.3 Finding the formula

Once you have decided on the right shape, the hard work is done: the calculator takes care of the rest.

1. Hit STAT to return again to the Statistics menu.

2. Hit the right-arrow key to go to **CALC**.
3. At this point, you have several choices. **LinReg** will give you a line that best fits your points. **QuadReg** will give you a quadratic function, aka a second-order polynomial. There are also options for "cubic" (third-order polynomial), "quartic" (fourth-order polynomial), logarithmic, or exponential curves. Choose the one you want and hit **ENTER**.

The calculator does not graph your curve for you, but it does tell you what the curve is. For instance, if I run a **QuadReg** on the data above, the calculator gives me:

```

QuadReg
y=ax2+bx+c
a=1.333333333
b=-7
c=9.666666667

```

Figure 11.1

This tells me that the best quadratic fit for my data is the curve $y = 1.33x^2 - 7x + 9.67$. One way to double-check this, of course, is to enter $Y1 = 1.33x^2 - 7x + 9.67$ and then graph it, and see how closely it approximates the points!

NOTE: Remember that whatever option you choose, it will operate on the points you have entered in the L1 and L2 lists, so make sure your data is correctly entered there!

11.5.1.3.1 Starting over

There's just one more thing you have to know: once you've done this once, how do you clear out the lists to enter new ones? Here is one way to do it.

1. Hit **MEM** (you do this by hitting the yellow **2nd** key, and then hitting **+**).
2. This brings up a menu. Choose **ClrAllLists**.
3. Then—after you return to the main screen—hit **ENTER** and the lists are emptied out.

Chapter 12

Conics

12.1 Conic Concepts¹

So far, we have talked about how to graph two shapes: lines, and parabolas. This unit will discuss parabolas in more depth. It will also discuss circles, ellipses, and hyperbolas. These shapes make up the group called the conic sections: all the shapes that can be created by intersecting a plane with a double cone.

¹This content is available online at <<http://cnx.org/content/m18265/1.1/>>.

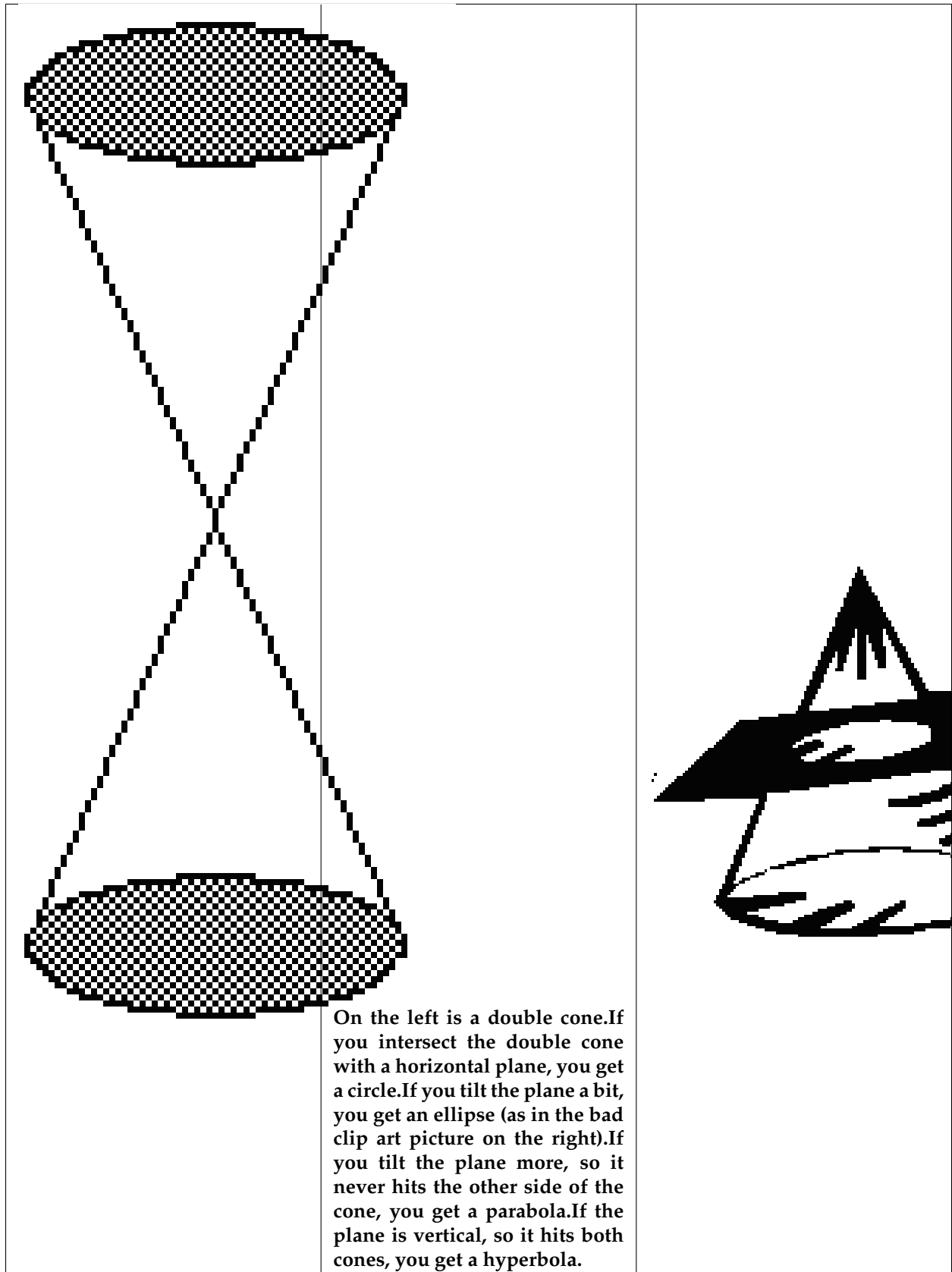


Table 12.1

We are going to discuss each of these shapes in some detail. Specifically, for each shape, we are going to provide...

- A formal definition of the shape, and
- The formula for graphing the shape

These two things—the definition, and the formula—may in many cases seem unrelated. But you will be doing work in the text exercises to show, for each shape, how the definition leads to the formula.

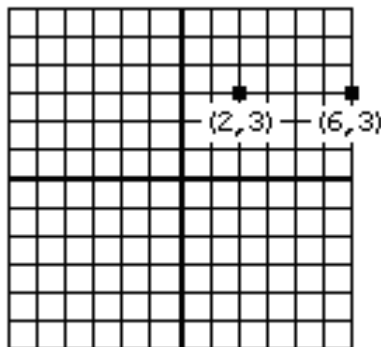
12.2 A Mathematical Look at Distance²

The key mathematical formula for discussing all the shapes above is the distance between two points.

Many students are taught, at some point, the “distance formula” as a magic (and very strange-looking) rule. In fact, the distance formula comes directly from a bit of intuition...and the Pythagorean Theorem.

The intuition comes in finding the distance between two points that have one coordinate in common.

12.2.1 The distance between two points that have one coordinate in common



The drawing shows the points $(2,3)$ and $(6,3)$. Finding the distance between these points is easy: just count! Take your pen and move it along the paper, starting at $(2,3)$ and moving to the right. Let's see...one unit gets you over to $(3,3)$; the next unit gets you to $(4,3)$...a couple more units to $(6,3)$. The distance from $(2,3)$ to $(6,3)$ is 4.

Of course, it would be tedious to count our way from $(2,3)$ to $(100,3)$. But we don't have to—in fact, you may have already guessed the faster way—we subtract the x coordinates.

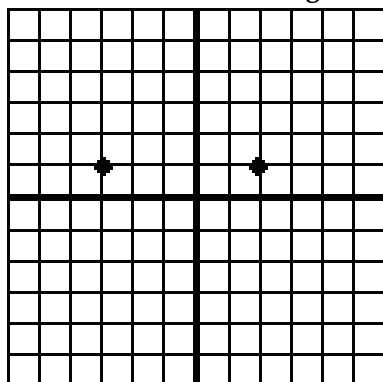
- The distance from $(2,3)$ to $(6,3)$ is $6 - 2 = 4$
- The distance from $(2,3)$ to $(100,3)$ is $100 - 2 = 98$

And so on. We can write this generalization in words:

NOTE: Whenever two points lie on a horizontal line, you can find the distance between them by subtracting their x -coordinates.

²This content is available online at <http://cnx.org/content/m18246/1.2/>.

This may seem pretty obvious in the examples given above. It's a little less obvious, but still true, if one of the x coordinates is negative.



The drawing above shows the numbers $(-3, 1)$ and $(2, 1)$. You can see that the distance between them is 5 (again, by counting). Does our generalization still work? Yes it does, because subtracting a negative number is the same as adding a positive one.

The distance from $(-3, 1)$ to $(2, 1)$ is $2 - (-3) = 5$

How can we express this generalization mathematically? If two points lie on a horizontal line, they have two different x -coordinates: call them x_1 and x_2 . But they have the same y -coordinate, so just call that y . So we can rewrite our generalization like this: “the distance between the points (x_1, y) and (x_2, y) is $x_2 - x_1$.” In our most recent example, $x_1 = -3$, $x_2 = 2$, and $y = 1$. So the generalization says “the distance between the points $(-3, 1)$ and $(2, 1)$ is $2 - (-3)$ ”, or 5.

But there's one problem left: what if we had chosen x_2 and x_1 the other way? Then the generalization would say “the distance between the points $(2, 1)$ and $(-3, 1)$ is $(-3) - 2$ ”, or -5. That isn't quite right: distances can never be negative. We get around this problem by taking the absolute value of the answer. This guarantees that, no matter what order the points are listed in, the distance will come out positive. So now we are ready for the correct mathematical generalization:

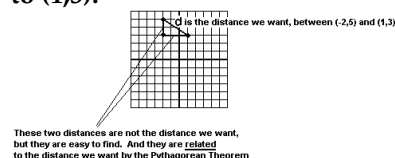
Distance Between Two Points on a Horizontal Line

The distance between the points (x_1, y) and (x_2, y) is $|x_2 - x_1|$

You may want to check this generalization with a few specific examples—try both negative and positive values of x_1 and x_2 . Then, to really test your understanding, write and test a similar generalization for two points that lie on a vertical line together. Both of these results will be needed for the more general case below.

12.2.2 The distance between two points that have no coordinate in common

So, what if two points have both coordinates different? As an example, consider the distance from $(-2, 5)$ to $(1, 3)$.



The drawing shows these two points. The (diagonal) line between them has been labeled d : it is this line that we want the length of, since this line represents the distance between our two points.

The drawing also introduces a third point into the picture, the point $(-2,3)$. The three points define the vertices of a right triangle. Based on our earlier discussion, you can see that the vertical line in this triangle is length $|5 - -3| = 2$. The horizontal line is length $|1 - -(-2)| = 3$.

But it is the diagonal line that we want. And we can find that by using the Pythagorean Theorem, which tells us that $d^2 = 2^2 + 3^2$. So $d = \sqrt{13}$

If you repeat this process with the generic points (x_1, y_1) and (x_2, y_2) you arrive at the distance formula:

Distance between any two points

If d is the distance between the points (x_1, y_1) and (x_2, y_2) , then $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$x_2 - x_1$ is the horizontal distance, based on our earlier calculation. $y_2 - y_1$ is the vertical distance, and the entire formula is simply the Pythagorean Theorem restated in terms of coordinates.

And what about those absolute values we had to put in before? They were used to avoid negative distances. Since the distances in the above formulae are being squared, we no longer need the absolute values to insure that all answers will come out positive.

12.3 Circles³

12.3.1 The Definition of a Circle

You've known all your life what a circle looks like. You probably know how to find the area and the circumference of a circle, given its radius. But what is the exact mathematical definition of a circle? Before you read the answer, you may want to think about the question for a minute. Try to think of a precise, specific definition of exactly what a circle is.

Below is the definition mathematicians use.

Definition of a Circle

The set of all points in a plane that are the same distance from a given point forms a circle. The point is known as the center of the circle, and the distance is known as the radius.

Mathematicians often seem to be deliberately obscuring things by creating complicated definitions for things you already understood anyway. But if you try to find a simpler definition of exactly what a circle is, you will be surprised at how difficult it is. Most people start with something like "a shape that is round all the way around." That does describe a circle, but it also describes many other shapes, such as this pretzel:



So you start adding caveats like "it can't cross itself" and "it can't have any loose ends." And then somebody draws an egg shape that fits all your criteria, and yet is still not a circle:



So you try to modify your definition further to exclude that... and by that time, the mathematician's definition is starting to look beautifully simple.

But does that original definition actually produce a circle? The following experiment is one of the best ways to convince yourself that it does.

Experiment: Drawing the Perfect Circle

³This content is available online at <<http://cnx.org/content/m18245/1.2/>>.

1. Lay a piece of cardboard on the floor.
2. Thumbtack one end of a string to the cardboard.
3. Tie the other end of the string to your pen.
4. Pull the string as tight as you can, and then put the pen on the cardboard.
5. Pull the pen all the way around the thumbtack, keeping the string taut at all times.

The pen will touch every point on the cardboard that is exactly one string-length away from the thumbtack. And the resulting shape will be a circle. The cardboard is the plane in our definition, the thumbtack is the center, and the string length is the radius.

The purpose of this experiment is to convince yourself that if you take all the points in a plane that are a given distance from a given point, the result is a circle. We'll come back to this definition shortly, to clarify it and to show how it connects to the mathematical formula for a circle.

12.3.2 The Mathematical Formula for a Circle

You already know the formula for a line: $y = mx + b$. You know that m is the slope, and b is the y -intercept. Knowing all this, you can easily answer questions such as: "Draw the graph of $y = 2x - 3$ " or "Find the equation of a line that contains the points (3,5) and (4,4)." If you are given the equation $3x + 2y = 6$, you know how to graph it in two steps: first put it in the standard $y = mx + b$ form, and then graph it.

All the conic sections are graphed in a similar way. There is a standard form which is very easy to graph, once you understand what all the parts mean. If you are given an equation that is not in standard form, you put it into the standard form, and then graph it.

So, to understand the formula below, think of it as the $y = mx + b$ of circles.

Mathematical Formula for a Circle

$(x - h)^2 + (y - k)^2 = r^2$ is a circle with center (h, k) and radius r

From this, it is very easy to graph a circle in standard form.

Example 12.1: Graphing a Circle in Standard Form

Graph $(x + 5)^2 + (y - 6)^2 = 10$	The problem. We recognize it as being a circle in standard form.
$h = -5$ $k = 6$ $r^2 = 10$	You can read these variables straight out of the equation, just as in $y = mx + b$. Question: how can we make our equation's $(x + 5)$ look like the standard formula's $(x - h)$? Answer: if $h = -5$. In general, h comes out the opposite sign from the number in the equation. Similarly, $(y - 6)$ tells us that k will be positive 6.
<i>continued on next page</i>	

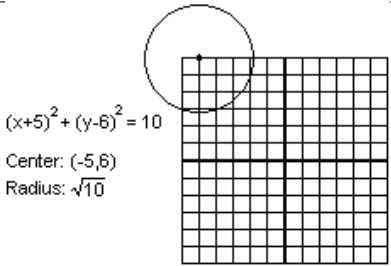
Center: $(-5,6)$ Radius: $\sqrt{10}$  $(x+5)^2 + (y-6)^2 = 10$ Center: $(-5,6)$ Radius: $\sqrt{10}$	Now that we have the variables, we know everything we need to know about the circle. And we can graph it! $\sqrt{10}$ is, of course, just a little over 3—so we know where the circle begins and ends.
--	---

Table 12.2

Just as you can go from a formula to a graph, you can also go the other way.

Example 12.2: Find the Equation for this Circle

Find the equation for a circle with center at $(15,-4)$ and radius 8.	The problem.
$(x - 15)^2 + (y + 4)^2 = 64$	The solution, straight from the formula for a circle.

Table 12.3

If a circle is given in nonstandard form, you can always recognize it by the following sign: it has both an x^2 and a y^2 term, and they have the same coefficient.

- $-3x^2 - 3y^2 + x - y = 5$ is a circle: the x^2 and y^2 terms both have the coefficient -3
- $3x^2 - 3y^2 + x - y = 5$ is not a circle: the x^2 term has coefficient 3, and the y^2 has -3
- $3x^2 + 3y = 5$ is not a circle: there is no y^2 term

Once you recognize it as a circle, you have to put it into the standard form for graphing. You do this by completing the square... twice!

Example 12.3: Graphing a Circle in Nonstandard Form

Graph $2x^2 + 2y^2 - 12x + 28y - 12 = 0$	The problem. The equation has both an x^2 and a y^2 term, and they have the same coefficient (a 2 in this case): this tells us it will graph as a circle.
<i>continued on next page</i>	

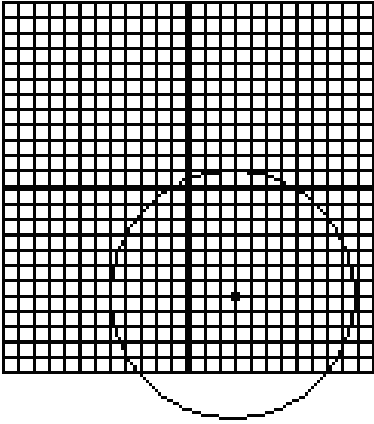
$x^2 + y^2 - 6x + 14y - 6 = 0$	Divide by the coefficient (the 2). Completing the square is always easiest without a coefficient in front of the squared tem.
$(x^2 - 6x) + (y^2 + 14y) = 6$	Collect the x terms together and the y terms together, with the number on the other side.
$(x^2 - 6x + 9) + (y^2 + 14y + 49) = 6 + 9 + 49$	Complete the square for both x and y .
$(x - 3)^2 + (y + 7)^2 = 64$	Rewrite our perfect squares. We are now in the correct form. We can see that this is a circle with center at (3,-7) and radius 8. (*Remember How the signs change on h and k !)
	
Once you have the center and radius, you can immediately draw the circle, as we did in the previous example.	

Table 12.4

12.3.3 Going From the Definition of a Circle to the Formula

If you're following all this, you're now at the point where you understand the definition of a circle...and you understand the formula for a circle. But the two may seem entirely unconnected. In other words, when I said $(x - h)^2 + (y - k)^2 = r^2$ is the formula for a circle, you just had to take my word for it.

In fact, it is possible to start with the definition of a circle, and work from there to the formula, thus showing why the formula works the way it does.

Let's go through this exercise with a specific example. Suppose we want to find the formula for the circle with center at $(-2, 1)$ and radius 3. We will start with the definition: this circle is the set of all the points that are exactly 3 units away from the point $(-2, 1)$. Think of it as a club. If a point is exactly 3 units away from $(-2, 1)$, it gets to join the club; if it is not exactly 3 units away, it doesn't get to join.

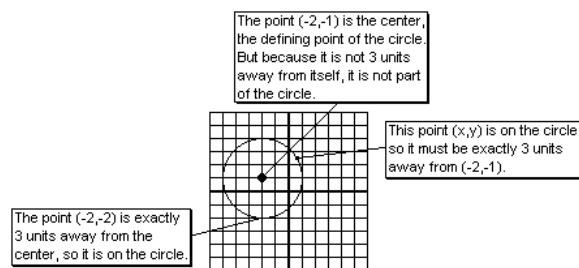


Figure 12.1

You already know what the formula is going to be, but remember, in this exercise we're not going to assume that formula—we're going to assume nothing but the definition, and work our way to the formula. So here is our starting point, the definition for this circle:

"The distance from (x, y) to $(-2, 1)$ is 3."

Any point (x, y) that meets this criterion is in our club. Using the distance formula that we developed above, we can immediately translate this English language definition into a mathematical formula. Recall that if xxx_d is the distance between the points (x_1, y_1) and (x_2, y_1) , then $(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$ (Pythagorean Theorem). So in this particular case,

$$(x + 2)^2 + (y - 1)^2 = 9$$

Note that this corresponds perfectly to the formula given above. In fact, if you repeat this exercise more generically—using (h, k) as the center instead of $(-2, 1)$, and r as the radius instead of 3—then you end up with the exact formula given above, $(x - -h)^2 + (y - -k)^2 = r^2$.

For each of the remaining shapes, I'm going to repeat the pattern used here for the circle. First I will give the geometric definition and then the mathematical formula. However, I will not take the third step, of showing how the definition (with the distance formula) leads to the formula: you will do this, for each shape, in the exercises in the text.

12.4 Parabolas⁴

12.4.1 The Definition of a Parabola

Based on the discussion of circles, you might guess that the definition of a parabola will take the form: "The set of all points that..." and you would be correct. But the definition of a parabola is more complicated than that of a circle.

Definition of a Parabola

Take a point (called the focus) and a horizontal line (the directrix) that does not contain that point. The set of all points in a plane that are the same distance from the focus as from the directrix forms a parabola.

⁴This content is available online at <<http://cnx.org/content/m18268/1.1/>>.

In the text, you begin with a specific example of this process. The focus is $(0,3)$ and the directrix is the line $y = -3$. If we use our “club” analogy again, we could say that this time, a point is a member of our club if its distance to $(0,3)$ is the same as its distance to $y = -3$.

The resulting shape looks something like this:

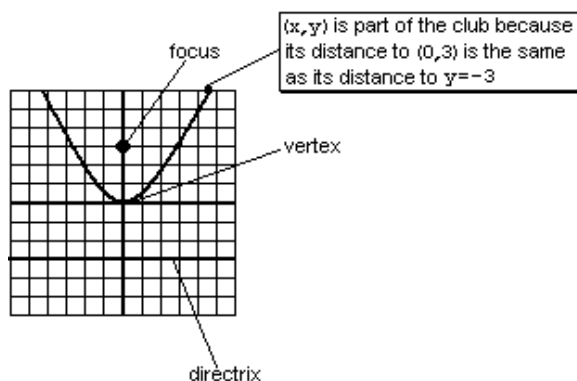
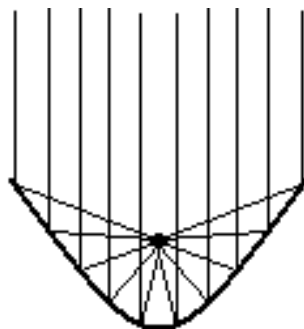


Figure 12.2

You may recall that a circle is entirely defined by its center—but the center is not, itself, a part of the circle. In a similar way, the focus and directrix define a parabola; but neither the focus, nor any point on the directrix, is a part of the parabola. The vertex, on the other hand—the point located directly between the focus and the directrix—is a part of the parabola.

One of the obvious questions you might ask at this point is—who cares? It’s pretty obvious that circles come up a lot in the real world, but parabolas? It turns out that parabolas are more useful than you might think. For instance, many telescopes are based on parabolic mirrors. The reason is that all the light that comes in bounces off the mirror to the focus. The focus therefore becomes a point where you can see very dim, distant objects.



Parabolic mirror in a telescope.
All parallel light beams converge at the focus.

Figure 12.3

12.4.2 The Formula of a Parabola

We've already graphed parabolas in a previous chapter. As you may recall, we began with the simplest parabola, $y = x^2$, and permuted it.

- $x^2 + k$ moves it up by k
- $(x - h)^2$ moves it to the right by h
- Multiplying by a number in front stretches the graph vertically
- Multiplying by a negative number turns the graph upside-down.

Putting it all together, we arrive at:

Mathematical Formula for a Vertical Parabola

$y = a(x - h)^2 + k$ is a parabola with vertex (h, k) . If a is positive, it opens up; if a is negative, it opens down.

Parabolas can also be horizontal. For the most part, the concepts are the same. The simplest horizontal parabola is $x = y^2$, which has its vertex at the origin and opens to the right—from there, you can permute it. The directrix in this case is a vertical line.

Mathematical Formula for a Horizontal Parabola

$x = a(y - k)^2 + h$ is a parabola with vertex (h, k) . If a is positive, it opens to the right; if a is negative, it opens to the left.

At this point, there are two useful exercises that you may want to try.

First, compare the two equations. How are they alike, and how are they different?

Second, consider the horizontal parabola equation as a set of permutations of the basic form $x = y^2$. What is k doing to the parabola, and why? How about h , and a ?

12.5 Ellipses⁵

12.5.1 The Definition of an Ellipse

An ellipse is a sort of squashed circle, sometimes referred to as an oval.

Definition of an Ellipse

Take two points. (Each one is a focus; together, they are the foci.) An ellipse is the set of all points in a plane that have the following property: the distance from the point to one focus, plus the distance from the point to the other focus, is some constant.

They just keep getting more obscure, don't they? Fortunately, there is an experiment you can do, similar to the circle experiment, to show why this definition leads to an elliptical shape.

Experiment: Drawing the Perfect Ellipse

1. Lay a piece of cardboard on the floor.
2. Thumbtack one end of a string to the cardboard.
3. Thumbtack the other end of the string, elsewhere on the cardboard. The string should not be pulled taut: it should have some slack.
4. With your pen, pull the middle of the string back until it is taut.
5. Pull the pen all the way around the two thumbtacks, keeping the string taut at all times.
6. The pen will touch every point on the cardboard such that the distance to one thumbtack, plus the distance to the other thumbtack, is exactly one string length. And the resulting shape will be an ellipse. The cardboard is the "plane" in our definition, the thumbtacks are the "foci," and the string length is the "constant distance."

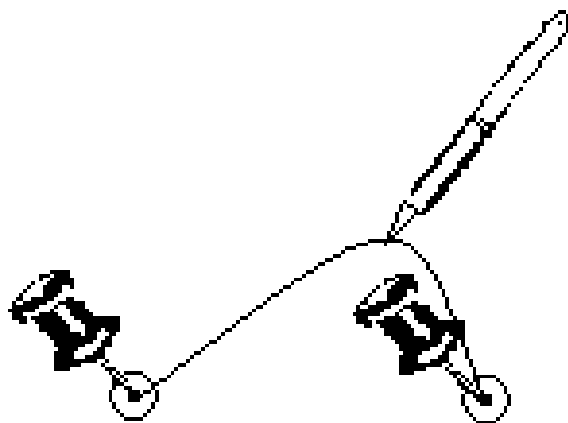


Figure 12.4

Do ellipses come up in real life? You'd be surprised how often. Here is my favorite example. For a long time, the orbits of the planets were assumed to be circles. However, this is incorrect: the orbit of a planet is actually in the shape of an ellipse. The sun is at one focus of the ellipse (not at the center). Similarly, the moon travels in an ellipse, with the Earth at one focus.

⁵This content is available online at <http://cnx.org/content/m18247/1.2/>.

12.5.2 The Formula of an Ellipse

With ellipses, it is crucial to start by distinguishing horizontal from vertical.

Mathematical Formula for an Ellipse with its Center at the Origin

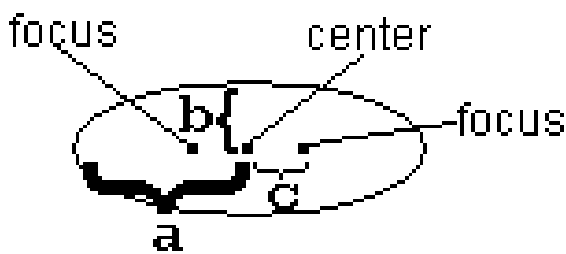
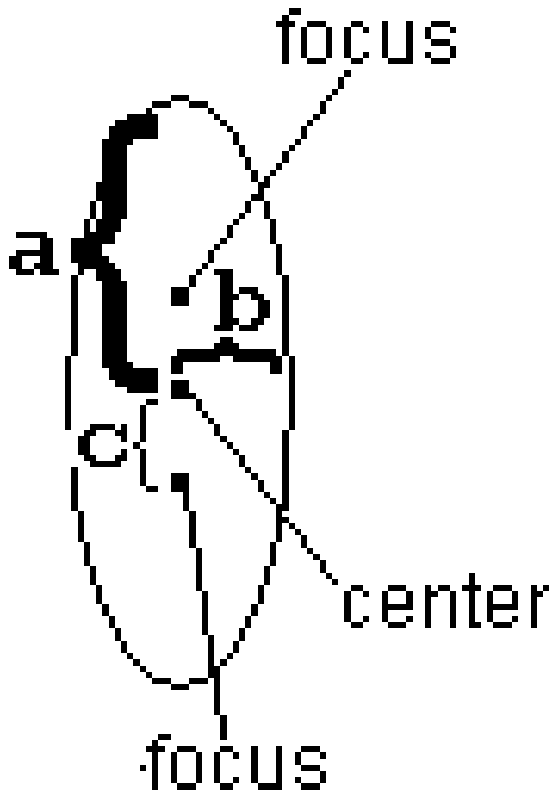
Horizontal	Vertical
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b)$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \ (a > b)$
	

Table 12.5

And of course, the usual rules of permutations apply. For instance, if we replace x with $x - h$, the ellipse moves to the right by h . So we have the more general form:

Mathematical Formula for an Ellipse with its Center at xxx(h,k)

Horizontal	Vertical
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \ (a > b)$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \ (a > b)$

Table 12.6

The key to understanding ellipses is understanding the three constants a , b , and c .

	Horizontal Ellipse	Vertical Ellipse
Where are the foci?	Horizontally around the center	Vertically around the center
How far are the foci from the center?	c	c
What is the “major axis”?	The long (horizontal) way across	The long (vertical) way across
How long is the major axis?	$2a$	$2a$
What is the “minor axis”?	The short (vertical) way across	The short (horizontal) way across
How long is the minor axis?	$2b$	$2b$
Which is biggest?	a is biggest. $a > b$, and $a > c$.	a is biggest. $a > b$, and $a > c$.
crucial relationship	$a^2 = b^2 + c^2$	$a^2 = b^2 + c^2$

Table 12.7

The following example demonstrates how all of these concepts come together in graphing an ellipse.

Example 12.4: Graphing an Ellipse

Graph $x^2 + 9y^2 - 4x + 54y + 49 = 0$	The problem. We recognize this as an ellipse because it has an x^2 and a y^2 term, and they both have the same sign (both positive in this case) but different coefficients (3 and 2 in this case).
$x^2 - 4x + 9y^2 + 54y = -49$	Group together the x terms and the y terms, with the number on the other side.
$(x^2 - 4x) + 9(y^2 + 6y) = -49$	Factor out the coefficients of the squared terms. In this case, there is no x^2 coefficient, so we just have to factor out the 9 from the y terms.
$(x^2 - 4x + 4) + 9(y^2 + 6y + 9) = -49 + 4 + 81$	Complete the square twice. Remember, adding 9 inside those parentheses is equivalent to adding 81 to the left side of the equation, so we must add 81 to the right side of the equation!
$(x - 2)^2 + 9(y + 3)^2 = 36$	Rewrite and simplify. Note, however, that we are still not in the standard form for an ellipse!
$\frac{(x-2)^2}{36} + \frac{(y+3)^2}{4} = 1$	Divide by 36. This is because we need a 1 on the right, to be in our standard form!
<i>continued on next page</i>	

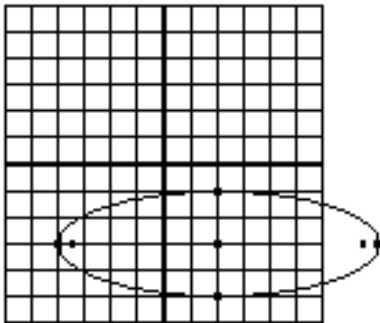
Center: (2,-3)	We read the center from the ellipse the same way as from a circle.
$a = 6$ $b = 2$	Since the denominators of the fractions are 36 and 4, a and b are 6 and 2. But which is which? The key is that, for ellipses, a is always greater than b . The larger number is a and the smaller is b .
Horizontal ellipse	Going back to the equation, we see that the a^2 (the larger denominator) was under the x , and the b^2 (the smaller) was under the y . This means our equation is a horizontal ellipse. (In a vertical ellipse, the a^2 would be under the y .)
$c = \sqrt{32} = 4\sqrt{2}$ (approximately $5\frac{1}{2}$)	We need c if we are going to graph the foci. How do we find it? From the relationship $a^2 = b^2 + c^2$ which always holds for ellipses.
	So now we can draw it. Notice a few features: The major axis is horizontal since this is a horizontal ellipse. It starts a to the left of center, and ends a to the right of center. So its length is $2a$, or 12 in this case. The minor axis starts b above the center and ends b below, so its length is 4. The foci are about $5\frac{1}{2}$ from the center.

Table 12.8

12.6 Hyperbolas⁶

12.6.1 The Definition of a Hyperbola

A hyperbola is the strangest-looking shape in this section. It looks sort of like two back-to-back parabolas. However, those shapes are not exactly parabolas, and the differences are very important.

Surprisingly, the definition and formula for a hyperbola are very similar to those of an ellipse.

Definition of a Hyperbola

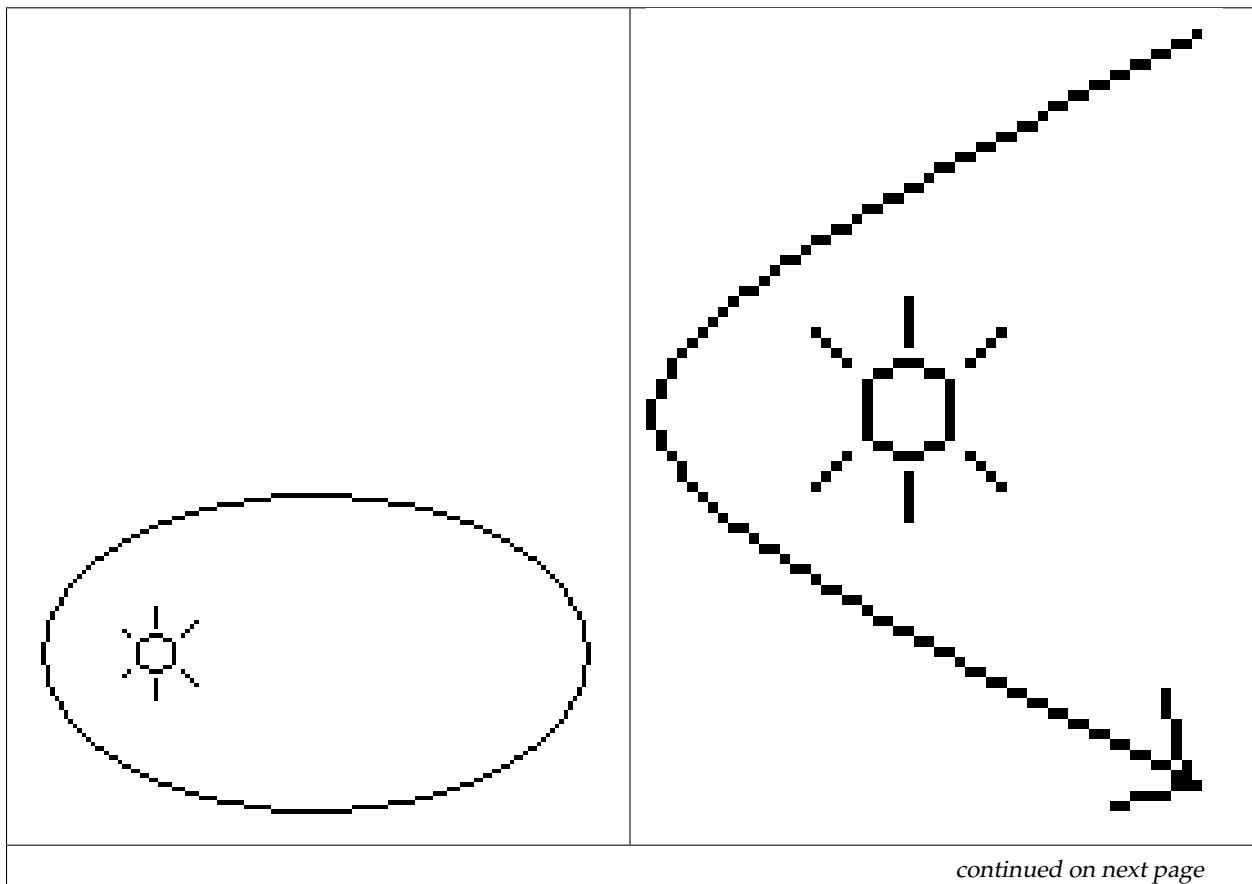
Take two points. (Each one is a focus; together, they are the foci.) A hyperbola is the set of all points in a plane that have the following property: the distance from the point to one focus, minus the distance from the point to the other focus, is some constant.

⁶This content is available online at <<http://cnx.org/content/m18249/1.2/>>.

The entire definition is identical to the definition of an ellipse, with one critical change: the word “plus” has been changed to “minus.”

One use of hyperbolas comes directly from this definition. Suppose two people hear the same noise, but one hears it ten seconds earlier than the first one. This is roughly enough time for sound to travel 2 miles. So where did the sound originate? Somewhere 2 miles closer to the first observer than the second. This places it somewhere on a hyperbola: the set of all points such that the distance to the second point, minus the distance to the first, is 2.

Another use is astronomical. Suppose a comet is zooming from outer space into our solar system, passing near (but not colliding with) the sun. What path will the comet make? The answer turns out to depend on the comet’s speed.



continued on next page

If the comet's speed is low, it will be trapped by the sun's gravitational pull. The resulting shape will be an elliptical orbit.	If the comet's speed is high, it will escape the sun's gravitational pull. The resulting shape will be half a hyperbola.
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Table 12.9

We see in this real life example, as in the definitions, a connection between ellipses and hyperbolas.

12.6.2 The Formula of an Hyperbola

With hyperbolas, just as with ellipses, it is crucial to start by distinguishing horizontal from vertical. It is also useful to pay close attention to which aspects are the same as ellipses, and which are different.

Mathematical Formula for a Hyperbola with its Center at the Origin

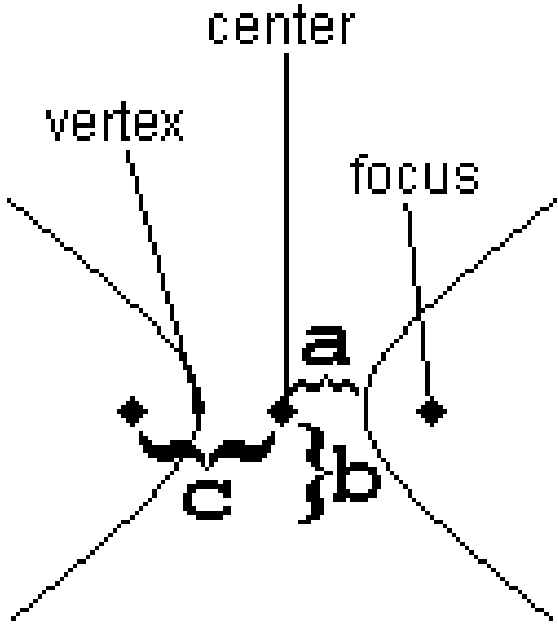
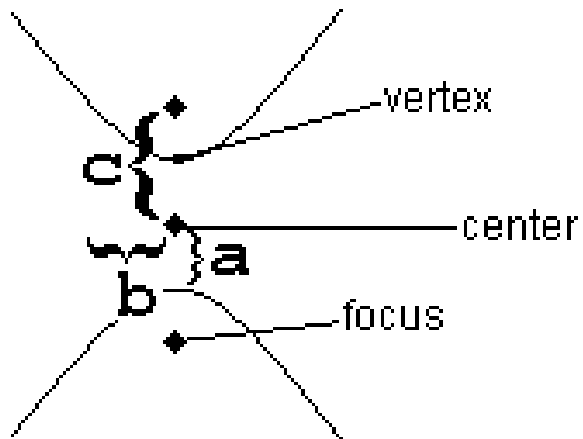
Horizontal	Vertical
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
	

Table 12.10

And of course, the usual rules of permutations apply. For instance, if we replace x with $x - h$, the hyperbola moves to the right by h . So we have the more general form:

Formula for a Hyperbola with its Center at (h,k)

Horizontal	Vertical
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Table 12.11

The key to understanding hyperbolas is understanding the three constants a , b , and c .

	Horizontal Hyperbola	Vertical Hyperbola
Where are the foci?	Horizontally around the center	Vertically around the center
How far are the foci from the center?	c	c
What is the “transverse axis”?	The (horizontal) line from one vertex to the other	The (vertical) line from one vertex to the other
How long is the transverse axis?	$2a$	$2a$
Which is biggest?	c is biggest. $c > a$, and $c > b$.	c is biggest. $c > a$, and $c > b$.
crucial relationship	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$

Table 12.12

Having trouble keeping it all straight? Let’s make a list of similarities and differences.

Similarities between Hyperbolas and Ellipses

- The formula is identical, except for the replacement of $a+$ with $a-$.
- The definition of a is very similar. In a horizontal ellipse, you move horizontally a from the center to the edges of the ellipse. (This defines the major axis.) In a horizontal hyperbola, you move horizontally a from the center to the vertices of the hyperbola. (This defines the transverse axis.)
- b defines a different, perpendicular axis.
- The definition of c is identical: the distance from center to focus.

Differences Between Hyperbolas and Ellipses

- The biggest difference is that for an ellipse, a is always the biggest of the three variables; for a hyperbola, c is always the biggest. This should be evident from looking at the drawings (the foci are inside an ellipse, outside a hyperbola). However, this difference leads to several other key distinctions.
- For ellipses, $a^2 = b^2 + c^2$. For hyperbolas, $c^2 = a^2 + b^2$.
- For ellipses, you tell whether it is horizontal or vertical by looking at which denominator is greater, since a must always be bigger than b . For hyperbolas, you tell whether it is horizontal or vertical by looking at which variable has a positive sign, the x^2 or the y^2 . The relative sizes of a and b do not distinguish horizontal from vertical.

In the example below, note that the process of getting the equation in standard form is identical with hyperbolas and ellipses. The extra last step—rewriting a multiplication by 4 as a division by $\frac{1}{4}$ —can come up with ellipses as easily as with hyperbolas. However, it did not come up in the last example, so it is worth taking note of here.

Example 12.5: Putting a Hyperbola in Standard Form

Graph $3x^2 - 12y^2 - 18x - 24y + 12 = 0$	The problem. We recognize this as a hyperbola because it has an x^2 and a y^2 term, and have different signs (one is positive and one negative).
<i>continued on next page</i>	

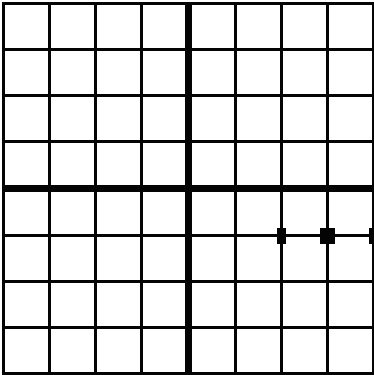
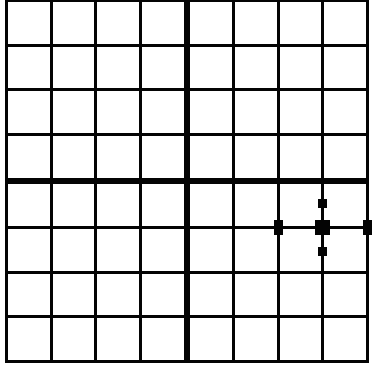
$3x^2 - 18x - 12y^2 - 24y = -12$	Group together the x terms and the y terms, with the number on the other side.
$3(x^2 - 6x) - 12(y^2 + 2y) = -12$	Factor out the coefficients of the squared terms. In the case of the y^2 for this particular equation, the coefficient is minus 12.
$3(x^2 - 6x + 9) - 12(y^2 + 2y + 1) = -12 + 27 - 12$	Complete the square twice. Adding 9 inside the first parentheses adds 27; adding 1 inside the second set subtracts 12.
$3(x - 3)^2 - 12(y + 1)^2 = 3$	Rewrite and simplify.
$(x - 3)^2 - 4(y + 1)^2 = 1$	Divide by 3, to get a 1 on the right. Note, however, that we are still not in standard form, because of the 4 that is multiplied by $(y + 1)^2$. The standard form has numbers in the denominator, but not in the numerator.
$(x - 3)^2 - \frac{(y + 1)^2}{\frac{1}{4}} = 1$	Dividing by $\frac{1}{4}$ is the same as multiplying by 4, so this is still the same equation. But now we are in standard form, since the number is on the bottom.

Table 12.13

However, the process of graphing a hyperbola is quite different from the process of graphing an ellipse. Even here, however, some similarities lurk beneath the surface.

Example 12.6: Graphing a Hyperbola in Standard Form

Graph $(x - 3)^2 - \frac{(y + 1)^2}{\frac{1}{4}} = 1$	The problem, carried over from the example above, now in standard form.
Center: (3,-1)	Comes straight out of the equation, both signs changed, just as with circles and ellipses.
$a = 1$ $b = \frac{1}{2}$	The square roots of the denominators, just as with the ellipse. But how do we tell which is which? In the case of a hyperbola, the a always goes with the positive term. In this case, the x^2 term is positive, so the term under it is a^2 .
<i>continued on next page</i>	

Horizontal hyperbola	Again, this is because the x^2 term is positive. If the y^2 were the positive term, the hyperbola would be vertical, and the number under the y^2 term would be considered a^2.
$c = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$	Remember that the relationship is different: for hyperbolas, $c^2 = a^2 + b^2$
	Now we begin drawing. Begin by drawing the center at (3,-1). Now, since this is a horizontal ellipse, the vertices will be aligned horizontally around the center. Since $a = 1$, move 1 to the left and 1 to the right, and draw the vertices there.
	In the other direction—vertical, in this case—we have something called the “conjugate axis.” Move up and down by b ($\frac{1}{2}$ in this case) to draw the endpoints of the conjugate axis. Although not part of the hyperbola, they will help us draw it.
<i>continued on next page</i>	

	Draw a box that goes through the vertices and the endpoints of the conjugate axis. The box is drawn in dotted lines to show that it is not the hyperbola.
	Draw diagonal lines through the corners of the box—also dotted, because they are also not the hyperbola. These lines are called the asymptotes, and they will guide you in drawing the hyperbola. The further it gets from the vertices, the closer the hyperbola gets to the asymptotes. However, it never crosses them.
	Now, at last, we are ready to draw the hyperbola. Beginning at the vertices, approach—but do not cross!—the asymptotes. So you see that the asymptotes guide us in setting the width of the hyperbola, performing a similar function to the latus rectum in parabolas.

Table 12.14

The hyperbola is the most complicated shape we deal with in this course, with a lot of steps to memorize.

But there is also a very important concept hidden in all that, and that is the concept of an asymptote. Many functions have asymptotes, which you will explore in far greater depth in more advanced courses. An asymptote is a line that a function approaches, but never quite reaches. The asymptotes are the

easiest way to confirm that a hyperbola is not actually two back-to-back parabolas. Although one side of a hyperbola resembles a parabola superficially, parabolas do not have asymptotic behavior—the shape is different.

Remember our comet? It flew into the solar system at a high speed, whipped around the sun, and flew away in a hyperbolic orbit. As the comet gets farther away, the sun's influence becomes less important, and the comet gets closer to its “natural” path—a straight line. In fact, that straight line is the asymptote of the hyperbolic path.

Before we leave hyperbolas, I want to briefly mention a much simpler equation: $y = \frac{1}{x}$. This is the equation of a diagonal hyperbola. The asymptotes are the x and y axes.

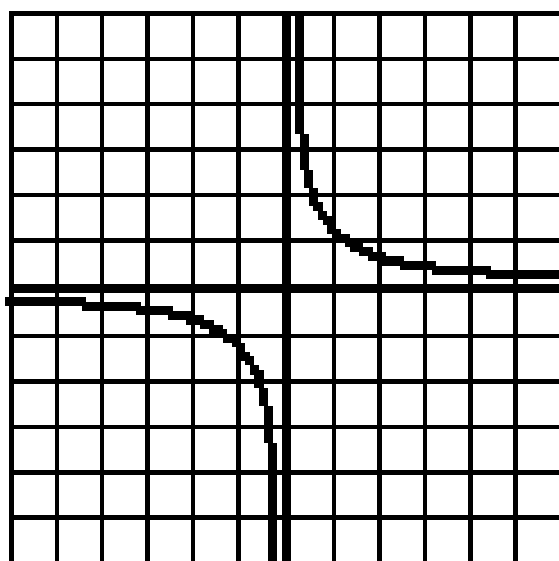


Figure 12.5: $y = \frac{1}{x}$

Although the equation looks completely different, the shape is identical to the hyperbolas we have been studying, except that it is rotated 45° .

12.7 A Brief Recap: How Do You Tell What Shape It Is?⁷

If it has...	Then it's a...	Example	Horizontal or Vertical?
<i>continued on next page</i>			

⁷This content is available online at <http://cnx.org/content/m18270/1.1/>.

No squared terms	Line	$2x + 3y = 7$	
One squared term	Parabola	$2x^2 - 10x + 7y = 9$	If you have an x^2 but no y^2 , you're a horizontal parabola. If you have a y^2 but no x^2 , vertical.
Two squared terms with the same coefficient	Circle	$3x^2 + 3y^2 + 6x + 3y = 2$	
Two squared terms with different coefficients but the same sign	Ellipse	$2x^2 + 3y^2 + 6x + 6y = 12$	The difference between vertical ellipses and horizontal is based on which squared term has the larger coefficient.
Two squared terms with different signs	Hyperbola	$3x^2 - 3y^2 + 6x + 3y = 2$	The difference between vertical hyperbolas and horizontal is based on which squared term is positive.

Table 12.15

Note that all of this is based only on the squared terms! The other terms matter in terms of graphing, but not in terms of figuring out what shape it is.

Chapter 13

Sequences and Series

13.1 Sequences¹

A sequence is a list of numbers: like 4,9,3,2,17.

An arithmetic sequence is a list where each number is generated by adding a constant to the previous number. An example is 10,13,16,19,22,25. In this example, the first term (t_1) is 10, and the “common difference” (d)—that is, the difference between any two adjacent numbers—is 3. Another example is 25,22,19,16,13,10. In this example $t_1 = 25$, and $d = (-3)$. In both of these examples, n (the number of terms) is 6.

A geometric sequence is a list where each number is generated by multiplying a constant by the previous number. An example is 2,6,18,54,162. In this example, $t_1 = 2$, and the “common ratio” (r)—that is, the ratio between any two adjacent numbers—is 3. Another example is 162,54,18,6,2. In this example $t_1 = 162$, and $r = \frac{1}{3}$. In both examples $n = 5$.

A recursive definition of a sequence means that you define each term based on the previous. So the recursive definition of an arithmetic sequence is $t_n = t_{n-1} + d$, and the recursive definition of a geometric sequence is $t_n = rt_{n-1}$.

An explicit definition of an arithmetic sequence means you define the n^{th} term without making reference to the previous term. This is more useful, because it means you can find (for instance) the 20th term without finding all the other terms in between.

To find the explicit definition of an arithmetic sequence, you just start writing out the terms. The first term is always t_1 . The second term goes up by d so it is $t_1 + d$. The third term goes up by d again, so it is $(t_1 + d) + d$, or in other words, $t_1 + 2d$. So we get a chart like this.

t_1	t_2	t_3	t_4	t_5
t_1	$t_1 + d$	$t_1 + 2d$	$t_1 + 3d$	$t_1 + 4d$

Table 13.1

...and so on. From this you can see the generalization that $t_n = t_1 + (n - 1)d$, which is the explicit definition we were looking for.

¹This content is available online at <<http://cnx.org/content/m19076/1.1/>>.

The explicit definition of a geometric sequence is arrived at the same way. The first term is t_1 ; the second term is r times that, or $t_1 r$; the third term is r times that, or $t_1 r^2$; and so on. So the general rule is $t_n = t_1 r^{n-1}$. Read this as: “ t_1 multiplied by r , $(n - 1)$ times.”

13.2 Series²

A series is a list of numbers—like a sequence—but instead of listing them, you add them all up. For instance, $4+9+3+2+17$. (This particular series adds up to 35.)

One way to compactly represent a series is with “summation notation,” which looks like this:

$$\sum_{n=3}^7 n^2 \quad (13.1)$$

The big funny-looking thing in the middle is the Greek letter uppercase Sigma, and it indicates a series. To “unpack” this notation, start counting at the bottom ($n = 3$), and stop when you reach the stop ($n = 7$). For each term, plug that value of n into the given formula (n^2). So this particular formula, which we can read as “the sum as n goes from 3 to 7 of n^2 ”, simply means:

$$3^2 + 4^2 + 5^2 + 6^2 + 7^2$$

13.2.1 Arithmetic Series

If you add up all the terms of an arithmetic sequence, you have an arithmetic series. For instance, $10 + 13 + 16 + 19 + 22 + 25 = 105$.

There is a “trick” that can be used to add up the terms of any arithmetic series. While this trick may not save much time with a 6-item series like the one above, it can be very useful if adding up longer series. The trick is to work from the outside in.

Consider the example given above: $10 + 13 + 16 + 19 + 22 + 25$. Looking at the first and last terms, $10 + 25 = 35$. Going in, to the second and next-to-last terms, $13 + 22 = 35$. Finally, the two inside numbers $16 + 19 = 35$. So we can see that the sum of the whole thing is $3 * 35$.

Pause here and check the following things.

- You understand the calculation that was done for this particular example.
- You understand that this “trick” will work for any arithmetic series.
- You understand that this trick will not work, in general, for series that are not arithmetic.

If we apply this trick to the generic arithmetic series, we get a formula that can be used to sum up any arithmetic series.

Every arithmetic series can be written as follows:

$$t_1 + (t_1 + d) + (t_1 + 2d) \dots (t_n - d) + t_n$$

If you add the first and last terms, you get $t_1 + t_n$. Ditto for the second and next-to-last terms, and so on. How many such pairs will there be in the whole series? Well, there are n terms, so there are $\frac{n}{2}$ pairs. So the sum for the whole series is $\frac{n}{2} (t_1 + t_n)$.

²This content is available online at <<http://cnx.org/content/m19074/1.1/>>.

13.2.2 Geometric Series

If you add up all the terms of a geometric sequence, you have a geometric series. The “arithmetic series trick” will not work on such a series; however, there is a different trick we can use. As an example, let’s find the sum $2 + 6 + 18 + 54 + 162$.

We begin by calling the sum of this series S :

$$S = 2 + 6 + 18 + 54 + 162$$

Now, if you multiply both sides of this equation by 3, you get the first equation I have written below. (The second equation below is just copied from above.)

$$3S = 6 + 18 + 54 + 162 + 486 \text{ (*confirm this for yourself!)}$$

$$S = 2 + 6 + 18 + 54 + 162$$

Here comes the key moment in the trick: subtract the two equations. This leaves you with:

$$2S = 486 - 2, \text{ so } S = 242$$

Once again, pause to convince yourself that this will work on all geometric series, but only on geometric series.

Finally—once again—we can apply this trick to the generic geometric series to find a formula. So we begin with $t_1 + t_1r + t_1r^2 + t_1r^3 \dots t_1r^{n-1}$ and write...

$$rS = t_1r + t_1r^2 + t_1r^3 + t_1r^4 \dots t_1r^{n-1} + t_1r^n \text{ (*confirm this!)}$$

$$S = t_1 + t_1r + t_1r^2 + t_1r^3 + \dots t_1r^{n-1}$$

Again, subtracting and solving, we get...

$$rS - S = t_1r^n - t_1$$

$$S(r - 1) = t_1(r^n - 1)$$

$$S = t_1 \frac{r^n - 1}{r - 1}$$

So there we have it: a general formula for the sum of any finite geometric series, with the first term t_1 , the common ratio r , and a total of n terms.

13.3 Proof by Induction³

“Induction” is a method of proving something. Once again, let’s start with an example.

Consider the sum $\sum_{i=1}^n \frac{1}{i(i+1)}$. In other words, $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)}$. This is neither arithmetic nor geometric, so none of our established tricks will work on it. How can we find the sum of such a series?

Students are often surprised to hear that mathematicians typically begin such problems by looking for a pattern. What does this series do for the first few terms?

- 1 term: $\frac{1}{1 \times 2} = \frac{1}{2}$

³This content is available online at <<http://cnx.org/content/m19075/1.1/>>.

- 2 terms: $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$
- 3 terms: $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$

At this point, you might already suspect the pattern. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$ could it be that the next term will be $\frac{4}{5}$? Let's find out.

- 4 terms: $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}$

It seems to work. The next term will probably be $\frac{5}{6}$, and then $\frac{6}{7}$, and so on. Stop for a moment and make sure you see the pattern. Then, see if you can express that pattern using mathematical notation instead of words. (Try this yourself before you keep reading!)

The pattern can be expressed like this:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Stop for a moment and make sure you know where we are. What we have done is figured out a pattern to the answers, and shown that the pattern works for $n = 1$, $n = 2$, $n = 3$, and $n = 4$. Based on this pattern, we expect that if we added up $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{100 \times 101}$, we would get $\frac{100}{101}$.

But we have not yet proven anything. Maybe the pattern breaks down for $n = 5$. Or maybe it works for all the n -values from 1 to 1000, and then suddenly stops working. We cannot possibly test all the values in the world, one by one.

This is where the proof by induction comes in. It gives us a way to prove that such a pattern will continue to hold forever.

An inductive proof, in general, consists of two steps. The first step is to show that the pattern holds when $n = 1$. The second step is to show that, whenever this pattern holds for some particular n , it will also hold for the next n . If it holds for $n = 5$, then it must hold for $n = 6$. If it works for $n = 99$, then it must also work for $n = 100$. Once we have proven that, in general, then we will have shown that it works for all n values.

Example 13.1: Proof by Induction

Inductive Proof of:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

First Step:

Show that it works for $n = 1$

1 term: $\frac{1}{1 \times 2} = \frac{1}{2}$

Second Step:

Show that it works for $n + 1$, assuming it works for some n

For $n + 1$, the left side of the equation looks like:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+1+1)}$$

and the right side of the equation looks like:

$$\frac{n+1}{n+1+1} \tag{13.2}$$

So we want to see if:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} \stackrel{?}{=} \frac{n+1}{n+2}$$

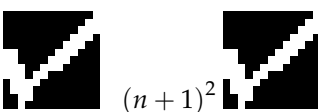
Now comes the key step.

If this pattern held for n , then the first n terms of the left side—that is, all but the last (new) term—must add up to $\frac{n}{n+1}$. So we do that substitution:

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \stackrel{?}{=} \frac{n+1}{n+2}$$

All that remains, now, is the algebra to show that equation is true. Get a common denominator:

$$\frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} \stackrel{?}{=} \frac{(n+1)^2}{(n+1)(n+2)}$$

$$n^2 + 2n + 1 \quad (n+1)^2$$


The algebra here all comes from our unit on rational expressions: you may want to take a moment to make sure you can follow it. But don't let the algebra distract you from the main point, which is what we proved in the second step. We proved that the formula works for $n+1$. But in the middle of that proof, we assumed that it works for the previous term, n . In doing so, we proved that if it works for 1, it must also work for 2; if it works for 2, it must also work for 3; and so on. This amounts, then, to a proof that the pattern holds forever.

Chapter 14

Probability

14.1 Probability Concepts¹

If you flip a coin, what is the chance of getting heads? That's easy: 50/50. In the language of probability, we say that the probability is $\frac{1}{2}$. That is to say, half the time you flip coins, you will get heads.

So here is a harder question: if you flip two coins, what is the chance that you will get heads both times? I asked this question of my son, who has good mathematical intuition but no training in probability. His immediate answer: $\frac{1}{3}$. There are three possibilities: two heads, one heads and one tails, and two tails. So there is a $\frac{1}{3}$ chance of getting each possibility, including two heads. Makes sense, right?

But it is not right. If you try this experiment 100 times, you will not find about 33 "both heads" results, 33 "both tails," and 33 "one heads and one tails." Instead, you will find something much closer to: 25 "both heads," 25 "both tails," and 50 "one of each" results. Why?

Because hidden inside this experiment are actually four different results, each as likely as the others. These results are: heads-heads, heads-tails, tails-heads, and tails-tails. Even if you don't keep track of what "order" the coins flipped in, heads-tails is still a different result from tails-heads, and each must be counted.

And what if you flip a coin three times? In this case, there are actually eight results. In case this is getting hard to keep track of, here is a systematic way of listing all eight results.

First Coin	Second Coin	Third Coin	End Result
Heads	Heads	Heads	HHH
		Tails	HHT
	Tails	Heads	HTH
		Tails	HTT
continued on next page			

¹This content is available online at <<http://cnx.org/content/m19073/1.2/>>.

Tails	Heads	Heads	THH
		Tails	THT
	Tails	Heads	TTH
		Tails	TTT

Table 14.1

When you make a table like this, the pattern becomes apparent: each new coin doubles the number of possibilities. The chance of three heads in a row is $\frac{1}{8}$. What would be the chance of four heads in row?

Let's take a slightly more complicated—and more interesting—example. You are the proud inventor of the SongWriter 2000tm.



Figure 14.1

The user sets the song speed ("fast," "medium," or "slow"); the volume ("loud" or "quiet"); and the style ("rock" or "country"). Then, the SongWriter automatically writes a song to match.

How many possible settings are there? You might suspect that the answer is $3 + 2 + 2 = 7$, but in fact there are many more than that. We can see them all on the following "tree diagram."

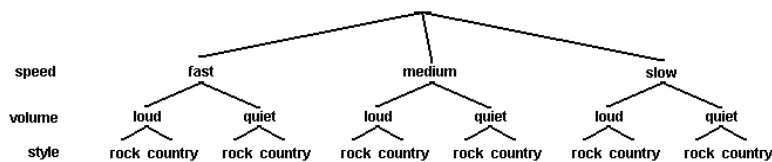


Figure 14.2

If you start at the top of a tree like this and follow all the way down, you end up with one particular kind of song: for instance, “fast loud country song.” There are 12 different song types in all. This comes from multiplying the number of settings for each knob: $3 \times 2 \times 2 = 12$.

Now, suppose the machine has a “Randomize” setting that randomly chooses the speed, volume, and style. What is the probability that you will end up with a loud rock song that is not slow? To answer a question like this, you can use the following process.

1. Count the total number of results (the “leaves” in the tree) that match your criterion. In this case there are 2: the “fast-loud-rock” and “medium-loud-rock” paths.
2. Count the total number of results: as we said previously, there are 12.
3. Divide. The probability of a non-slow loud rock song is $2/12$, or $1/6$.

Note that this process will always give you a number between 0 (no results match) and 1 (all results match). Probabilities are always between 0 (for something that never happens) and 1 (for something that is guaranteed to happen).

But what does it really mean to say that “the probability is $1/6$?” You aren’t going to get $1/6$ of a song. One way to make this result more concrete is to imagine that you run the machine on its “Randomize” setting 100 times. You should expect to get non-slow loud rock songs 1 out of every 6 times; roughly 17 songs will match that description. This gives us another way to express the answer: there is a 17% probability of any given song matching this description.

14.1.1 The multiplication rule

We can look at the above problem another way.

What is the chance that any given, randomly selected song will be non-slow? $\frac{2}{3}$. That is to say, 2 out of every three randomly chosen songs will be non-slow.

Now...out of those $\frac{2}{3}$, how many will be loud? Half of them. The probability that a randomly selected song is both non-slow and loud is half of $\frac{2}{3}$, or $\frac{1}{2} \times \frac{2}{3}$, or $\frac{1}{3}$.

And now, out of that $\frac{1}{3}$, how many will be rock? Again, half of them: $\frac{1}{2} \times \frac{1}{3}$. This leads us back to the conclusion we came to earlier: $1/6$ of randomly chosen songs will be non-slow, loud, rock songs. But it also gives us an example of a very general principle that is at the heart of all probability calculations:

When two events are independent, the probability that they will both occur is the probability of one, multiplied by the probability of the other.

What does it mean to describe two events as “independent?” It means that they have no effect on each other. In real life, we know that rock songs are more likely to be fast and loud than slow and quiet. Our machine, however, keeps all three categories independent: choosing “Rock” does not make a song more likely to be fast or slow, loud or quiet.

In some cases, applying the multiplication rule is very straightforward. Suppose you generate two different songs: what is the chance that they will both be fast songs? The two songs are independent of each other, so the chance is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

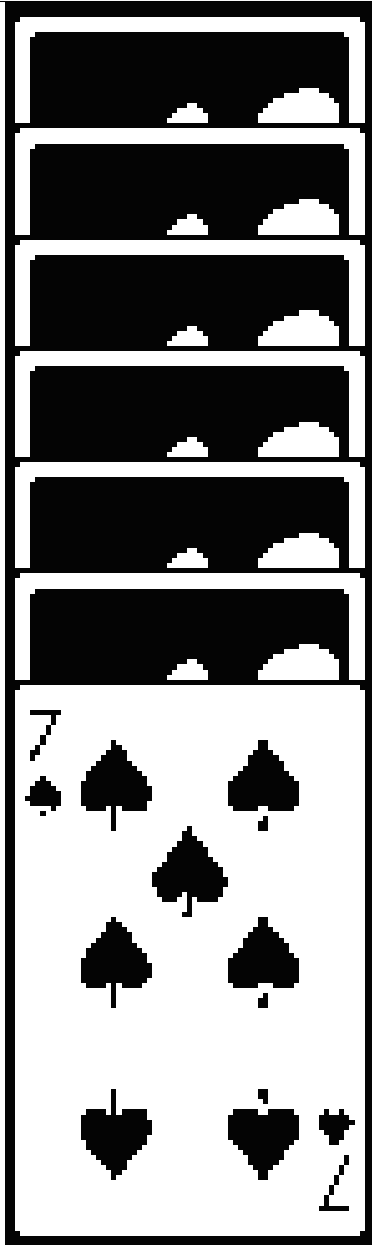
Now, suppose you generate five different songs. What is the chance that they will all be fast? $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$, or $\left(\frac{1}{3}\right)^5$, or 1 chance in 243. Not very likely, as you might suspect!

Other cases are less obvious. Suppose you generate five different songs. What is the probability that none of them will be a fast song? The multiplication rule tells us only how to find the probability of “this and that”; how can we apply it to this question?

The key is to reword the question, as follows. What is the chance that the first song will not be fast, and the second song will not be fast, and the third song will not be fast, and so on? Expressed in this way, the question is a perfect candidate for the multiplication rule. The probability of the first song being non-fast is $\frac{2}{3}$. Same for the second, and so on. So the probability is $(\frac{2}{3})^5$, or 32/243, or roughly 13%.

Based on this, we can easily answer another question: if you generate five different songs, what is the probability that at least one of them will be fast? Once again, the multiplication rule does not apply directly here: it tells us “this and that,” not “this or that.” But we can recognize that this is the opposite of the previous question. We said that 13% of the time, none of the songs will be fast. That means that the other 87% of the time, at least one of them will!

14.2 Permutations²



In the game of “Solitaire” (also known as “Patience” or “Klondike”), seven cards are dealt out at the beginning, as shown to the left: one face-up, and the other six face-down. (A bunch of other cards are dealt out too, but let’s ignore that right now.) A complete card deck has 52 cards. Assuming that all you know is the 7 of spades showing, how many possible “hands” (the other six cards) could be showing underneath? What makes this a “permutations” problem is that order matters: if an ace is hiding somewhere in those six cards, it makes a big difference if the ace is on the first position, the second, etc. Permutations problems can always be addressed as an example of the multiplication rule, with one small twist.

Table 14.2

- Question - How many cards might be in the first position, directly under the showing 7?
- Answer - 51. That card could be anything except the 7 of spades.
- Question - For any given card in first position, how many cards might be in second position?
- Answer - 50. The seven of spades, and the next card, are both “spoken for.” So there are 50 possibilities left in this position.
- Question - So how many possibilities are there for the first two positions combined?
- Answer - 51×50 , or 2,550.
- Question - So how many possibilities are there for all six positions?
- Answer - $51 \times 50 \times 49 \times 48 \times 47 \times 46$, or approximately 1.3×10^{10} ; about 10 billion possibilities!

This result can be expressed (and typed into a calculator) more concisely by using factorials.

A “factorial” (written with an exclamation mark) means “multiply all the numbers from 1 up to this number.” So $5!$ means $1 \times 2 \times 3 \times 4 \times 5 = 120$.

What is $\frac{7!}{5!}$? Well, it is $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}{1 \times 2 \times 3 \times 4 \times 5}$, of course. Most of the terms cancel, leaving only $6 \times 7 = 42$.

And what about $\frac{51!}{45!}$? If you write out all the terms, you can see that the first 45 terms cancel, leaving $46 \times 47 \times 48 \times 49 \times 50 \times 51$, which is the number of permutations we want. So instead of typing into your calculator six numbers to multiply (or sixty numbers or six hundred, depending on the problem), you can always find the answer to a permutation problem by dividing two factorials. In many calculators, the factorial option is located under the “probability” menu for this reason.

14.3 Combinations³

Let’s start once again with a deck of 52 cards. But this time, let’s deal out a poker hand (5 cards). How many possible poker hands are there?

At first glance, this seems like a minor variation on the Solitaire question above. The only real difference is that there are five cards instead of six. But in fact, there is a more important difference: order does not matter. We do not want to count “Ace-King-Queen-Jack-Ten of spades” and “Ten-Jack-Queen-King-Ace of spades” separately; they are the same poker hand.

To approach such question, we begin with the permutations question: how many possible poker hands are there, if order does matter? $52 \times 51 \times 50 \times 49 \times 48$, or $\frac{52!}{47!}$. But we know that we are counting every possible hand many different times in this calculation. How many times?

The key insight is that this second question—“How many different times are we counting, for instance, Ace-King-Queen-Jack-Ten of spades?”—is itself a permutations question! It is the same as the question “How many different ways can these five cards be rearranged in a hand?” There are five possibilities for the first card; for each of these, four for the second; and so on. The answer is $5!$ which is 120. So, since we have counted every possible hand 120 times, we divide our earlier result by 120 to find that there are $\frac{52!}{(47!)(5!)}$, or about 2.6 Million possibilities.

This question—“how many different 5-card hands can be made from 52 cards?”—turns out to have a surprisingly large number of applications. Consider the following questions:

- A school offers 50 classes. Each student must choose 6 of them to fill out a schedule. How many possible schedules can be made?

³This content is available online at <<http://cnx.org/content/m19071/1.1/>>.

- A basketball team has 12 players, but only 5 will start. How many possible starting teams can they field?
- Your computer contains 300 videos, but you can only fit 10 of them on your iPod. How many possible ways can you load your iPod?

Each of these is a combinations question, and can be answered exactly like our card scenario. Because this type of question comes up in so many different contexts, it is given a special name and symbol. The

last question would be referred to as “300 choose 10” and written $\binom{300}{10}$. It is calculated, of course,

as $\frac{300!}{(290!)(10!)}$ for reasons explained above.

Glossary

D Domain

The domain of a function is all the numbers that it can successfully act on. Put another way, it is all the numbers that can go into the function.

I Inverse Function

$f^{-1}(x)$ is defined as the inverse function of $f(x)$ if it consistently reverses the $f(x)$ process. That is, if $f(x)$ turns a into b , then $f^{-1}(x)$ must turn b into a . More concisely and formally, $f^{-1}(x)$ is the inverse function of $f(x)$ if $f(f^{-1}(x)) = x$.

L Linear Function

A function is said to be “linear” if every time the independent variable increases by 1, the dependent variable increases or decreases by the same amount.

R Range

The range of a function is all the numbers that it may possibly produce. Put another way, it is all the numbers that can come out of the function.

T The Vertical Line Test

If you can draw any vertical line that touches a graph in two places, then that graph violates the rule of consistency and therefore does not represent any function.

Index of Keywords and Terms

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Advanced Algebra II: Teacher's Guide

By:
Kenny Felder

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C O N N E X I O N S

Rice University, Houston, Texas

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The Long Rambling Philosophical Introduction¹

What you're holding in your hand is much closer to a set of detailed lesson plans than to a traditional textbook. As you read through it, your first reaction may be "Who does he think he is, telling me exactly what to say and when to say it?"

Please don't take it that way. Take it this way instead.

Over a period of time, I have developed a set of in-class assignments, homeworks, and lesson plans, that work for me and for other people who have tried them. If I give you the in-class assignments and the homeworks, but not the lesson plans, you only have 2[U+2044]3 of the story; and it may not make sense without the other third. So instead, I am giving you everything: the in-class assignments and the homeworks (gathered together in the student book), the detailed explanations of all the concepts (the other student book), and the lesson plans (this document). Once you read them over, you will know exactly what I have done.

What do you do then? You may choose to follow my plan exactly, for a number of reasons—because it worked for me, or because it looks like a good plan to you, or just because you have enough other things to do without planning a lesson that I've already planned. On the other hand, you may choose to do something quite different, that incorporates my ideas in some form that I never imagined. This book is not a **proscription**, in other words, but a **resource**.

OK, with that out of the way...suppose you decide that you do want to follow my plan, exactly or pretty closely. Here's what you do.

- Right now, you read this whole introduction—despite the title, it really does contain useful information about these materials.
- Before beginning each new unit, you read my "conceptual explanation" of that unit, so you know what I'm trying to achieve.
- Each day before class, you carefully read over my lesson plan (in this document), and the in-class assignment and homework (in the student book), so you know what I'm doing and why I'm doing it.

A Typical Day in Mr. Felder's Class (...and why you care)

At the risk of repeating myself, let me emphasize—I'm not trying to insult you by suggesting that my way is the only right way to run a class. But it will help you understand these materials if you understand how I use them.

I begin each day by taking questions on **last night's** homework. I answer any and all questions. This may take five minutes, or it may take the entire class period: I don't stop until everyone is perfectly comfortable with last night's homework.

Why is that so important? Because, very often, the homework introduces new concepts that the students have **never seen in class before**. For instance, very early in the first unit, I introduce the idea of

¹This content is available online at <<http://cnx.org/content/m19503/1.1/>>.

“permuting” graphs: for instance, if you add 3 to any function, the graph moves up by three units. This concept never comes up in class, in any form—it is developed entirely on a homework. So it’s vitally important to debrief them the next day and make sure that they got, not only the right answers, but the point.

After the homework is covered, I begin a new topic. This is almost (almost!) never done in a long lecture. Sometimes it happens in a class discussion; sometimes it happens in a TAPPS exercise (more on that when we do our first one); most often, it happens in an **in-class assignment**. These assignments should almost always be done in pairs or groups of three, very rarely individually. They generally require pretty high-level thinking. On a good day I can hear three or four heated arguments going on in different groups. Most of my class time is spent moving between different groups and helping them when they are stuck. In general, there is some particular **point** I want them to get from the exercise, and they will need that point to do the homework—so a lot of my job in class is to make sure that, before they leave, they got the point.

Timing

If you read through this entire document (which I do not recommend at one sitting), you get the illusion that I have everything planned down to the day. If I say “do this assignment in class, then do this homework,” they had better get that done in one day, or they will fall irretrievably behind.

Well, suppose you add it all up that way. Every “1-day assignment” (with homework) counts as one day, and what the heck, let’s allocate two days for every test (one day for preparation, using the “Sample Test”—and one day for the actual test). If you add it up that way, you will get a total of 91 days, or thereabouts. There are 180 days in the school year.

So what does that mean? Does it mean you will be done in one semester? No, of course not. It means, take your time and do it right.

For one thing, I believe in building in a lot of time for review. Ideally, two weeks before mid-terms and another two weeks before finals. (What I do during this time is cover one topic a day, with the students teaching each class.)

But even leaving that aside, one apparent day’s worth of material will sometimes take you two days to get through. You spend the whole day reviewing last night’s homework and you don’t even get to the new assignment. Or, you get to the end of the class and you realize that most of the groups are only half-way through the in-class assignment. Don’t rush it! It’s much more important to get today’s concept, and really make sure everyone has it, then to rush on to tomorrow. The way I see it, you have three reasonable choices.

1. If **most** of the class is **mostly** finished with the in-class assignment, it may make sense to say “Finish the in-class assignment tonight, and also do the homework.”
2. If most of the class is only half-way done, it may make sense to say “Finish the in-class assignment tonight, and we will do the homework in class tomorrow.” This puts you a half-day “behind” which is fine. **However, some in-class assignments really cannot be done at home...**they require too much group work or help from you. So...
3. Sometimes you just say “We’ll finish the in-class assignment tomorrow.” This puts you one day “behind” which is also fine.

Of course you need to pace yourself. But do it by tests, not by days. There are sixteen tests. If you are going at a clip that will get you through more or less that many tests by the end of the year, you’re doing fine. And even that isn’t exact—of course, some units will take longer than others. Personally, I would much rather skip the unit on Conics (the last unit) entirely, than lose the entire class by trying to rush through Exponents. (However, in real life, I do make it through the entire syllabus.)

Tests

At the end of every unit I have a “Sample Test.” This is for the students’ benefit as much as for yours: it makes a great study guide and/or homework. If you say “The homework tonight is the sample test.

Tomorrow we will go over any questions you have on the sample test, and on the topic in general—that will be your last chance to ask me questions! The next day will be our actual test,” then you are giving the students a great chance to bone up before the test. Doing this has dramatically improved my classes.

So what about your actual test? Of course, you may (or may not) want to **base** your test on mine. In that case, however, be careful about timing—some of my “Sample Tests” are actually too long to be a real test. But they are made up of actual questions that I have used on actual tests in the past—and in any case, calling them “Sample Tests” gets students’ attention better than calling them “Review Questions.”

By the way, although I do not generally recommend using **exactly** my questions—you want to change the numbers at least—it is sometimes OK to use **exactly** my extra credit. Even if they just did it, it often has enough real learning in it that it is worth giving them a few points if they took the time to look it over and/or ask about it.

Chapter 1

Functions

1.1 Introduction¹

This is the most important unit in the year, because it introduces many of the major “themes” that will run through the entire class. These themes include:

- What is a function?
- How are functions used to model things in the real world?
- What does it mean for two functions to be “equal”?
- How does a functional description $f(x) = 3x$ relate to a graph?

More detailed topics include the vertical line test (the “rule of consistency”), the “dependent” and “independent” variables, domain and range, composite functions, and inverse functions.

Also included in this unit is a quick review of graphing lines. It is assumed that students are mostly familiar with this topic from Algebra I.

1.2 The Function Game²

This game is an introduction to the idea of a function.

Begin by breaking the students into groups of three. In each group, one student is designated as the **leader** and another as the **recorder**. (Do this quickly and arbitrarily: “The shortest person is the leader and the tallest is the recorder” or some such. Assure them that the roles will rotate.) Go over the instructions (which are in the student packet): walk through a sample session, using the function “add five,” to make sure they understand who does what and what gets written down. In particular, make sure they understand how “add five” can be represented as “ $x + 5$ ”: that is, that x is being used to designate the number that comes **in**. You also want to mention **domain** in particular, and the idea that if you are doing $\frac{1}{x}$ and someone gives you a 0, it is “not in your domain.”

Then they can start. Your job is to circle around, keeping them on task, and helping students who are stuck by giving hints: “Do you notice anything in common about all the numbers you’ve gotten back?” or “Why are all negative numbers outside the domain?” or even “Try a 2 and see what happens.” Also, at 10 or 15 minute intervals, instruct them to switch roles.

Toward the end of the class period, interrupt briefly to talk about the word **function**. What is the leader representing? He is not a number. He is not a variable. He is a **process** that turns one number into another. That’s all a function is—a mechanical process that takes one number in, and spits a different number back

¹This content is available online at <<http://cnx.org/content/m19335/1.2/>>.

²This content is available online at <<http://cnx.org/content/m19342/1.2/>>.

out. Use the analogy of a little machine (you can draw it) with an input and an output. Functions are going to be the main focus of the entire year in Algebra II.

If some groups don't finish all the problems, that's OK: as long as they did enough to get the idea. If a group finishes early, tell them to start making up their own functions—challenge them to stump you!

Homework:

"Homework: The Function Game"

1.3 "The Real World"³

Begin by going over the homework. There are two key points to bring out, which did not come up in class yesterday.

1. If two functions always give the same answer, we say they are equal. This just cannot be stressed enough. Put this on the board:

$$3(x - 5) = 3x - 15$$

They've all seen this. Most of them know it is the "distributive property." But do they know what it **means**? Explain it very carefully. $3(x - 5)$ is a function that says "subtract 5, then multiply by 3." $3(x - 15)$ is a function that says "multiply by 3, then subtract 15." They are very different processes! But we say they are "equal" because no matter what number you plug in, they yield the same answer. So in the function game, there is no possible question you could ask that would tell you if the person is doing "subtract 5, then multiply by 3" or "multiply by 3, then subtract 5." Have the class give you a few numbers, and show how it works—for negative numbers, fractions, zero, **anything**.

2. The last problem on the homework brings up what I call the "rule of consistency." It is perfectly OK for a function to give the same answer for different questions. (For instance, x^2 turns both 3 and -3 into 9.) But it is **not** OK to give different answers to the **same** question. That is, if a function turns a 3 into a 9 once, then it will **always** turn a 3 into a 9.

Once those two points are very clearly made, and all questions answered, you move on to today's work. Note that there is no "in-class assignment" in the student book: this is a day for interacting with the students.

Remind them that a function is simply a process—any process—that takes one number in, and spits a different number out. Then explain that functions are so important because they model relationships in the real world, where one number **depends** on a different number. Give them a few examples like the following—no math, just a verbal assertion that one number depends on another:

- The number of toes in class depends on the number of feet in class.
- Which, in turn, depends on the number of people in class.
- The number of points you make in basketball depends on how many baskets you make.
- The amount I pay at the pump depends on the price of gas.

Have them brainstorm in pairs (for 1-2 minutes at most) to come up with as many other examples as they can. Since this is the first "brainstorming" exercise in class, you may want to take a moment to explain the concept. **The goal of brainstorming is quantity, not quality.** The object is to come up with as many examples as you can, no matter how silly. But although they may be silly, they must in this case be **valid**. "The color of your shirt depends on your mood" is not a function, because neither one is a number. "The number of phones in the class depends on the number of computers" is not valid, because it doesn't. By the end of a few minutes of brainstorming and a bit more talk from you, they should be able to see how easy it is to find numbers that depend on other numbers.

Then—after the intuitive stuff—introduce formal functional notation. Suppose you get two points per basket. If we let p represent the number of points, and b represent the number of baskets, then $p(b) = 2(b)$. If we say $p(c) = 2c$ that is not a different function, because they are both ways of expressing the idea that " p doubles whatever you give it" (relate to the function game). So if you give it a 6, you get $12 : p(6) = 12$. If you give it a duck, you get two ducks: $p(\text{duck}) = 2\text{ducks}$. It doubles whatever you give it.

³This content is available online at <<http://cnx.org/content/m19331/1.2/>>.

Key points to stress:

- This notation, $p(b)$, does **not** indicate that p is being multiplied by b . It means that p **depends on** b , or (to express the same thing a different way), p **is a function of** b .
- You can plug any numbers you want into this formula. $p(6) = 12$ meaning that if you get 6 baskets, you make 12 points. $p(2\frac{1}{2}) = 5$ is valid mathematically, but in the “real world” you can’t make $2\frac{1}{2}$ baskets. Remind them that p is a **function**—a process—“double whatever you are given.”
- Also introduce at this point the terminology of the **dependent** and **independent** variables.
- Stress clearly defining variables: not “ b is baskets” but “ b is the number of baskets you make.” (“Baskets” is not a **number**.)
- Finally, talk about how we can use this functional notation to ask questions. The question “How many points do I get if I make 4 baskets?” is expressed as “What is $p(4)$?” The question “How many baskets do I need in order to get 50 points?” is expressed as “ $p(b) = 50$, solve for b .”

For the rest of class—whether it is five minutes or twenty—the class should be making up their own functions. The pattern is this:

1. Think of a situation where one number depends on another. (“Number of toes depends on number of feet.”)
2. Clearly label the variables. (t =number of toes, f =number of feet.)
3. Write the function that shows how the **dependent** variable depends on the **independent** variable ($t(f) = 5f$.)
4. Choose an example number to plug in. (If there are 6 feet, $t(f) = 5(6) = 30$. 30 toes.)

Encourage them to think of problems where the relationship is a bit more complicated than a simple multiplication. (“The area of a circle depends on its radius, $A(r) = \pi r^2$ ”).

Homework:

“Homework: Functions in the Real World.”

When going over this homework the next day, one question that is almost **sure** to come up is #4g: $f(f(x))$. Of course, I’m building up to the idea of composite functions, but there is no need to mention that at this point. Just remind them that $f(\text{anything}) = \text{anything}^2 + 2\text{anything} + 1$. So the answer to (e) is $f(\text{spaghetti}) = \text{spaghetti}^2 + 2\text{spaghetti} + 1$. And the answer to this one is $f(f(x)) = f(x)^2 + 2f(x) + 1$. Of course, this can (and should) be simplified, but the point right now is to stress that idea that you can plug anything you want in there.

1.4 Algebraic Generalizations⁴

This is the real fun, for me.

Start by telling the students “Pick a number. Add three. Subtract the number you started with. You are left with...three!” OK, no great shock and surprise. But let’s use algebra to express what we have just discovered. $x + 3 - x = 3$. The key is recognizing what that sentence mean. x can be any number. So when we write $x + 3 - x = 3$ we are indeed asserting that if you take **any number**, add three and then subtract the number, you get three in the end.

Here’s a harder one. Pick a number, add three, multiply by four, subtract twelve, divide by the number you started with. Everyone started with different numbers, but everyone has 4 in the end. Ask the students to find a generalization to represent **that**, and see if they can work their way to $\frac{4(x+3)-12}{x} = 4$. See also if they can guess what number this trick will **not** work with.

Now, have them work on the in-class assignment “Algebraic Generalizations,” in groups of three.. Most of the class period should be spent on this. This is hard!!! After the first couple of problems (which very directly echo what you already did in class), most groups will need a lot of help.

⁴This content is available online at <<http://cnx.org/content/m19332/1.2/>>.

Here are some of the answers I'm looking for—I include this to make sure that the purpose of the assignment is clear to teachers.

- In #3, the object is to get to $2^{x+1} = 2 \bullet 2^x$ (or, equivalently, $2^x = 2 \bullet 2^{x-1}$. Talk through this very slowly with individual groups. “Let’s call this number x . So what is this number? (2^x) And how can we represent this number? ($x+1$).” etc... The real goal is to get them to see how, once you have written $2^{x+1} = 2 \bullet 2^x$, you have said in one statement that 2^8 is twice 2^7 , and also that 2^5 is twice 2^4 , and also that 2^{11} is twice 2^{10} , and so on. It is a “generalization” because it is one statement that represents many separate facts.
- In #4, the object is to get to $x^a x^b = x^{a+b}$. Again, it will take a lot of hand-holding. It isn’t important for them to do it entirely on their own. It is critically important that, by the time they are done, they see how those numbers lead to that generalization; and how that generalization leads to those numbers.
- If all that works, they should be able to do #6 and come up with something like $(x-1)(x+1) = x^2 - 1$ pretty much on their own. (Or, of course, $x^2 = (x-1)(x+1) + 1$, which is a bit more unusual-looking but just as good.)

Homework:

“Homework: Algebraic Generalizations”

1.5 Graphing⁵

Make sure, on the homework, that they reached something like $(x-a)(x+a) = x^2 - a^2$. Of course, someone may have used completely different letters; and someone else may have said that $(x-a)(x+a) + a^2 = x^2$. Point out that these are just as good—they look different, but they say the same thing. Finally, show them how they could have arrived at $(x-a)(x+a) = x^2 - a^2$ through FOIL.

Once all questions are satisfied, on to graphing. This may actually be two days worth of material. Don’t rush it! And once again, no in-class assignment, but a lot of interaction.

Start by putting the following points on the board: (6,5)(3,6)(2,9)(5,7)(1,10)(4,8). Ask for a general description—if these represent the number of pushups you do each day, what’s the trend? Then graph them, and you can see a definite downward trend with an odd spike in the middle. Moral: we can see things in **shapes** that we can’t see in **numbers**.

Now, let’s jump to the idea of graphing functions. Draw a U-shape, the equation $f(x) = x^2$, and say “This drawing is the graph of that function—but what does that **mean**? What does this drawing actually have to do with that function?” Get to the point where the following ideas have come out. Every time you “do” the function, one number goes in and another comes out. When we graph it, the “in” function is always x , and the “out” function is always y —in other words, **every time we graph a function, we are always graphing $y = f(x)$** . To put it another way, we are graphing all the points that have this particular relationship to each other. (Take as much time as you need on this point.)

Have the students graph $|x|$ (individually at their desks) by plotting points. This is not intended to teach them about absolute value, but to reinforce the ideas I just made about what it means to graph a function.

What can we tell by looking at a graph? Draw the graphs of x^2 and x^3 on the board. Talk about the things we can tell about these two **functions** by looking at the **graphs**. They both have one zero; they are similar on the right, except that x^3 rises faster; they are completely different on the left; they both have unlimited domains, but only one has an unlimited range. (Make sure to connect this back to “domain” and “range” from the function game!)

Now, draw this on the board.

⁵This content is available online at <<http://cnx.org/content/m19334/1.2/>>.

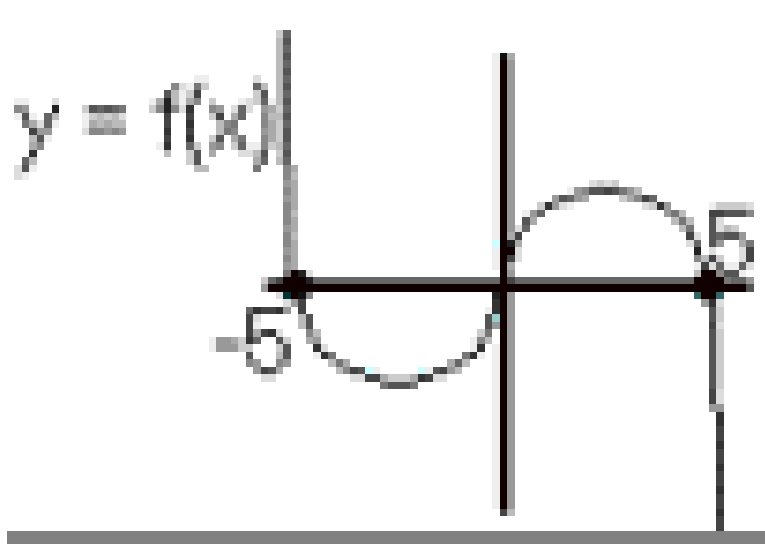


Figure 1.1

Time for another...brainstorming exercise! (Remind them that the object is quantity, not quality!) Each pair of students has to list as many things as it can tell about this function $f(x)$ by looking at the graph. Key points I want to bring out are:

- The three zeros. (Talk about the word “zeros.”)
- Places where the function is negative and places where it is positive.
- Places where it is increasing and places where it is decreasing. (Talk about slope!)
- What happens for very low and very high values of x .
- For the experts, odd symmetry: $f(-x) = -f(x)$.

Talk more about domain and range. This is unrestricted in both. We saw that x^2 has an unrestricted domain, but a restricted range. Show graphs of $\sqrt{x}$ and $1/x$ for restricted domains.

Remind them of the “rule of consistency” that we discussed earlier: a function can never take one **input** and generate two different **outputs**. Ask them to discuss in pairs, for one minute, how this rule manifests on a graph. Then have class discussion until you have reached the vertical line test.

Come back to the question we started with: why do we graph things? As we demonstrated earlier, graphs enable us to see things visually that are very hard to see in numbers. Draw several different jaggy, shaky graphs, and suggest that they represent the price of gas—ask for verbal descriptions of what each one tells us. Talk again about domain, range, positive, negative.

Throw something straight up into the air and catch it. Tell them there is a function $h(t)$ that represents the height of that object, as a function of time. (Make sure they get this.) Give them one minute to sketch the graph of that function. Then show them that it is an upside-down parabola (you don’t need to use that word). Emphasize that this does **not** mean the object traveled in an arc shape: it traveled straight up and down. The horizontal axis is **time** and the vertical axis is height.

If you have a bit of extra time, explain how to generate graphs (and set the window) on the calculator.

Homework:

“Homework: Graphing”

1.6 Permutations⁶

The end of the “Graphing” homework sets this topic up. You want to talk for a while about **vertical permutations** (which tweak the y -value) and **horizontal permutations** (which tweak the x -value).

With relatively little preamble, hand them the worksheet “Horizontal and Vertical Permutations.” It looks long, but it’s not really a huge deal, and should not take the whole period. Afterwards, when you most-mortem it, you make sure they really get the point. If we know what $y = x^2$ looks like, we don’t have to plot a whole bunch of new points to find what $y = x^2 + 3$ looks like; we can see from first principles that it will look like the original graph, but moved up three. **This should not be a memorized fact: it should be obvious, if they understand what graphing means.** Less obviously, $y = (x + 3)^2$ will look like the original graph, moved to the left by 3. The “Conceptual Explanations” contains my second-best explanation of why this is. My very-best explanation is the worksheet itself: plot a bunch of points and see what happens.

If you haven’t already done so, introduce graphing on the calculator, including how to properly set the window. It only takes 5-10 minutes, but it is necessary for the homework.

1.7 Test Preparation⁷

Follow up carefully on the homework from the day before. In #8, make sure they understand that losing money is **negative** profit. But mostly talk about #9. Make sure they understand how, and why, each modification of the original function changed the graph. When we make a statement like “Adding two to a function moves the graph **up** by 2” this is not a new rule to be memorized: it is a common-sense result of the basic idea of graphing a function, and should be understood as such.

Ask what they think $-(x + 1)^2 - 3$ would look like, and then show them how it combines three of the modifications. (The $+1$ moves it to the left, the -3 moves it down, and the $-$ in front turns it upside-down.)

Talk about the fact that you can do those generalizations to any function, eg $-|x + 1| - 3$ (recalling that they graphed absolute value yesterday in class).

Draw some random squiggly $f(x)$ on the board, and have them draw the graph of $f(x) + 2$. Then, see if they can do $f(x + 2)$.

Go back to the idea of algebraic generalizations. We talked about what it means for two functions to be “equal”—what does that look like, graphically? They should be able to see that it means the two functions have exactly the same graph. But this is an opportunity for you to come back to the main themes. If two functions are equal, they turn every x into the same y . So their graphs are the same because a graph is all the (x, y) pairs a function can generate!

I also like to mention at this point that we actually use the $=$ sign to mean some pretty different things. When we say $x + 3 = 5$ we are asking **for what value of x is this true?** Whereas, when we say $2(x - 7) = 2x - 14$ we are asserting that **this is true for all values of x .** Mention, or ask them to find, statements of equality that are not true for **any** value of x (eg $x = x + 1$).

Now, at this point, you hand out the “sample test.” Tell them this test was actually used for a past class; and although the test you give will be different, this test is a good way of reviewing. I have found this technique—handing out a real test from a previous year, as a review—to be tremendously powerful. But I have to say a word here about how to use it. **Sometimes** I say “Work on it in class if you have time, glance it over tonight if it helps you study; the test is tomorrow.” And **sometimes** I say “This is the homework. Tonight, do the sample test, and also look over all the materials we have done so far. Tomorrow, we will go over the sample test and any questions you have on the material, in preparation for the test, which will be the day after tomorrow.” It all depends on timing, and on how prepared you think the class is.

Homework:

“Sample Test: Functions I”

⁶This content is available online at <http://cnx.org/content/m19339/1.2/>.

⁷This content is available online at <http://cnx.org/content/m19340/1.2/>.

I should say a word about #2 here, just to be clear about what I'm looking for. We have a function $c(s)$. In part (b) I supply $s = 20$ and ask for c : so the question in function notation is $c(20)$. (Then you plug a 20 into the formula and go from there.) In part (c) I supply $c = 35$ and ask for $c(20)$: so this question in function notation is $c(s) = 35$. (Then you set the formula equal to 35 and solve.) Students have a lot of difficulty with this asymmetry.

1.7.1 Now give a test of your own on functions I

You may use something very similar to my test, except changing numbers around. Or you may do something quite different.

I will tell you, as a matter of personal bias, that I feel very strongly about question #7. Many students will do quite poorly on this question. For instance, they may give you two variables that do not in fact depend on each other (number of guitarists and number of drummers). Or they may give you variables that are in fact constants (let n equal the number of notes in an octave). The answers don't have to be complicated, although sometimes they are—sometimes a very simple answer gets full credit. ("CDs cost \$12 apiece, I spend d dollars on c CDs. $d(c) = 12c$ " is a full credit answer to parts a-c.) But they have to show that they can clearly articulate what the variables are, and how they depend on each other. If they cannot, spend the time to explain why it is wrong, and work them through correct answers. In my opinion, this skill is the best measure of whether someone really understands what a function is. And it is a skill like any other, in the sense that it develops over time and practice.

(Also, I don't believe in surprises. Tell them in advance that this question will definitely be on the test—the only thing that will change is the topic.)

1.8 Lines⁸

This is largely review: if there is one thing the students **do** remember from Algebra I, it's that $y = mx + b$ and m is the slope and b is the y -intercept. However, we're going to view this from the viewpoint of **linear functions**.

Start by giving an example like the following: I have 100 markers in my desk. Every day, I lose 3 markers. Talk about the fact that you can write a function $m(d)$ that represents the number of markers I have as a function of day. It is a **linear** function because it **changes by the same amount every day**. If I lose three markers one day and four the next, there is still a function $m(d)$, but it is no longer a **linear** function. (If this is done right, it sets the stage for exponential functions later: linear functions **add** the same amount every day, exponential functions **multiply** by the same amount every day. But I wouldn't mention that yet.)

So, given that it changes by the same amount every day, what do you need to know? Just two things: how much it changes every day (-3), and where it started (100). These are the slope and the y -intercept, respectively. So we can say $y = -3x + 100$ but I actually prefer to write the b first: $y = 100 - 3x$. This reads very naturally as "start with 100, and then subtract 3, x times."

Hammer this point home: a linear function is one that adds the same amount every time. Other examples are: I started with \$100 and make \$5.50 each hour. (Money as a function of time.) I start on a 40' roof and start piling on bricks that are $\frac{1}{3}$ ' each. (Height as a function of number of bricks.)

Then talk more about slope—that slippery concept that doesn't tell you **how high** the function is at all, but just **how fast it's going up**. With a few quick drawings on the board, show how you can look at a line and guesstimate its slope: positive if it's going up, negative if it's going down, zero for horizontal. You can't necessarily tell the difference between a slope of 3 and a slope of 5, but you can immediately see the difference between 3 and $\frac{1}{2}$. Emphasize that when we say "going up" and "going down" we always mean **as you go from left to right**: this is a very common source of errors.

Talk about the strict definition of slope. Actually, I always give two definitions. One is: every time x increases by 1, y increases by the slope. (Again: if the slope is negative, y decreases.) The other is: for

⁸This content is available online at <<http://cnx.org/content/m19337/1.2/>>.

any two points on the line, the slope is $\frac{\Delta x}{\Delta y}$ (“rise over run”). Show that this ratio is the same whether you choose two points that are close, or two points that are far apart. Emphasize that this is only true for lines.

Finally, why is it that in $y = mx + b$, the b is the y -intercept? Because the y -intercept is, by definition, the value of y when $x = 0$. If you plug $x = 0$ into $y = mx + b$ you get $y = b$.

All this may take all day, or more than one day. Or, it may go very quickly, since so much of it is review. When you’re done, have them work the in-class assignment “Lines” in groups.

Homework:

“Homework: Graphing Lines”

1.9 Composite Functions⁹

OK, it’s getting hard again: this one may take a couple of days. We start with discussion.

There are a number of ways to look at composite functions. It’s important to be able to use **all** of these ways, and to see how they relate.

1. All the way back to the function game. Let one student be the function $4x + 6$, and another student be the function $x/2$. You give a number to the first student, who spits out a number at the second student, who spits out a number back at you. For instance, if you give the first student a 3, his output is an 18; the second student takes this 18 and comes out with a 9. Do this for a while until everyone has the hang of it. See if anyone realizes that what’s going on is, in the end, the function $2x + 3$ is being done to your number.
2. Now, talk about a factory. One box turns garbage into gloop; the next box turns gloop into shlop; the final box turns shlop into food. Each box can be represented by a function that says “If this much goes in, that much goes out.” The entire factory is a gigantic composite function, where the output of each box is the input of the next, and the composite function says “If this much garbage goes in, this much food goes out.” (Draw it!)
3. In general, composite functions come up with **this variable depends on that variable, which in turn depends on the other variable**. The amount of taxes you pay depends on the amount of money you make, which in turn depends on the number of hours you work. Have them come up with a few examples. Be very careful to distinguish composite functions from multivariate functions, **e.g.** the number of kids in the class depends on the number of boys, and the number of girls. That is not a function, because those two variables don’t depend on each other.
4. Finally, there is the formalism, $f(g(x))$. Remind them that this is mechanical. If $g(x) = 4x + 6$ and $f(x) = x/2$, then what is $f(g(x))$? Well, $f(\text{anything}) = \text{anything}/2$. So $f(g(x)) = g(x)/2$, which is $(4x + 6)/2$ or $2x + 3$. Note that this is completely different from $g(f(x))$! Take a moment to connect this mechanical process with the idea of a composite function that you have already discussed.

Now, have them work through the in-class assignment on “Composite Functions” in groups. Make sure they do all right on #4.

#6 is a build-up to inverse functions, although you don’t need to mention that. If anyone asks for help, help them see that if $h(x) = x - 5$, then $h(\text{anything}) = \text{anything} - 5$, so $h(i(x)) = i(x) - 5$. So $i(x) - 5 = x$, and we can solve this to find $i(x) = x + 5$.

Homework:

“Homework: Composite Functions”

1.10 Inverse Functions¹⁰

This is definitely two days, possibly three.

⁹This content is available online at <<http://cnx.org/content/m19333/1.2/>>.

¹⁰This content is available online at <<http://cnx.org/content/m19336/1.2/>>.

Just as with composite functions, it is useful to look at this three different ways: in terms of the function game, in terms of real world application, and in terms of the formalism.

1. Ask a student to triple every number you give him and then add 5. Do a few numbers. Then ask another student to **reverse what the first student is doing**. This is very easy. You give the first student a 2, and he gives you an 11. Then you give the second student an 11, and he gives you a 2. Do this a few times until everyone is comfortable with what is going on. Then ask what function the **second** student is doing. With a little time, everyone should be able to figure this out—he is reversing what the first student did, so he is subtracting five, then dividing by 3. These two students are “inverses” of each other—they will always reverse what the other one does.
2. Give a few easy functions where people can figure out the inverse. The inverse of $x + 2$ is $x - 2$ (and vice-versa: it is always symmetrical). The inverse of x^3 is $\sqrt[3]{x}$.

The key thing to stress is how you **test an inverse function**. You try a number. For instance...

$10 \rightarrow x + 2 \rightarrow 12 \rightarrow x - 2 \rightarrow 10$
$-5 \rightarrow x + 2 \rightarrow -3 \rightarrow x - 2 \rightarrow -5$

Table 1.1

The point is that you take any number and put it into the first function; put the answer in the second function, and you should get back to your original number. Testing inverses in this way is more important than finding them, because it shows that you know what an inverse function **means**.

3. Ask for the inverse of x^2 . Trick question: it doesn't have any! Why not? Because x^2 turns 3 into 9, and it also turns -3 into 9. It's allowed to do that, it's still a function. But an inverse would therefore have to turn 9 into both 3 and -3 , and a function is **not** allowed to do that—rule of consistency. So x^2 is a function with no inverse. See if the class can come up with others. (Some include $y = |x|$ and $y = 3$.)
4. Now ask them for the inverse of $10 - x$. They will guess $10 + x$ or $x - 10$; make sure they test! They have to discover for themselves that these don't work. The answer is $10 - x$; it is its own inverse. (It turns 7 into 3, and 3 into 7.) Ask for other functions that are their own inverses, see if they can think of any. (Other examples include $y = x$, $y = -x$, $y = 20/x$.)
5. In practice, inverse functions are used to go backwards, as you might expect. If we have a function that tells us “If you work this many hours, you will get this much money,” the inverse function tells us “If you want to make this much money, you have to work this many hours.” It reverses the x and the y , the dependent and independent variables. Have the class come up with a couple of examples.
6. Formally, an inverse function is written $f^{-1}(x)$. This does not mean it is an exponent, it is just the way you write “inverse function.” The strict definition is that $f(f^{-1}(x)) = x$. This definition utilizes a composite function! It says that if x goes into the inverse function, and then the original function, what comes out is... x . This is a hard concept that requires some talking through.

OK, at this point, you have them start working on the in-class exercise “Inverse Functions.” Note that you have not yet given them a way of **finding** inverse functions, except by noodling around! Let them noodle. Even for something like $y = \frac{2x+3}{7}$, they should be able to get there, with a bit of hand-holding, by reversing the steps: first multiply by 7, then subtract 3, then divide by 2. If they ask about $\sqrt[3]{x}$, make sure they try a few things (such as $\sqrt[3]{x}$) and **test them**—they will discover they don't work. Explain that, in fact, we have no inverse function of 2^x right now, so we're going to make one up later in the year and call it a “logarithm.” They can then leave this one blank.

After they have finished noodling their way through the most of the exercises, interrupt the class and say “Now, I'm going to give you a formal method of finding inverse functions—you will need this for the homework.” The formal way is: first, reverse the x and the y , then solve for y . For instance, from $y = \frac{2x+3}{7}$ we first write $x = \frac{2y+3}{7}$. Then solve for y to get $y = \frac{7x-2}{3}$.

Homework:

“Homework: Inverse Functions”

But wait! We’re not done with this topic!

What happens the next day is, they come in with questions. Whatever else they did or didn’t get, they got stuck on #10. (If they got stuck on #9, point out that it is the same as #2; make sure they understand why.) #10 is hard because they cannot figure out how to solve for y . This brings us, not to a new conceptual point, but to a very important algebraic trick, which we are going to learn by doing a TAPPS exercise.

TAPPS (Thinking Aloud Pair Problem Solving) is a powerful learning tool, and here’s how it goes. The students are broken into pairs.

One person in each pair is the **teacher**. His job is to walk through the following solution, step by step, explaining it. For each step, explain two things. Why am I **allowed** to do that, and why did I **want** to do that? Your explanation should make perfect sense to a normal Algebra I student. You should never skip steps—go line, by line, by line, explaining each one.

The other person is the **student**. He also has two jobs. First, whenever the teacher says something that is not perfectly clear, stop him! Even if you understand it, say “Wait, that didn’t make perfect sense.” Keep pushing until the explanation is completely bullet-proof. Second, keep the teacher talking. If he pauses to think, say “Keep talking. What are you thinking?” The teacher should “think out loud” until he comes up with something.

If the students are stuck on a line, they should raise their hands and ask you.

Take the time to carefully explain the process—we will do other TAPPS exercises. And one more thing—warn them that after everyone has completed the exercise, you will be calling on individuals to explain tricky steps. **You will not call for volunteers.** After they are done, everyone in the class should be able to answer any question about anything in this derivation. So you will just pick people and ask them questions like “Why did I do that?”

After they are done, call on individuals and ask questions like “Why did we subtract $2xy$ from both sides?” and “How did we get $y(1 - 2x)$?” Point out the general strategy is only two steps: get all the y things on one side, then pull out the y . Check their answer to the question at the end.

1.10.1 Time for another test!

Once again, there is the sample test—you will probably want to assign it as a homework, and tell them to do that and also study everything since the last test. The next day, go over the homework and any questions. Then give the test.

Congratulations, you’re through with your first unit! If things went well, you have laid the groundwork for the entire year. Onward and upward from here!

Chapter 2

Inequalities and Absolute Values

2.1 Introduction¹

I like to have this unit early in the year, because it introduces another of my main themes: some of these problems **require you to think**. There is almost no “mechanical” way to solve them. I want to set up that expectation early.

Of course, this is really two very different topics, and both of them contain an element of review as well as some new material. But by the time we get to combining them in problems like $2|3x + 4| < 7$, it is new to everybody. Those problems are harder than you might suppose!

2.2 Inequalities²

This is pretty easy, isn't it? You can just start by having them work through “Inequalities” with no preamble at all.

After they've worked on it for a while, you might want to interrupt the class to talk about it for a while. #1 is obviously an attempt to get at the old “you have to reverse the inequality when you multiply or divide by a negative number” thing. But stress as loudly as you can, the idea that they **shouldn't just take anyone's word for it**. The question we're trying to get at is, **why** do you have to switch the inequality, then and only then? I usually illustrate this point by drawing a number line and showing how, on the left of the zero (in “negative land”) the numbers are going **backward** (an observation that every second grader notices, but that they have forgotten by high school). So you can visually show how $2 < 3$ becomes $-2 > -3$ when you move it to the other side.

Also, tell them how important it is to distinguish carefully between ANDs and ORs—this will become a major issue later. I have two pet peeves on this topic.

One pet peeve is people who memorize a facile rule (such as “less-than problems become AND and greater than problems become OR”) without having the slightest idea what they are doing. I go out of my way to create problems that frustrate such rules (which isn't hard to do). They have to see and understand what these conjunctions **mean**.

The other pet peeve is $x > \pm 4$. This is, for all intents and purposes, meaningless. I want them to realize that on #9, and then I warn them that I will always take off points if they answer any question this way. (This comes up in the context of $|x| > 4$ and is a good example of how wrong you go if you answer mechanically instead of thinking.) x can **equal** ± 4 , if you know what that means (shorthand for $x = 4$ or $x = -4$), but it cannot be **greater than**, or **less than**, $\pm$ anything.

Homework:

“Homework: Inequalities”

¹This content is available online at <http://cnx.org/content/m19432/1.2/>.

²This content is available online at <http://cnx.org/content/m19430/1.2/>.

2.3 Inequality Word Problems³

The difficulty with this is that there is really nothing to say about it at all: it's just something I want them to see. So there is this homework assignment, which you can give at any time, before or after anything else. Stick it in when you find yourself, at the end of a day, not quite ready to give out the next homework, or something like that.

Homework:

"Inequality Word Problems."

In one problem, they have to **make up** an inequality word problem. You might think that, just because they have made up so many word problems and functions by now, they would knock this one down easily—but it ain't so. I've had students who could create functions all day long (by this time) who could not create a good inequality to save their lives. They create scenarios like "I eat three bowls of cereal a day, how many do I eat in a week?" There is nothing **unequal** there. Just like everything else, this takes practice. But I do want them to see that inequality relationships are all around us.

2.4 Absolute Value Equations⁴

Now it gets tough. But once again, little or no preamble is needed: just have them start working on "Absolute Value Equations."

Here's the thing. Problems 1–9 really contain all the math: all the concepts they need to get. And, for the most part, they will get them right—although you need to check this before they go on any further.

But when it comes to the more complicated-**looking** problems in the second half of the assignment, they panic. They stop thinking, revert to rules, and start getting wrong answers. If they are diligently checking, they will realize that their answer to #12 doesn't work. But they may need you to point out that this is because it is analogous to #6 and has no answer.

So, around this time, I spend a lot of time insisting "Think, think, think!" The way to think it through is this. Once you have solved for the absolute value, go back to the kind of thinking you did in the first page. For instance, when you have $|x + 3| = -1$, cover up the $x + 3$ and ask yourself the question like this: "The absolute value of **something** is -1 . What is the something?" The answer, of course, is "nothing." Think, and you will get it right. Plug and chug, and you will get it wrong.

OK, if $|x + 3| = 7$ has two answers, and $|x + 3| = 0$ has one, and $|x + 3| = -4$ has none, then what about $|x - -2| = 2x - -10$? The answer is, **you don't know until you try**. You begin by splitting it the same way you did before: $x - -2 = 2x - -10$ or $x - -2 = - (2x - -10)$. Find both answers. But then check them: **even if you did the math right, they may not work!** Don't tell them this up front, but make sure to discuss this with them toward the end of class, in the context of #13; they will need to know this for the homework.

Homework:

"Homework: Absolute Value Equations"

2.5 Absolute Value Inequalities⁵

They are going to work on the assignment "Absolute Value Inequalities" in class. You may want to begin by reminding them that they have already been solving absolute value inequalities. On the previous assignment they turned "the **absolute value of my number is less than 7**" into an inequality and solved it, by trying a bunch of numbers. These are no different. We are going to use the same sort of thinking process as before: confronted with $|3x - 1| > 10$ we will say "OK, the absolute value of **something** is greater than 10. What

³This content is available online at <http://cnx.org/content/m19428/1.2/>.

⁴This content is available online at <http://cnx.org/content/m19426/1.2/>.

⁵This content is available online at <http://cnx.org/content/m19431/1.2/>.

could the **something** be? <think, think> OK, the **something** must be greater than 10 (like 11,12,13) or else less than -10 (like -11,-12,-13).” So we write $3x - 1 > 10$ or $3x - 1 < -10$ and go from there.

“Gee, why are you making it so hard? My Algebra I teacher taught me that if it’s greater than, just make it an “or” and if it’s less than, just make it an “and.””

OK, let’s try a slight variation: $|3x - 1| > -10$. Now what? “OK, the absolute value of **something** is greater than -10. What could the **something** be? <think, think> OK, the **something** can be...anything!” The absolute value of anything is greater than -10, so any x -value will work!

“That’s not what my Algebra I teacher taught me.”

Fine, then, let’s try that same problem your way. Then, let’s test our answers, by plugging into the original inequality and see which answer works.

You get the idea? The whole point of this unit (to me) is to—very early in the year—establish a pattern that the only way to solve math problems is by thinking about them. This unit is great for that.

Homework:

“Homework: Absolute Value Inequalities”

2.6 Graphing Inequalities and Absolute Values⁶

This is a two-day topic, possibly three.

Start by putting the function $y = x^2 - 1$ on the board. Now, distinguish between two different kinds of questions.

1. Solve (or graph the solution of) $x^2 - 1 < 0$. This should remind the students of problems we did in the last unit, where we asked “For what x -values is this function negative?” The answer is $-1 < x < 1$; it could be graphed on a number line.
2. Graph $y < x^2 - 1$. This is a completely different sort of question: it is not asking “For what x -values is this true?” It is asking “For what (x, y) pairs is this true?” The answer cannot live on a number line: it must be a shaded region on a two-dimensional graph. Which region? Well, for every point **on** this curve, the y -value is **equal** to $x^2 - 1$. So if you go **up** from there, then y is **greater than**...but if you go **down** from there, then y is **less than**... So you shade below it.

It is important to be able to solve both types of problems, but it is even more important, I think, to distinguish between them. If you answer the first type of question with a shaded area, or the second type on a number line, then you aren’t just wrong—you’re farther than wrong—you’re not even thinking about what the question is asking. (“ $2 + 2 = 5$ ” is wrong, but “ $2 + 2 = \text{George Washington}$ ” is worse.)

With that behind you, get them started on the assignment “Graphing Inequalities and Absolute Values.” They should get mostly or entirely finished in class, and they can finish it up and also do the homework that evening.

Homework:

“Homework: Graphing Inequalities and Absolute Values”

If they come in the next day asking about #4, by the way, just tell them to turn it into $y < -2|x|$ and then it is basically like the other ones.

Second day, no worksheet. After going over the homework, stressing the ways we permute graphs, warn them that here comes a problem that they cannot solve by permuting. Challenge them—first person with the correct shape is the winner, no calculators allowed—and then put on the board $y = x + |x|$. Give them a couple of minutes to plot points. Then let someone who got it right put it on the board—both the points, and the resultant shape.

Now, you point out that this shape is really a combination of two different lines: $y = 2x$ on the right, and $y = 0$ on the left. This odd two-part shape is predictable, **without** plotting points, if you understand absolute values in a different way. This is our lead-in to the **piecewise definition of the absolute value**:

⁶This content is available online at <<http://cnx.org/content/m19433/1.2/>>.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

This takes a whole lot of explaining: it is just one of those things that students find difficult. Here are a few ways to explain it (use all of them).

1. Just try numbers. If $x = 3$, then $|x| = 3$, so $|x| = x$. Same for $x = 4$, $x = 52$, and even $x = 0$. But if $x = -3$, then $|x| = 3$, so $|x| \neq x$ (they are not the same)! Instead, $|x| = -x$. Why, because $-x$ in this case is $-(-3)$ which is 3 which is indeed $|x|$.

But how can $|x| = -x$ when $|x|$ is never negative? Well, that brings us to...

2. Putting a $-$ sign in front of a number does not make it negative: it **switches the sign**. It makes positive numbers negative, and negative numbers positive. So you can read that piecewise definition as “if x is negative, then the absolute value switches the sign.”
3. Finally, come back to the graph of good old $y = |x|$. Point out that it is, indeed, the graph of $y = x$ on the right, and the graph of $y = -x$ on the left.

Now, how does all this relate to our original problem? When $x < 0$, we replace $|x|$ with $-x$ so our function becomes $y = x - x = 0$. When $x \geq 0$, we replace $|x|$ with x so our function becomes $y = x + x = 2x$. That’s why the graph came out the way it did.

Why is this important? It’s an important way to understand what absolute value means. But it’s also our first look at piecewise functions (one of the only looks we will get) so take a brief timeout to talk about why piecewise functions are so important. Throw an object into the air and let it drop, and talk about the function $h(t)$. We previously discussed this function only **during** the flight. But to get more general, you have to break it into three different functions: $h = 3$ before you throw it (assuming it was in your hand 3’ above the ground), $h = 16 - t^2$ or something like that during the flight, and $h = 0$ after it hits the ground. Do a few more examples to get the idea across that piecewise functions come up all the time because conditions change all the time.

OK, back to our friend the absolute value. The students should now graph $y = \frac{x}{|x|}$ on their own (individually, not in groups), not by plotting points, but by breaking it down into three regions: $x < 0$, $x = 0$, and $x > 0$. (It is different in all three.) Get the right graph on the board.

Now, hopefully, you have at least 10-15 minutes left of class, because now comes the hardest thing of all. You’re going to graph $|x| + |y| = 4$. Since x is under the absolute value, we have to break it into two pieces—the left and the right—just as we have been doing. Since y is under the absolute value, we also have to break it vertically. So what we wind up doing is looking at **each quadrant separately**. For instance, in the second quadrant, $x < 0$ (so we replace $|x|$ with $-x$) and $y > 0$ (so we replace $|y|$ with y). So we have $y - x = 4$ which we then put into $y = mx + b$ format and graph, but **only in the second quadrant**. You do all four quadrants separately.

Explain this whole process—how to divide it up into the four quadrants, and how to rewrite the equation in the second quadrant. Then, set them going in groups to work the problem. Walk around and help. By the end of the class, most of them should have a diamond shape.

After they are all done, you may want to mention to them that this exact problem is worked out in the “Conceptual Explanations” at the very end of this chapter. So they can see it again, with explanations.

Homework:

Graph $|x| - 2|y| < 4$. This requires looking at each quadrant as a separate **inequality** and graphing them all in the appropriate places!

2.6.1 Time for another test!

Once again, there is the sample test—you will probably want to assign it as a homework, and tell them to do that and also study everything since the last test. The next day, go over the homework and any questions. Then give the test.

Chapter 3

Simultaneous Equations

3.1 Introduction to Simultaneous Equations¹

Like all our topics so far, this unit reviews something the students covered in Algebra I—but it goes deeper.

Begin by having them work their way through the assignment “Distance, Rate, and Time” in pairs. Most of it should be pretty easy, including getting to the general relation $d = rt$. It should not take much time.

Until the last question, that is. Let them work on this for a while. Some will get all the way, some will not get very far at all. But after they’ve been at it for a while, tell them to stop working and pull them back to a classwide discussion. Show them how to set up $d = rt$ for each case, and make sure they understand. Maybe come up with another problem or two along the same lines, including one where the **times** are the same and the **distances** are different. (A train leaves Chicago and a train leaves New York, when do they crash...) Get them comfortable with **setting up** the two equations—we’re not really focused on solving them.

Toward the end, see if they remember that there are three ways of solving these equations. Two of them, Substitution and Elimination, will be covered tomorrow. Tonight, on homework, we are going to solve by graphing. Your big job is to drive home the point that since a graph represents all the points where a particular relationship is true, therefore the place where the two graphs intersect is the point where both relationships are true. Also talk about the fact that graphing is not 100% accurate (you sort of eyeball a point and say “it looks like around this”) and how to check your answer (plug it back into both equations).

Homework:

“Homework: Simultaneous Equations by Graphing”

3.2 Simultaneous Equations²

At the start of the second day, explain the two techniques of substitution and elimination. This is review, so they should get it with just a few examples.

Then have them do the assignment “Simultaneous Equations.”

At the end of class, if they are mostly done, they can finish that for homework and also do the homework. If they are not mostly done, they can just finish it for homework, and you can have them do the “homework” the next day in class.

Homework:

“Homework: Simultaneous Equations”

¹This content is available online at <<http://cnx.org/content/m19497/1.2/>>.

²This content is available online at <<http://cnx.org/content/m19498/1.2/>>.

3.3 The "Generic" Simultaneous Equation³

This should be done in class as another TAPPS exercise. Remind them of the ground rules, and especially of the fact that you will be asking them questions afterwards to make sure they got it.

3.3.1 Time for another test!

Once again, there is the sample test. If everyone finishes the TAPPS exercise early, and they all seem pretty comfortable with the material, you may not want to do the “do the sample test tonight and we’ll go over it tomorrow and then have the test the next day”—tomorrow may be pretty boring! Instead, it may be OK (depending on the class) to just say “Now work in class on the sample test, use it to help you study tonight, and we will have a real test tomorrow.”

³This content is available online at <<http://cnx.org/content/m19499/1.2/>>.

Chapter 4

Quadratics

4.1 Introduction¹

There are really three separate pieces of this unit: factoring, solving quadratic equations, and graphing quadratic functions. The first piece is vital and important, but small. Nonetheless, you may want to add a small quiz between that section and the next. I have included two sample tests—the first on factoring and solving quadratic equations, the second on graphing.

4.2 Multiplying Binomials²

Sounds trivial, doesn't it? But this is one of the most important days in the year.

What they **do** know, from Algebra I, is how to FOIL. This takes two seconds of review and you're done. However, there are two points that their Algebra I teacher never made.

1. When we say $(x + 3)(x + 4) = x^2 + 7x + 12$, we are asserting the equality of two functions—that is, if I plug any number into $(x + 3)(x + 4)$, and plug that same number into $x^2 + 7x + 12$, it should come out the same. It's an **algebraic generalization**.
2. FOIL leaves you high and dry if you have to multiply $(x + 2)(x + y + 3)$. The **real** algorithm for multiplying polynomials is to multiply **everything on the left** by **everything on the right**. Walk through an example of this on the board. Show them how FOIL is just a special case of this rule, with both things are binomials.

At this point, they can start working in pairs on the exercise “Multiplying Binomials.” They should have no problem with the first few. As you are walking around, your main job is to make sure that they are doing #5 correctly. They should **not** be multiplying these out explicitly (so that $(x + 4)(x + 4)$ becomes $x^2 + 4x + 4x + 16$ and then combining the middle terms. They should instead be using the **formula** that they just developed, $(x + a)^2 = x^2 + 2ax + a^2$ to jump straight to the right answer. A lot of them will find this very confusing. I always explain it this way: x and a are both placeholders that could represent **anything**.

So when we say:

$$(x + a)^2 = x^2 + 2ax + a^2$$

what we're really saying is:

$$(\text{something} + \text{something_else})^2 = \text{something}^2 + 2(\text{something})(\text{something_else}) + \text{something_else}^2$$

Maybe walk them through the first one as an example.

One point of this exercise is to get them to the point where they can see immediately, **with no in-between steps**, that $(x + 4)^2 = x^2 + 8x + 16$. Some of them will think that this new, confusing method

¹This content is available online at <<http://cnx.org/content/m19469/1.2/>>.

²This content is available online at <<http://cnx.org/content/m19472/1.2/>>.

may be faster, but they can just go right on doing it the “old way” with FOIL. I always explain to them that, in a few days, we’ll be learning a technique called **completing the square** that involves **reversing** this formula, and therefore cannot possibly be done with FOIL. They need to know the formula.

Another point is to get our three formulae on the table: $(x + a)^2$, $(x - a)^2$, and $x^2 - a^2$. There are very few things I ask the class to memorize during the year, but these three formulae should all be committed to memory.

But the larger point is to give them a new understanding and appreciation for what variables do—to understand that x and a represent **anything**, so that once you have a formula for $(x + a)^2$ you can use that formula directly to find $(2y + 6z)^2$.

Homework:

“Homework: Multiplying Binomials”

4.3 Factoring³

Begin class by reminding them of what they already know: factoring means turning $x^2 + 7x + 12$ into $(x + 3)(x + 4)$. Then ask—how would we **check** that? There are two ways. First, we can multiply it back (using FOIL for instance). Second, we can try a number (since we are making the claim that these two functions are “equal,” remember?). Stress this very heavily: they have to know **both ways** of checking. Why? Because if you don’t know both ways of checking, then you don’t really understand what factoring is, even if you get the right answer.

OK, on to...how do you do it? There are three steps to factoring.

1. Pull out common terms. **This is always the first step!**
2. Use the formulae from yesterday. For instance, given $x^2 - 9$, you recognize it as the difference between two squares. Given $x^2 - 6x + 9$, you recognize it as $(x - 3)^2$.
3. When all else fails, plain old-fashioned factoring. Do a few examples on the board, just to refresh their memory. $x^2 + 7x + 12$, $x^2 - 7x + 12$, $x^2 + x - 12$, $x^2 - x - 12$ are good starting examples, and give you an opportunity to talk about what effect negative numbers have, both in the middle term (not much) and in the last term (lots). The thing I always stress is to **start with the last term**—find all the pairs of numbers that multiply to give you the last term, and then see if any of them add to give you the middle term. The real test is when you are faced with something like $x^2 + 4x + 8$; it should **not take long** to determine that it cannot be factored!

Now they are ready to start on the “Factoring” assignment during class.

Homework:

“Homework: Factah Alla Dese Heah Sprepressions”

As I mentioned, this might be a good place to break and have a quiz. Or it might not. What do I know? Anyway, on to quadratics...

4.4 Introduction to Quadratic Equations⁴

Get them started on the assignment “Introduction to Quadratic Equations” with little or no preamble. Then, after a few minutes—after everyone has gotten through #5—stop them.

Make sure they all got the right answers to numbers 2 and 3, and that they understand them. If $xy = 0$ then **either** x , **or** y , **must equal zero**. There is no other way for it to happen. On the other hand, if $xy = 1$, that doesn’t tell you much—either one of them could be anything (except zero).

Now, show how this relates to quadratic equations. They do remember how to solve quadratic equations by factoring. $x^2 + x - 12 = 0$, $(x + 4)(x - 3) = 0$, $x = -4$ **or** $x = 3$. But that last step is taken as a random

³This content is available online at <<http://cnx.org/content/m19466/1.2/>>.

⁴This content is available online at <<http://cnx.org/content/m19470/1.2/>>.

leap, “because they told me so.” The thing I want them to realize is that, when they write $(x + 4)(x - 3) = 0$, they are in fact asserting that “these two numbers multiply to give zero,” so one of them has to be zero. This helps reinforce the idea of the previous lesson, that x and y can mean **anything**: $(x + 4)(x - 3) = 0$ is in fact a special case of $xy = 0$.

The acid test is, what do you do with $(x + 5)(x + 3) = 3$? The ones who don’t get it will turn it into $x + 5 = 3$, $x + 3 = 3$. And get two wrong answers. Instead you have to multiply it out, **then** get everything on one side so the other side is 0, and **then** factor.

Now they can keep going. Many of them will need help with #6—talk them through it if they need help, but make as much of it as possible come from them. This is a very standard sort of “why we need quadratic equations” type of problem.

The last four problems are a sneaky glimpse ahead at completing the square. For #11, many students will say $x = 3$; remind them that it can also be -3 . For #12, this is yet another good example of the “ x can be anything” rule, and should remind them in some ways of the work we did with absolute values: something² = 9, so something = ± 3 . #13 is obviously #12 rewritten, and #14 can be turned into #13 by adding 16 to both sides.

Homework:

“Homework: Introduction to Quadratic Equations.”

4.5 Completing the Square⁵

When you’re going over the homework, talk for a while about the throwing-a-ball-into-the-air scenario. It will come up again, and I really want people to understand it. The particular point I try to make is how the math reflects the reality. You have a function $h(t)$ where if you plug in any t at all, you will get an h . You’re using it backward, specifying h and asking for t (as in, “when will the ball hit the ground?”). What kind of answers would you expect? Well, suppose you throw the ball 16 ft in the air. If you ask “When will it be at 20ft?” you would expect to get no answer at all. If you ask “When will it be at 5 ft?” you would expect two answers—one on the way up, and one on the way down. If you ask “When will it be at 16 ft?” you would expect exactly one answer. In all three cases, the math gives you exactly what you expect.

On the other hand, suppose you ask “When will it be at -3 ft?” (That is, under the ground.) You might expect no answer at all, since the ball never is under the ground. But the math doesn’t know that—it thinks the ball is following the same function forever. So you get two answers. One is **after** the ball hits the ground. The other is before it left—a negative time! This is where you have to use common sense to find the “real” answer, as distinct from the answer the math gave you.

I spend a good half-period, at least, talking through this. I think it is an incredibly important point about the way we use math to model the world. See this webpage⁶ for an exercise you can use just on this.

Anyway, onward...the assignment “Completing the Square” pretty much speaks for itself. Probably the only preamble you need is to point out that many quadratic equations, which **do have solutions**, cannot be factored. So we are going to learn another technique which has the advantage that it can **always** be used. (Factoring is still easier and faster when it works.)

Now you can just get them started on it, and then wander around and help. Just make sure that before the class is done, everyone gets the technique. You may also want to point out to them that they already did this on yesterday’s assignment.

On #4 make sure they get two answers, not just one!

Homework:

“Homework: Completing the Square”. The hard ones here, that you will get questions on the next day, are #9 and #10. Note that, on #9, I am **not** looking for the discriminant and the quadratic formula and stuff; just the obvious fact, based on completing the square, that if $c < 0$ we have no real answers, if $c = 0$ we have one, and if $c > 0$ we have two. #10 is worth looking at closely if there are questions, because it leads to the next day.

⁵This content is available online at <<http://cnx.org/content/m19465/1.2/>>.

⁶<http://www.ncsu.edu/felder-public/kenny/papers/physicist.html>

4.6 The "Generic" Quadratic Equation⁷

Begin by reminding them of what we did with simultaneous equations. First, we learned how to solve them (using substitution or elimination). Then we used those exact same techniques to solve the **generic** version—that is, simultaneous equations where all the numbers were replaced by letters. This, in turn, gave us a **formula** that could instantly be used to solve any pair of simultaneous equations.

Now we are going to do that same thing with quadratic equations. The “generic” quadratic equation is, of course, $ax^2 + bx + c = 0$. Now, we have learned two different ways of solving such equations. The “generic” version is hard to solve by factoring (although it is possible); we are going to do it by completing the square.

Make sure they look over my example of completing the square; this might be a good opportunity for a quick TAPPS exercise. There are other examples in the “Conceptual Explanations” so you could do two TAPPS exercises—that way everyone gets a chance to be the teacher.

Then have them work through the sheet. They should derive the quadratic formula, and then use it.

By the time they are done, they should have two things. They should have the quadratic formula, and a bit of practice using it—so now we have **three** different techniques for solving quadratic equations. They should also have **derived** the formula. I always warn them that I will ask for this derivation on the next test: it is not enough to know the formula (although that too is good), you have to be able to derive it.

At the end of class, you may want to talk for just a couple of minutes about the discriminant, in reference to #11. It should be fairly obvious by that point to most of them.

Homework:

“Homework: Solving Quadratic Equations”

4.6.1 Time for another test!

As always, there is the sample test, which may or may not be assigned as a homework. Then there is the test—on multiplying polynomials, on factoring, and mostly on solving quadratic equations. Make it shorter than my sample [U+263A]

4.7 Graphing Quadratic Functions⁸

OK, we’re done solving quadratic equations—we already have three techniques and that’s enough. But—one thing I say a million times throughout my class—you never really understand a function until you graph it.

So, they can do the exercise “Graphing Quadratic Functions.” It doesn’t require any buildup, they can do it right now. Note that we are not introducing any of the formal machinery of parabolas (focus, directrix, *etc.*)—all that will come much later, in the unit on conics. We are graphing both horizontal and vertical parabolas the way we did in the very first unit on functions—by taking an initial starting point ($y = x^2$ or $x = y^2$) and moving it up and down and left and right and stretching it and turning it upside-down. None of this should require a calculator.

It might be worth mentioning that a horizontal parabola is not a function. But we can still talk about it and graph it.

Homework:

Finish the in-class assignment. That was a long one, wasn’t it?

4.8 Graphing Quadratic Functions II⁹

The beginning of the “Graphing Quadratic Functions II” exercise is review of yesterday. After letting them work on it together, you may want to interrupt and have them do the thing in the middle as a TAPPS

⁷This content is available online at <<http://cnx.org/content/m19480/1.2/>>.

⁸This content is available online at <<http://cnx.org/content/m19468/1.2/>>.

⁹This content is available online at <<http://cnx.org/content/m19467/1.2/>>.

exercise. The key things you need to ask them about are how this is the same as, and different from, the way we completed the square before. For instance, we used to add nine to both sides (because, let's face it, we **had** two sides). Now we only have one side, so we add 9 to it, **and** subtract 9 from it, at the same time. This gives us what we want (the perfect square) without changing the function.

Homework:

"Homework: Graphing Quadratic Functions II." You will get questions the next day about #8 (which is really a line) and #11 (which they just flat can't graph at this point). These lead nicely into #12. If it has an x^2 but no y^2 , it's a vertical parabola. If it has a y^2 but no x^2 , it's a horizontal parabola.

4.9 Solving Problems by Graphing Quadratic Functions¹⁰

Now, at long last, we see a **use** for all this graphing we've been doing.

In the throwing-a-ball scenario, we have an $h(t)$ that can be used to answer two kinds of questions. "I know the time, but what is the height?" (easy, plug in) and "I know the height, what is the time?" (harder, requires solving a quadratic equation). But there is a third kind of question, very important in the real world, which is: "How high does it go?" Now we don't know the time **or** the height! But if we graph it, and find the vertex, we can find both.

Now they can work a while on the in-class assignment. Many of them will get stuck dead on #3. This is where you have to pull back and lecture a bit more. Help them draw it, and set up the function $A(x)$. But more importantly, talk about what that function **means**. You plug in any x (length) and you get back an A (area). So, if the graph looks like this $\cap$ what does that tell us? Well, at the peak there, that is the highest A ever gets on our graph—that is, the highest the area ever gets. Find the vertex, and you will find the x that maximizes A !

This is worth a lot of time to make sure people really get it. It comes all the way back to week 1, and the idea of graphing a function. On one level, it's incredibly abstract—we are drawing an upside-down parabola that somehow represents the "possibility spaces" for a bunch of rectangles. But if you understand the idea of graphing a function, it is really very simple. Every point on that parabola pairs an x (length) with an A (area). Every point represents one farm that our farmer could create. It's obvious, looking at it, that this point at the top here represents the one with the **highest area**.

This is one of those cases where the in-class assignment and the homework, together, could easily take two days instead of one. Let it take that, if it does. Make up more problems, if you have to. But don't let them get away with thinking "I understand everything else, I just don't get the word problems."

Homework:

"Homework: Solving Problems by Graphing Quadratic Functions"

4.10 Quadratic Inequalities¹¹

For some reason, this is one of the hardest topics in the course. It shouldn't be hard. There is nothing hard about it. But students get incredibly tied in knots on this, by trying to take short cuts. The hardest part is convincing them that they have to think about it graphically.

So, begin by simply putting these two problems on the board.

$$x^2 - 3x - 4 > 0$$

$$x^2 - 3x + 3 > 0$$

Allow them to work in pairs or groups. Offer a piece of candy or a bit of extra credit or some such to anyone who can find the answer to both problems. Give them time to really work it. Almost no one will get it right, and that's the point. It's very hard to think about a problem like this algebraically. It's very easy if you think about it the right way: by graphing.

¹⁰This content is available online at <<http://cnx.org/content/m19479/1.2/>>.

¹¹This content is available online at <<http://cnx.org/content/m19473/1.2/>>.

So, we're going to graph both of those functions. But strangely enough, we're going to do it without completing the square or finding the vertex. In each case, we're only going to ask two questions: what are the **zeros** of the function, and which **direction** does it open in? These two questions are all we need to answer the inequality.

In the first case, by factoring, we find two zeros: 4, and -1 . In the second case, we find with the quadratic formula that there are no zeros. Both graphs open up. (Why? Because **the coefficient of the x^2 term is positive.**) So the graphs look something like this.

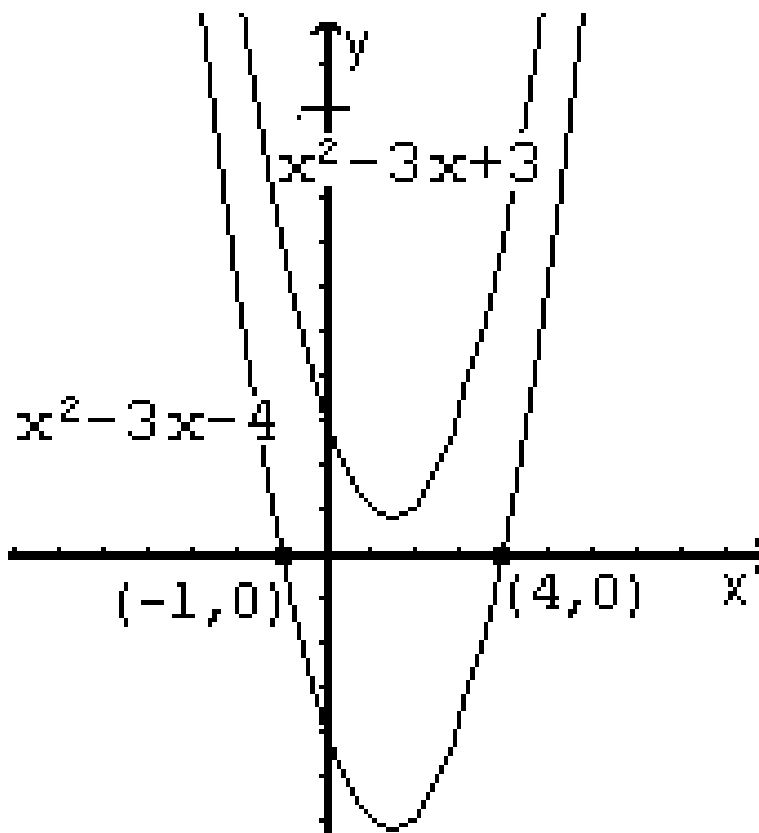


Figure 4.1

What are the vertices, exactly? We don't know. If we wanted to know that, we would have to complete the square, just as we did before.

But if all we want to know is **where each graph is positive**, we now have it. The graph of the first function should make it clear that all numbers to the **right of 4** work, as do the numbers to the **left of -1** , but the numbers in between don't work. (Quick review: how can we write that answer with inequalities? With set notation?) The graph of the second function makes it clear that **all numbers work**.

Check this by trying numbers in the original inequalities.

Now, challenge them to find a quadratic inequality that is in the form $f(x) > 0$ where $f(x)$ is a quadratic function and the solution is **nothing works**. The key is, of course, it has to be an upside-down quadratic.

Homework:

“Homework: Quadratic Inequalities”

4.10.1 Time for another test II!

Congratulations, kids! We are done with our entire unit on quadratic equations—probably the biggest unit in the course. (Certainly the only thing I remember from my own Algebra II course.)

Chapter 5

Exponents

5.1 Introduction¹

After all the incredibly **new** stuff we've been doing, it's a nice break to get back to something with a large element of review in it.

But it is also a problem. Many of the kids know already that $2^3 2^5 = 2^8$. A fair number of them even know that $2^{-7} = 1/27$. But they don't know **why**. It's vital to keep reminding them that it isn't enough to know it, they have to know why—and this will indeed be reflected on the test.

5.2 Rules of Exponents²

Yes, a lot of this assignment was already done, verbatim, in the unit on Functions. But there are a lot of reasons for bringing it back (just as we did in Quadratics). First, they (and you) can discover that they have gotten better at finding generalizations. But more importantly, back when we did this in functions, we were only interested in the process of finding generalizations. Now we are focused on creating, memorizing, and using the three rules of exponents.

2d is really pushing them another step toward “the way mathematicians think”—seeing that $2(2^x) = 2^{x+1}$ is really just a special case of $2^a 2^b = 2^{a+b}$, where $a = 1$.

In #3, I really want to get at the idea that $\frac{3^{12}}{3^{10}} = 3^2$, and $\frac{3^{10}}{3^{12}} = \frac{1}{3^2}$. In other words, the whole thing can and should be done without negative exponents. Why? Because we haven't yet defined what they mean—and why.

Homework:

“Homework: Rules of Exponents”

5.3 Extending the Idea of Exponents³

This is one of my favorite class discussions. You're going to talk almost the entire class, and just give out an assignment toward the end.

So far, we have only talked about exponents in the context of positive integers. The base can be anything: for instance, we can find $(-3)^4$ or $(\frac{1}{2})^3$. But when we say that 2^3 means $2 \bullet 2 \bullet 2$, that definition is really only meaningful if the exponent is a positive integer. We can't multiply 2 by itself “— 3 times” or “ $\frac{1}{2}$ times.” Or “0 times” for that matter.

¹This content is available online at <<http://cnx.org/content/m19325/1.2/>>.

²This content is available online at <<http://cnx.org/content/m19327/1.2/>>.

³This content is available online at <<http://cnx.org/content/m19318/1.2/>>.

So, let's plop ourselves down in an imaginary point in history where exponents are only defined for positive integers. We are the king's mathematicians. The king has just walked in and demanded that we come up with some sort of definition for what 2^{-3} means. "Zero and negative numbers have rights too," he growls. "They must be treated equally, and given equal rights to be in the exponent."

So we start with a brainstorming exercise. The object is to come up with as many possible things as you can think of, for 2^{-3} to mean. As always, this should be done in groups of 2 or 3, and remind them that the object is quantity, not quality: let's get creative. If you can't think of more than two or three definitions, you're not trying hard enough.

By the way, somewhere in class there is a smart-aleck who knows the right answer and therefore won't plan. "It's $1/2^3$ " he insists proudly. "Why are we doing this?" To which you reply: "The people in the class are coming up with dozens of things it could mean. Can you give them a good argument as to why it should mean that, instead of all the others?" And he weakly answers "Well, my Algebra I teacher told me..." and you've won. The point, you explain, is not to parrot what your teacher told you, but to understand several things. The first is that our old definition of an exponent ("multiply by itself this many times") just doesn't apply here, and so we need quite literally a new definition. The second is that there are a ton of definitions we could choose, and it frankly seems arbitrary which one we pick. The third is that there really is a good reason for choosing one and only one definition. But if he doesn't know what that is, how about if he gets with the program and brainstorms? End of discussion.

OK, so after a few minutes, you start collecting ideas from the groups. 2^{-3} means... 2^3 , only negative (so it's -8). It means 2 divided by itself three times, $2/2/2$ (so it's either 2 or $\frac{1}{2}$, depending on how you parenthesize it). And so on. With a bunch of ideas on the board, you say, now we have to choose one. How do we do that? That is, what criteria do we use to decide that one definition is better than the others? (silence)

The answer is—the definition should be as consistent as possible with the one we already have. Of course it won't mean the same thing. But it should behave mathematically consistently with the rules we have: for instance, it should still obey our three laws of exponents. That sort of consistency is going to be the guideline that we use to choose a definition for the king.

And hey, what exactly do negative numbers mean anyway? One way to look at them is, they are what happens when you take the positive numbers and keep going down. That is, if you go from 5 to 4 to 3 and just keep going, you eventually get to 0 and then negative numbers. This alone is a very powerful way of looking at negative numbers. You can use this to see, for instance, why positive-times-negative-equals-negative and why negative-times-negative-equals-positive.

$3 \bullet 5 = 15$	$3 \bullet -5 = -15$
$2 \bullet 5 = 10$	$2 \bullet -5 = -10$
$1 \bullet 5 = 5$	$1 \bullet -5 = -5$
$0 \bullet 5 = 0$	$0 \bullet -5 = 0$
$-1 \bullet 5 = -5$	$-1 \bullet -5 = 5$
$-2 \bullet 5 = -10$	$-2 \bullet -5 = 10$

Table 5.1

OK, you may or may not want to get into this, but I think it's cool, and it does help pave the way for where we're going. (It also helps reinforce the idea that even the rules you learned in second grade have reasons.) The numbers I've written on the left there show what happens to "multiply-by-5" as you count down. Clearly, looking at the positive numbers, the answers are going down by 5 every time. So if that trend continues as we dip into negative numbers, then we will get -5 and then -10 on the bottom: negative times positive equals negative.

On the right, we see what happens to "multiply-by-5" as you count down. Since we already know (just proved) that positive-times-negative-equals-negative, we know that $3 \bullet -5 = -15$ and so on. But what is

happening to these answers as we count down? They are going up by 5. So, continuing the trend, we find that $-1 \bullet -5 = 5$ and so on.

As I say, you may want to skip that. What is essential is to get across the point that we need a new definition that will cover negative exponents, and that we are going to get there by looking for consistency with the positive ones. Then they are ready for the in-class assignment: it shouldn't take long. Do make sure to give them 10 minutes for it, though—you want them to finish it in class, and have time to ask you questions, so you know they are ready for the homework.

Homework:

“Homework: Extending the Idea of Exponents”

5.4 Fractional Exponents⁴

Start by reminding them of where we are, in the big picture. We started with nothing but the idea that exponents mean “multiply by itself a bunch of times”—in other words, 7^4 means $7 \bullet 7 \bullet 7 \bullet 7$. We went from there to the rules of exponents— $x^a x^b = x^{a+b}$ and so on—by common sense. Then we said, OK, our definition only works if the exponent is a positive integer. So we found new definitions for zero and negative exponents, but extending down from the positive ones.

Now, we don't have a definition for fractional exponents. Just as with negative numbers, there are lots of definitions we could make up, but we want to choose one carefully. And we can't get there using the same trick we used before (you can't just count and “keep going” and end up at the fractions). But we still have our rules of exponents. So we're going to see what sort of definition of fractional exponents allows us to keep our rules of exponents.

From there, you just let them start working. I can summarize everything on the assignment in two lines.

1. The rules of exponents say that $(x^{\frac{1}{2}})^2 = x$. So whatever $x^{\frac{1}{2}}$ is, we know that when we square it, we get x . Which means, by definition, that it must be $\sqrt{x}$. Similarly, $x^{\frac{1}{3}} = \sqrt[3]{x}$ and so on.
2. The rules of exponents say that $(x^{\frac{1}{3}})^2 = x^{\frac{2}{3}}$. Since we now know that $x^{\frac{1}{3}} = \sqrt[3]{x}$, that means that $x^{\frac{2}{3}} = (\sqrt[3]{x})^2$. So there you have it.

Thirty seconds, written that way. A whole class period to try to get the students to arrive their on their own, and even there, many of them will require a lot of help to see the point. Toward the end, you may just call the class's attention to the board and write out the answers. But by the time they leave, you want them to have the following rule: for fractional exponents, the denominator is a root and the numerator is an exponent. And they should have some sense, at least, that this rule followed from the rules of exponents.

There is one more thing I really want them to begin to get. If a problem ends up with $\sqrt{25}$, you shouldn't leave it like that. You should call it 5. But if a problem ends up with $\sqrt{2}$, you should leave it like that: don't type it into the calculator and round off. This is also worth explicitly mentioning toward the end.

Homework:

“Homework: Fractional Exponents”. Mostly this is practicing what they learned in class. The inverse functions are a good exercise: it forces them to review an old topic, but also forces them to practice the current topic. For instance, to find the inverse function of $y = x^2$ [U+2044]3, you write:

$$x = y^{\frac{2}{3}} = \sqrt[3]{y^2}. \quad x^3 = y^2. \quad y = \sqrt{x^3} = x^{\frac{3}{2}}.$$

After you go through that exercise a few times, you start to see the pattern that the inverse function actually inverts the exponent. The extra fun comes when you realize that x^0 has no inverse function, just as this rule would predict.

At the end of the homework, they do some graphs—just by plotting points—you will want to make sure they got the shapes right, because this paves the way for the next topic. When going over the homework the next day, sketch the shapes quickly and point out that, on the graph of 2^x , every time you move on to the right, y doubles. On the graph of $(\frac{1}{2})^x$, every time you move one to the right, y drops in half.

⁴This content is available online at <http://cnx.org/content/m19322/1.2/>.

5.5 “Real Life” Exponential Curves⁵

This is another one of those topics where the in-class exercise and the homework may take a total of two days, combined, instead of just one. This is a difficult and important topic.

We begin with a lecture something like the following:

Earlier this year, we talked about “linear functions”: they add a certain amount every time. For instance, if you gain \$5 every hour, then the graph of your money vs. time will be a line: every hour, the total will add 5. The amount you gain each hour (5 in this case) is the slope.

Can a line also subtract every day? Sure! That isn’t a different rule, because adding is the same as subtracting a negative number. So if Mr. Felder is losing ten hairs a day, and you graph his hairs vs. time, the graph will be a line going down. The total subtracts 10 every day, but another way of saying that is, it adds -10 every day. The slope is -10 . This is still a linear function.

So why am I telling you all this? Because “exponential functions” are very similar, except that they multiply by the same thing every time. And, just as linear functions can subtract (by adding negative numbers), exponential functions can divide (by multiplying by fractions: for instance, multiplying by $\frac{1}{3}$ is the same as dividing by 3). The amount you multiply by is called...well, come to think of it, it doesn’t have a cool name like “slope.” I guess we could call it the “base.”

Then they can begin to work on the assignment. They will make it through the table all right. But when it comes to finding the formula for the n th day, many will fall down. Here is a way to help them. Go back to the table and say: “On day 3, let’s not write “4”—even though it is 4 pennies. It is 2 times the previous amount, so let’s just write that: 2×2 . On day 4, it’s 2 times that amount, or $2 \times 2 \times 2$. On day 5, it’s 2 times that amount, or $2 \times 2 \times 2 \times 2$. This is getting tedious...what’s a shorter way we can write that?” Once they have expressed every answer in powers of 2, they should be able to see the 2^{n-1} generalization. If they get the wrong generalization, step them through to the next paragraph, where they test to see if they got the right answer for day 30.

You go through the same thing on the compound interest, only harder. A lot of hand-holding. If you end one year with x then the bank gives you $.06x$ so you now have a total of $x + .06x$ which is, in fact, $1.06x$. So, hey, your money is multiplying by 1.06 every year! Which means if you started with \$1000 then the next year you had $\$1000 \bullet 1.06$. And the next year, you multiplied that by 1.06, so then you had $\$1000 \times 1.06 \times 1.06$. And the year after that....

Toward the end of class, put that formula, $\$1000 \times 1.06^n$, on the board. Explain to them that they can read it this way: “Just looking at it, we can see that it is saying you have \$1000 multiplied by 1.06, n times.” This is always the way to think about exponential functions—you are multiplying by something a bunch of times.

The assignment is also meant to bring out one other point that you want to mention explicitly at the end. When we developed our definitions of negative and fractional exponents, we wanted them to follow the rules of exponents and so on. But now they are coming up in a much more practical context, and we have a new need. We want $x^{2\frac{1}{2}}$ to be bigger than x^2 and smaller than x^3 , right? After all, after $2\frac{1}{2}$ years, you certainly expect to have more money than you had at the beginning of the year! It isn’t obvious at all that our definition, $x^{\frac{5}{2}} = \sqrt{x^5}$, will have that property: and if it doesn’t, it’s useless in the real world, even if it makes mathematicians happy. Fortunately, it does work out exactly that way.

Homework:

“Homework: ‘Real life’ exponential curves”

5.5.1 Time for Another Test!

The sample test will serve as a good reminder of all the topics we’ve covered here. It will also alert them that knowing why $x^{\frac{1}{2}}$ is defined the way it is **really does count**. And it will give them a bit more practice (much-needed) with compound interest.

⁵This content is available online at <<http://cnx.org/content/m19329/1.2/>>.

Chapter 6

Logarithms

6.1 Introduction¹

I talk to a surprising number of math teachers who are really uncomfortable with logs. There's something about this topic that just makes people squeamish in Algebra, in the same way that "proving a series converges" makes people squeamish in Calculus.

It doesn't have to be hard. It is not intrinsically more complicated than a radical. When you see $\sqrt[3]{x}$ you are seeing a mathematical question: "What number, raised to the 3rd power, gives me x ?" When you see $\log_3 x$ you are seeing a question which is quite similar: "3, raised to what power, gives me x ?" I say this about a hundred times a day during this section. My students may forget the rules of logs and they may forget what a common log is and they will almost certainly forget e , but none of them will forget that $\log_2 8$ means the question "2 to what power is 8?" You may want to show them the "Few Quick Examples" at the beginning of the Conceptual Explanations chapter to drive the point home.

It is possible to take any arbitrary logarithm on a standard scientific or graphing calculator. I deliberately never mention this fact to my students, until the entire unit (including the test) is over. Faced with $\log_2 8$ I want them to think it through and realize that the answer is 3 because $2^3 = 8$. The good news is, none of them will figure out how to do that problem on the calculator, if you don't tell them.

6.1.1 Introduction to Logarithms

This is a pretty short, self-explanatory exercise. There isn't anything you need to say before it. But you do need to do some talking after the assignment. Introduce the word "log" and explain it, as I explained it above: $\log_2 8$ means "2 to what power is 8?" Also discuss the fact that the log is always the inverse of the exponential function.

After they have done the assignment, and heard your explanation of the word log, then they are ready for the homework. It wouldn't hurt if that happens in the middle of the class, so they can get started on the homework in class, and finish it up at home. The in-class exercise is short, the homework is long.

"Homework: Logs"

When going over the homework the next day, #20 can be explained two ways. First: 5 to what power is 5^4 ? When asked that way, it's easy, isn't it? You don't have to find what 5^4 is, to see that the answer is 4! But there is also another way to explain it, which gets back to the idea of 5^x and $\log_5 x$ being inverse functions. The first function turns 4 into 5^4 . So the second one has to reverse this process, and turn 5^4 back into 4. This way is harder to understand, but it makes it a lot easier to see why #21 also has to be 4.

Then, there is the graph—as always, make sure they get the right general shape. Point out that the most salient feature of this graph is that it grows...incredibly...slowly as you go farther out to the right. (Every time x doubles, the graph just goes up by 1.) This is a lot of what makes logs useful, as we will see.

¹This content is available online at <<http://cnx.org/content/m19436/1.2/>>.

6.2 Properties of Logarithms²

This is very standard stuff. Using the in-class exercise in groups of 2 or 3, they should be able to find—in some cases with a bit of help from you—some rules of logarithms. In the homework, they practice using those rules.

One thing I don't do in the worksheets is formally prove the rules. However, I have been known to “throw in” the proofs sometimes in class, either for a group that finishes early, or for the whole class if enough people are interested. One of the proofs is provided as an example in the “Conceptual Explanations” along with guidelines for the other two.

But what I really care about is giving them an intuitive grasp of why the rules work, rather than the proof. The intuitive grasp is what comes from the exercise, from realizing that the logarithm is essentially a counter. Once you see that $\log_2 8$ is asking how many 2s there are in 8, then it's obvious that $\log_2 (8 \bullet 16)$ will add up all the 2s in 8, and in 16.

Homework:

“Homework: Properties of Logarithms”

6.3 Using the Laws of Logarithms³

Once you have gone through the laws of logarithms, you can spend five minutes working a couple of problems on the board, like:

$$\log_5(x) = \log_5(3)$$

and then

$$\log_5(x+1) + \log_5(x-1) = \log_5(8)$$

The first establishes that if you have $\log_{(\text{this})} = \log_{(\text{that})}$, then this must equal that. The second shows how you have to use the laws of logs to get into that form. (The OK students will answer 3. The better students will answer ± 3 . Only the very best will get ± 3 and then realize that the -3 is, after all, invalid! But all of that is a detail, of course.)

Anyway, then there is the worksheet full of problems like that, which also gives good review of a number of old topics.

The thing is, this really isn't a whole day. Sneak it in when you have 15-20 minutes left in class. It doesn't matter whether it comes before, after, or in the middle of the next topic.

6.4 So What Are Logarithms Good For, Anyway?⁴

As always, things get harder when we get into word problems. There are a few things I want them to take away here.

First—logs are used in a wide variety of real world situations.

Second—logs are used because they compress scales. In other words, because they grow so slowly, we use logarithmic scales whenever we want to work with a function that, by itself, grows too quickly. Or, to put it another way, we use logarithms whenever something varies so much that you don't care exactly what is, just what the power of 10 is. Don't say all this before they start working, but hopefully they will come up with something like this on #6.

Homework:

“Homework: What Are Logarithms Good For, Anyway?”

In addition to following up on the in-class work, the homework here also introduces the common and natural logs. It's a bit of a weak connection, but I had to stick them somewhere.

²This content is available online at <http://cnx.org/content/m19438/1.1/>.

³This content is available online at <http://cnx.org/content/m19440/1.1/>.

⁴This content is available online at <http://cnx.org/content/m19439/1.2/>.

6.4.1 Time for Another Test!

The sample test is actually pretty important here. It pulls together a lot of ideas that have been covered pretty quickly.

The extra credit is just a pun. The answer is log cabin or, better yet, natural log cabin. Who says math can't be fun?

According to my reckoning, you are now approximately halfway through the curriculum. Mid-terms are approaching. If there are a couple of weeks before mid-terms, I would not recommend going on to radicals—spend a couple of weeks reviewing. Each topic (each test, really) can stand a whole day of review. It may be the most important time in the whole class!

Chapter 7

Rational Expressions

7.1 Introduction¹

We’ve talked about the word “rational”—it doesn’t mean “sane,” it means a “ratio” or, in other words, a fraction. A rational expression is just a fraction with variables.

This section is unique, perhaps, in the fact that it introduces practically no new skills. They have to be able to factor; they have to know the rules of exponents; they have to be able to work with fractions; they even have to be able to do long division. There is nothing new in any of that. It’s just putting it all together to simplify, and work with, rational expressions.

Part of the benefit of this unit is that there are always a few kids in class—maybe more than a few—who have a lingering, secret fraction-phobia. They are hoping that no one will ever notice because the calculator will always rescue them. You can spot these people because they always answer everything—including “what is 2 divided by 3?”—in decimals. But this unit will flush them out. You can’t get through rational expressions unless you know how to do fractions, and your calculator will not help you. (I always point this out, very explicitly, several times.) In the “Conceptual Explanations” I begin each section by working plain-old-number-fraction problems (simplifying them, multiplying them, adding them, and so on); tell them they can look there if they want a quick review.

Because of the nature of this unit—no new concepts, and fraction phobia—it has fewer “creative thinking” types of problems, and more “drill and practice,” than any other unit. It gets boring for you, but don’t let them see that. For a few students at least, this has the potential to break down a barrier that they have been struggling with since the third grade.

7.2 Rational Expressions²

Begin by explaining what rational expressions are, and making the points I made above—we are going to put together some of our old skills in a new way, which will require us to be good at fractions. So, we’re going to start by reviewing how to work with fractions.

Then pair the students up. We’re going to do a sort of do-it-yourself TAPPS exercise. One partner is the student, one is the teacher. The teacher’s job is to add $\frac{1}{2} + \frac{1}{3}$ —not just to come up with the answer, but to walk through the process, explaining what he is doing and why he is doing it at every step, all on paper. The student asks for clarifications of any unclear points. By the time they are done, they should have a written, step-by-step instruction guide for adding fractions.

Then they switch roles. The former student becomes the new teacher, and gives two lessons: how to multiply $(\frac{1}{2})(\frac{1}{3})$ and how to divide $\frac{1}{2}/\frac{1}{3}$. Once again, they should wind up with a step-by-step guide.

¹This content is available online at <http://cnx.org/content/m19486/1.1/>.

²This content is available online at <http://cnx.org/content/m19488/1.1/>.

The original teacher takes over again, and shows how to simplify a fraction.

Finally, you give a brief lesson on multiplying fractions—they’ve already done that, but the key point to emphasize here is that you can cancel before you multiply. For instance, if you want to multiply $\frac{7}{8}$ times $\frac{10}{21}$, you could say:

$$\frac{7}{8} \times \frac{10}{21} = \frac{70}{168}$$

and then try to simplify that. But it’s a lot easier to simplify before you multiply. The $\frac{7}{21}$ becomes $\frac{1}{3}$, the $\frac{10}{8}$ becomes $\frac{5}{4}$, so we have:

$$\frac{\cancel{7}^1}{\cancel{8}_4} \bullet \frac{\cancel{10}^5}{\cancel{21}_3} = \frac{5}{12}$$

Figure 7.1

Of course, you can only do this trick—canceling across different fractions—when you are multiplying. Never when you are adding, subtracting, or dividing!

So why are we going through all this? Because, even though they know how to do it with numbers, they are going to get confused when it comes to doing the exact same thing with variables. So whenever they ask a question (“What do I do next?” or “Do I need a common denominator here?” or some such), you refer them back to their own notes on how to handle fractions. I have had a lot of students come into the test and immediately write on the top of it:

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

They did this so they would have a “template” to follow when adding rational expressions—it’s a very smart move.

Other than basic fraction manipulation, there is only one other big thing to know about rational expressions—always factor first. Factoring shows you what you can cancel (especially when multiplying), and how to find the least common denominator (when adding or subtracting).

So, walk through some sample problems for them on the blackboard. Put up the problem, and ask them what the first step is...and what the second step is...and so on, until you have something like this on the blackboard.

$$\frac{1}{x} + \frac{2}{y} = \frac{y}{xy} + \frac{2x}{xy} = \frac{y+2x}{xy}$$

Emphasize over and over that this is just the same steps you would take to add $\frac{1}{2} + \frac{2}{3}$. But at the end, point out one other thing—just as we did in the very first week of class, we have asserted that two functions are equal. That means they should come out exactly the same for any x and y . So have everyone choose an x -value and a y -value, and plug them into both $\frac{1}{x} + \frac{2}{y}$ and $\frac{y+2x}{xy}$, and make sure they come out with the same number.

Now walk through something harder on the blackboard, like this:

$\frac{x}{x^2+5x+6} - \frac{2x}{x^3-9x}$	The original problem
<i>continued on next page</i>	

$\frac{x}{(x+3)(x+2)} - \frac{2x}{x(x+3)(x-3)}$	Always factor first!
$\frac{x}{(x+3)(x+2)} - \frac{2}{(x+3)(x-3)}$	Simplify (cancel the “ x ” terms on the right).
$\frac{x(x-3)}{(x+3)(x+2)(x-3)} - \frac{2(x+2)}{(x+3)(x-3)(x+2)}$	Get a common denominator. This step requires a lot of talking through. You have to explain where the common denominator came from, and how you can always find a common denominator once you have factored.
$\frac{(x^2-3x)-(2x+4)}{(x+2)(x+3)(x-3)}$	Now that we have a common denominator, we can combine. This step is a very common place to make errors—by forgetting to parenthesize the $(2x+4)$ on the right, students wind up adding the 4 instead of subtracting it.
$\frac{x^2-5x-4}{(x+2)(x+3)(x-3)}$	Done! Of course, we could multiply the bottom through, and many students want to. I don’t mind, but I don’t recommend it—there are advantages to leaving it factored.

Table 7.1

Once again, have them try numbers (on their calculators) to confirm that $\frac{x}{x^2+5x+6} - \frac{2x}{x^3-9x}$ gives the same answer as $\frac{x^2-5x-4}{(x+2)(x+3)(x-3)}$ for any x . But they should also remember (from week 1) how to find the domain of a function—and in this case, the two are not quite the same. The function we ended up with excludes $x = -2$, $x = -3$, and $x = 3$. The original function excludes all of these, but also $x = 0$. So in that one case, the two are not identical. For all other cases, they should be.

Whew! OK, you’ve been lecturing all day. If there are 10 minutes left, they can begin the exercise. They should work individually (not in groups or pairs), but they can ask each other for help.

Homework:

Finish the in-class exercise and do “Homework—Rational Expressions”

7.3 Rational Equations³

After you have answered all the questions on the previous homework, they can just get started on this assignment immediately—it should explain itself pretty well.

However, after about 5 minutes—when everyone has gotten past the first two problems—pull them back and talk to the whole class for a moment, just to make sure they get the point. The point is that if the denominators are the same, then the numerators must be the same (#1); and if the denominators are not the same, then you make them the same (#2). It’s pretty straightforward with these two problems, but it may be deceptively easy. The real thing to make sure they “get” is that, having established these two principals with these easy problems, they are now going to apply them in much more complicated ones.

Then they can get back to it, and you just float around and help. In #3, the only trick is remembering that if $x^2 = 25$, then $x = \pm 5$ (not just 5). Numbers 4 is straightforward. Give them time to struggle with #5 before pointing out that they should factor-and-simplify first—always factor first! #6 is really what all this is building up to: to solve rational equations in general, you must be able to solve quadratic equations!

Homework:

“Homework: Rational Expressions and Equations”

³This content is available online at <http://cnx.org/content/m19489/1.1/>.

7.4 Polynomial Division⁴

Half the class, maybe the whole class, will be lecture today—you have to show them how to do this. The lecture goes something like this.

Today we're going to talk about everybody's favorite topic...long division!

Before we do that, I have to start by pointing out some very important cases where you don't have to use long division. For instance, suppose you have this:

$$\frac{36x^3 + 8x^2 + 5x + 10}{2x} \quad (7.1)$$

That problem doesn't require any hard work—you should be able to divide it on sight. (Have them do this.) You should have gotten:

$$18x^2 + 4x + 2\frac{1}{2} + \frac{5}{x}$$

To take another example, how about this?

$$\frac{x^3 - 6x^2 + 5x}{x^2 - 5x} \quad (7.2)$$

That one isn't quite as easy. What do we do first? (factor!) Oh yes, let's do that! So it is $\frac{x(x-5)(x-1)}{x(x-5)}$. Oh, look...it's just $x - 1$! Once again, no long division necessary.

OK, but suppose we had this?

$$\frac{6x^3 - 8x^2 + 4x - 2}{2x - 4} \quad (7.3)$$

How can we simplify it? This is where we're going to have to use...long division.

So, let's start the same way we started with fractions: by remembering how to do this with numbers. Everyone, at your seats, work out the following problem on paper.

$$\frac{4327}{11} \quad (7.4)$$

(Here you pause briefly while they work it on paper—then you work it on the blackboard.) OK, you should have gotten 393 with a remainder of 4. So the actual answer is $393 \frac{4}{11}$. How could we check that? That's right...we would want to make sure that $393 \frac{4}{11}$ times 11 gives us back 4327. Because that's what multiplication is—it's division, backward.

Now, let's go back to that original problem. OK, kids...I'm going to leave my number long division over here on the blackboard, and I'm going to work this rational expressions long division next to it on the blackboard, so you can see that all the steps are the same.

We'll start here:

$$2x-4 \overline{) 6x^3-8x^2+4x-2}$$

Figure 7.2

From there, develop the whole thing on the blackboard, step by step. (I do this exact problem in the "Conceptual Explanations." I would not suggest you tell them to look it up at this point; instead, I would recommend that you go through it on the blackboard, explaining steps as you go, and continually reinforcing

⁴This content is available online at <<http://cnx.org/content/m19487/1.2/>>.

the analogy to what you did with numbers.) In the end you conclude that $\frac{6x^3-8x^2+4x-2}{2x-4}$ is $3x^2 + 2x + 6$ with a remainder of 22: or, to put it another way, $3x^2 + 2x + 6 + \frac{22}{2x-4}$.

So then you ask: OK, how could we test that? Hopefully they will come up with one answer, then you say “Good, how else?” and they come up with the other one. One way is to plug a number—any number—into $\frac{6x^3-8x^2+4x-2}{2x-4}$, and the same number into $3x^2 + 2x + 6 + \frac{22}{2x-4}$, and make sure you get the same thing. The other way is to multiply back $3x^2 + 2x + 6 + \frac{22}{2x-4}$ by $2x - 4$ and make sure you get back to $6x^3 - 8x^2 + 4x - 2$. Make sure they understand both ways. If you don’t understand the first way, you don’t understand function equality; if you don’t understand the second way, you don’t understand what division is!

Finally, you ask: how could we have made all that a bit easier? The answer, of course, is that we should have divided the top and bottom by 2 before we did anything else. This brings us back to our cardinal rule of rational expressions: always factor first!

If there is still time left in class, let them get started on the homework.

Homework:

“Dividing Polynomials”

When going over it, see how many people did #1 the “hard way.” Remind them that if the bottom is only one term, you can just do the whole thing quickly and painlessly!

Optional Exercise:

If you have extra time—if some students get way ahead and you want to give them an extra assignment, or if you want to spend more time on this topic—here is a good exercise that brings things together. Suppose you want to solve the equation $6x^3 - 5x^2 - 41x - 30 = 0$. You can use the “Solver” on the calculator and it will find one answer: most likely, $x = -1$. (This requires a 5-minute introduction to the “Solver.”) How do you find the other answers?

Well, we recall from our study of quadratic equations that if $x = -1$ is an answer, then the original function must be expressible as $(x + 1)$ (something). How do you find the something? With long division! (Maybe have them try it both ways.) What you end up with, after you divide, is $6x^2 - 11x - 30$. You can factor that the “old-fashioned” way (which takes a bit of time) and you get $(2x + 3)(3x - 10)$ which gives you the other two roots.

7.4.1 Time for Another Test!

And we’re done with yet another unit.

On #6 of the sample test, stress that they will get no credit without showing work. They can check their answer either way—multiplying back, or trying a number—but they have to show their work.

The extra credit is a good problem that I like to get in somewhere. I generally give one point for the obvious pairs (0,0) and (2,2), and two points for the equation $xy = x + y$. But what I really want to see is if they remember how to solve that for y and get $y = \frac{x}{x-1}$. Finally, from that, they should be able to see that $x = 1$ has no pairing number (which is obvious if you think about it: nothing plus one gives you the same thing times one!).

Chapter 8

Radicals

8.1 Introduction¹

Well, this is easier, isn't it? They know what a radical is. But they're going to go places with them that they have definitely never been before...

This looks long, but a lot of it is very fast. You don't have to do much setup, except to remind them what a square root is. I would explain it by analogy to the way we explained logs. $\log_2 8$ asks the question "2 to what power is 8?" Well, $\sqrt{9}$ also asks a question: "What squared is 9?"

But then, there is an important distinction—one that I like to make right away, and then repeat several times. The question "what squared is 9?" actually has two answers. So if we defined square root as the answer to that question, square root would not be a function—9 would go in, and both 3 and -3 would come out (the old "rule of consistency" from day 1). So we somewhat arbitrarily designate the $\sqrt{}$ symbol to mean the positive answer, so that it is a function. So if you see $x^2 = 9$ you should properly answer $x = \pm 3$. But if you see $x = \sqrt{9}$ then you should answer only $x = 3$. This is a subtle distinction, but I really want them to get it, and to see that it is nothing inherent in the math—just a definition of the square root, designed to make it single-valued. This is why if you see $x^2 = 2$ you have to answer $x = \pm\sqrt{2}$, to get both answers.

So, on to the assignment. It starts with a couple of word problems, just to set up the idea that radicals really are useful (which is not obvious). After everyone is done with that part, you may want to ask them to make up their own problems that require square roots as answers (they are not allowed to repeat #1). Get them to realize that we square things all the time, and that's why we need square roots all the time, whenever we want to get back. (We'll be returning to this theme a lot.)

Then there are problems with simplifying radicals. For many of them, they have seen this before—they know how to turn $\sqrt{8}$ into $2\sqrt{2}$. But they don't realize that they are allowed to do that because of the general rule that $\sqrt{ab} = \sqrt{a}\sqrt{b}$. So it's important for them to get that generalization, but it's also important for them to see that it allows them to simplify radicals. And it's equally important for them to see that $\sqrt{a+b}$ is not $\sqrt{a} + \sqrt{b}$.

The final question is a trap, of course—many of them will answer x^4 . But they should know they can test their answer by squaring back, and oops, $(x^4)^2$ is not x^{16} .

There are two ways to look at this problem correctly. One is that $\sqrt{x^{16}}$ asks a question: "What number, squared, is x^{16} ?" The rules of exponents are enough to answer this with x^8 . The other way to look at it is to remember that raising something to the $\frac{1}{2}$ power is the same as taking a square root. So $\sqrt{x^{16}}$ is the same as $(x^{16})^{\frac{1}{2}}$ which, again by the rules of exponents, is x^8 . (I prefer the first way.)

Homework:

"Homework: Radicals"

¹This content is available online at <http://cnx.org/content/m19484/1.2/>.

When going over the homework the next day, make sure to talk about the last few problems (the inverse functions). Make sure they tested them! There are several points that you want to make sure they got.

- x^2 has no perfect inverse. $\sqrt{x}$ works only if we confine ourselves to positive numbers. On the other hand, $\sqrt[3]{x}$ is a perfect inverse of x^3 . I always take a moment here to talk about what $\sqrt[3]{x}$ is, and also to make sure they understand that it is not the same thing as $3\sqrt{x}$, so it's vitally important to be careful in how you write it.
- x^3 and $3x$ are completely different functions. This is why exponents really have two different inverses, logs and radicals. Our last unit was on one of them, this unit will be on the other.

8.2 A Bunch of Other Stuff About Radicals²

Yeah, it's sort of a grab bag—a miscellaneous compilation of word problems, review from yesterday, and so on. You can just get them started working on the assignment after you're done going over yesterday's homework.

Toward the end, however, they are going to start running into trouble. This is when you introduce rationalizing the denominator. You may want to bring the whole class together to see who can figure out how to rationalize $\frac{1}{\sqrt{3}+1}$. It's a great opportunity to review our rules of multiplying binomials: $(a+b)^2 = a^2 + 2ab + b^2$ which is why multiplying by $\sqrt{3}+1$ doesn't work; $(a-b)^2 = a^2 - b^2$ which is why multiplying by $\sqrt{3}-1$ does.

But please be very careful here, because this particular topic has a very subtle danger. A lot of teachers communicate the idea that denominators should always be rationalized, “just because”—because I said so, or because somehow $\frac{\sqrt{2}}{2}$ is “simpler” than $\frac{1}{\sqrt{2}}$. This is one of the best ways to convince students that math just doesn't make sense.

What I'm trying to do with this exercise is demonstrate a real practical benefit of rationalizing the denominator, which is that it helps you add and subtract fractions. It's difficult or impossible to come up with a common denominator without doing this first!

And of course, we have the “you never understand a function until you've graphed it” question. Talk a bit about the graph after they get it. They should be able to see that the domain and range are both ≥ 0 and why this must be so. They should see that for large values, it grows very slowly (like a log), but without the drastic behavior that the log shows in the $0 \leq x \leq 1$ range.

Homework:

“Homework: A Bunch of Other Stuff About Radicals”

8.3 Radical Equations³

If you read over the assignment carefully, I think it's pretty self-explanatory. Encourage the students to read it carefully as they go (not just skip to the equations).

In my experience, most the trouble with this section comes from trying to make easy problems, hard (that is, squaring both sides when you don't need to); or from trying to make hard problems, easy (neglecting to square both sides when you should). Make sure they are clear on the distinction—if there is a variable under the radical, you will need to square; otherwise, you won't.

Also, it can't hurt to say this about a hundred times: whenever you square both sides, you have to check your answers—they may not work even if you did all your math right! Make sure they understand when this rule applies, and also why squaring both sides can introduce false answers. (I work through that explanation pretty carefully in the “Conceptual Explanations.”)

Homework:

“Homework: Radical Equations”

²This content is available online at <http://cnx.org/content/m19483/1.1/>.

³This content is available online at <http://cnx.org/content/m19485/1.2/>.

8.3.1 Time for another test!

Not much to say here, except that the real point of the extra credit is to see if they realize that the behavior will be very similar on the right, but it will extend down to the left as well.

Chapter 9

Imaginary Numbers

9.1 Introduction¹

This is an interesting unit in several ways, both good and bad.

The good news is, it's fun. It's like a game, and I always try to present it that way.

The bad news is, it's incredibly abstract. It's abstract because it's hard to understand these numbers-that-aren't-numbers, and it's also abstract because, to save my life, I can't come up with any good explanation of why imaginary numbers are useful. Of course, they are useful—invaluable even—but how can I explain that to an Algebra II student? Here are a few things I do always say (several times).

1. These numbers are indeed useful, and they are used in the real world, even if I can't do a great job of explaining why to you right now.
2. Nothing in the real world is imaginary. That is, you will never have i tomatoes, or measure a brick that is $5i$ feet long, or wait for $3i + 2$ seconds. So why are these useful? Because there are very often problems where the problem is real, and the answer is real, but in between, as you get from the problem to the answer, you have to use imaginary numbers. Repeat this several times. You have a problem, or real-world situation, which (of course) involves all real numbers. You do a bunch of math, which includes imaginary numbers. In the end, you wind up with the answer, which (of course) involves all real numbers again. But it would have been difficult or impossible to find that answer, if you didn't have imaginary numbers.
3. One example is electrical engineering. In an electric circuit you have resistors, capacitors, and inductors. They all act very differently in the circuit. When you model the circuit mathematically, to determine how it will behave (what current will flow through it), the inductor has an inductance and the capacitor has a capacitance and the resistor has a resistance and they are all very different, which makes the math really hairy. However, you can define a complex quantity called impedance which makes resistors, capacitors, and inductors all look mathematically the same. The disadvantage is that you are now working with a complex number instead of all real numbers. The advantage is that resistors, capacitors, and inductors now look the same in the equations, which makes life a whole lot simpler. So this is a good example of how you use complex numbers to make the math easier. (As a side note, electrical engineers call the imaginary number j whereas everyone else on the planet calls it i . I think kids like that bit of trivia.)
4. Imaginary numbers are also used in many other applications, such as quantum mechanics.

It's all very hand-wavy, and I admit that up front, and I don't hold the kids responsible for it. But I want them to know that this is really useful, and at the same time, I want to explain why we aren't going to have any “real world” problems in this unit.

¹This content is available online at <http://cnx.org/content/m19424/1.2/>.

This lecture, by the way, usually comes toward the end of day 1, or during day 2—not at the very beginning of day 1. In the beginning, I prefer to treat it as a game—“What if there were a square root of -1 ? Just suppose, what if there were?” With one class I went so far in treating it as a game that it was day 3 before they realized I wasn’t making the whole thing up.

Oh yeah, one more thing. The calculators will do imaginary numbers for them. I never tell them this. If they figure it out, more power to them. But I literally don’t tell them until after the test that the calculator knows anything at all about imaginary numbers! I want them to be able to do these things on their own.

9.2 Introduction to Imaginary Numbers²

This is a fun day, or possibly two days. The first exercise is something that, in theory, they could walk all the way through on their own. But it sets up all the major themes in imaginary numbers.

In practice, of course, some groups will have problems, and will need help at various points. But beyond that, almost no groups will see the point of what they have done, even if they get it right. So a number of times in class, you are going to interrupt them and pull them back together into a classwide discussion, and discuss what they have just done. The ideal time to do this is after everyone in the class has reached a certain point—for instance, after they have all done #2 (or struggled with it in vain), you pull them back and talk about #2. All my suggested interruptions are described below.

Before you start, remind them that the equation $x^2 = - - 1$ has no answer, and talk about why. Then explain that we are going to pretend it has an answer. The answer is, of course, an “imaginary” number, so we will call it i . The definition of i is therefore $i = \sqrt{-1}$ or, equivalently, $i^2 = - - 1$.

There are two ways to play this. One is to go into the whole “why i is useful” spiel that I spelled out above. The other approach, which is the one I take, is to treat it as a science fiction exercise. I always start by telling the class that in good science fiction, you start with some premise: “What if time travel were possible?” or “What if there were a man who could fly?” or something like that. Then you have to follow that premise rigorously, exploring all the ramifications of that one false assumption. So that is what we are going to do with our imaginary number. We are going to start with one false premise: “What if you could square something and get -1 ?” And we are going to follow that premise logically, using all the rules of math, and see where it would lead us.

Then they get started. And in #2, they get stopped in their tracks. So you give them a minute to struggle, and then walk it through on the board like this.

- $- - i$ means $- - 1 \bullet i$ (*Stress that this is not anything unusual about i , it is a characteristic of -1 . We could just as easily say -2 means $- - 1 \bullet 2$, and so on. So we are treating i just like any other number.)
- So $i(- - i)$ is $i \bullet - - 1 \bullet i$.
- But we can rearrange that as $i \bullet i \bullet - - 1$. (You can always rearrange multiplication any way you want.)
- But $i \bullet i$ is -1 , by definition. So we have $- - 1 \bullet - - 1$, so the final answer is 1.

The reason to walk through this is to get across the idea of what I meant about a science fiction exercise. Everything we just did was simply following the rules of math—except the last step, where we multiplied $i \bullet i$ and got -1 . So it illustrates the basic way we are going to work: assume that all the rules of math work just like they always did, and that $i^2 = -1$.

The next few are similar. Many of them will successfully get $\sqrt{-25}$ on their own. But you will have to point out what it means. So, after you are confident that they have all gotten past that problem (or gotten stuck on it), call the class back and talk a bit. Point out that we started out by just defining a square root of -1 . But in doing so, we have actually found a way to take the square root of any negative number! There are two ways to see this answer. One is (since we just came off our unit on radicals) to write $\sqrt{-25} = \sqrt{25 \cdot -1} = \sqrt{25}\sqrt{-1} = 5i$. The other—which I prefer—is to say, $\sqrt{-25}$ is asking the question “What number squared is -25 ? The answer is $5i$. How do you know? Try it! Square $5i$ and see what you get!”

²This content is available online at <http://cnx.org/content/m21990/1.1/>.

Now remind them of the subtle definition of square root as the positive answer. If you see the problem $x^2 = -25$ you should answer $x = \pm 5i$ (take a moment to make sure they all got the right answer to #4, so they see why $(-5i)^2$ gives -25). On the other hand, $\sqrt{-25}$ is just $5i$.

Next we move on to the cycle of powers. Again, they should be able to do this largely on their own. If a group needs a hint, remind them that if we made a similar table with powers of 2 (2^1 , 2^2 , 2^3 , and so on), we would get from each term to the next one by multiplying by 2. So they should be able to figure out that in this case, you get from each term to the next by multiplying by i , and they should be able to do the multiplication. They will see for themselves that there is a cycle of fours. So then you can ask the whole class what i^{40} must be, and then i^{41} and so on, and get them to see the general algorithm of looking for the nearest power of 4. (I have tried mentioning that we are actually doing modulo 4 arithmetic and I have stopped doing this—it just confuses things. I do, however, generally mention that the powers of -1 go in a cycle of 2, alternating between 1 and -1 , so this is just kind of like that.)

#13 is my favorite “gotcha” just to see who falls into the trap and says it’s 9–16.

After they do #18, remind them that this is very analogous to the way we got square roots out of the denominator. And this is not a coincidence— i is a square root, after all, that we are getting out of the denominator! You may want to introduce the term “complex conjugate” even at this stage, but the real discussion of complex numbers will come later.

Homework:

“Homework: Imaginary Numbers”

When going over this homework the next day, make sure they got the point. Our “pattern of fours” can be walked backward as well as forward. It correctly predicts that $i^0 = 1$ which it should anyway, of course, since anything⁰ = 1. It correctly predicts that $i^{-1} = -i$ which is less obvious—but remind them that, just yesterday, they showed in class that $\frac{1}{i}$ simplifies to $-i$!

9.3 Complex Numbers³

The first thing you need to do is define a complex number. A complex number is a combination of real and imaginary numbers. It is written in the form $a + bi$, where a and b are both real numbers. Hence, there is a “real part” (a) and an “imaginary part” (bi). For instance, in $3 + 4i$, the real part is 3 and the imaginary part is $4i$.

At this point, I like to try to put this in context, by talking about all the different kinds of numbers we have seen. We started with counting numbers: 1, 2, 3, 4, and so on. If you are counting pebbles, these are the only numbers you will ever need.

Then you add zero, and negative numbers. Are negative numbers real things? Can they be the answers to real questions? Well, sure...depending on the question. If the question is “How many pebbles do you have?” or “How many feet long is this stick?” then the answer can never be -2 : negative numbers are just not valid in these situations. But if the answer is “What is the temperature outside?” or “How much money is this company worth?” then the answer can be negative. This may seem like an obvious point, but I’m building up to something, so make sure it’s clear—we have invented new numbers for certain situations, which are completely meaningless in other situations. I also stress that we have gone from the counting numbers to a more general set, the integers, which includes the counting numbers plus other stuff.

Then we add fractions, and the same thing applies. If the question is “How many pebbles do you have?” or “How many live cows are on this farm?” the answer can never be a fraction. But if the question is “How many feet long is this stick?” a fraction may be the answer. So again, we have a new set—the rational numbers—and our old set (integers) is a subset of it. And once again, these new numbers are meaningful for some real life questions and not for others. I always mention that “rational numbers” (“rational” not meaning “sane,” but meaning rather a “ratio”) are always expressible as the ratio of two integers, such as $\frac{1}{2}$ or $\frac{-22}{7}$. So since we have already defined the integers, we can use them to help define our larger set, the rational numbers.

³This content is available online at <http://cnx.org/content/m19423/1.2/>.

But some numbers are not rational—they cannot be expressed as the ratio of two integers. These are the irrational numbers. Examples are π , e , and $\sqrt{2}$ (or the square root of any other number that is not a perfect square).

If we add those to our collection—put the rational and irrationals together—we now have all the real numbers. You can draw a number line, going infinitely off in both directions, and that is a visual representation of the real numbers.

And now, finally, we have expanded our set even further, to the complex numbers, $a + bi$. Just as we piggybacked the definition of rational numbers on top of our definition of integers, we are piggybacking our definition of complex numbers on top of our definition of real numbers. The complex numbers are, as far as I know, the broadest set—it doesn't get any more general than this.

All that may sound unnecessary, and of course, it is. But some students really get into it. I have had students draw the whole thing into a big Venn diagram—which I did not ask them to do. (*It is a good extra credit assignment, though.) My own diagram is at the very end of this unit in the “Conceptual Explanations,” under the heading “The World of Numbers.” Many students like seeing all of math put into one big structure. And it helps make the point that complex numbers—just like each other generalization—are valid answers to some questions, but not to others. In other words, as I said before, you will never measure a brick that is 5i inches long. (It never hurts to keep saying this.)

The complex numbers are completely general—any number in the world can be expressed as $a + bi$. This is not obvious! There are plenty of things you can write that don't look like $a + bi$. One example is $\frac{1}{i}$ which does not look like $a + bi$ but can be put into that form, as we have already seen. Other examples are 2^i and $\ln(i)$, which we are not going to mess with, but they are worth pointing out as other examples of numbers that don't look like $a + bi$, but take-my-word-for-it you can make them if you want to. And then there is $\sqrt{i}$, which we are going to tackle tomorrow.

That is probably all the setup you need. They can do the in-class exercise on Complex Numbers and see for themselves that whether you add, subtract, multiply, or divide them, you get back to a complex number. Also make sure they get the point about what it means for two complex numbers to be equal: this will be very important as we move on. One other thing I like to mention at some point (doesn't have to be now) is that there are no inequalities with imaginary numbers. You cannot meaningfully say that $1 > i$ or that $1 < i$. Because they cannot be graphed on a number line, they don't really have “sizes”—they can be equal or not, but they cannot be greater than or less than each other.

Homework:

“Homework: Complex Numbers”

When going over this homework, make a special point of talking about #12. This helps reinforce the most important point of the year, about generalizations. Once you have found what happens to $(a + bi)$ when you multiply it by its complex conjugate, you have a general formula which can be used to multiply any complex number by its complex conjugate, without actually going through the work. Show them how #6, 8, and 10 can all be solved using this formula. Also, remind them that a and b are by definition real—so the answer $a^2 + b^2$ is also real. That is, whenever you multiply a number by its complex conjugate, you get a real answer. This is why there is no possible answer to #14.

9.4 Me, Myself, and the Square Root of i ⁴

This is arguably the most advanced, difficult thing we do all year. But I like it because it contains absolutely nothing they haven't already done. It's not here because it's terribly important to know $\sqrt{i}$, or even because it's terribly important to know that all numbers can be written in $a + bi$ format. It is here because it reinforces certain skills—squaring out a binomial (always a good thing to practice), working with variables and numbers together, setting two complex numbers equal by setting the real part on the left equal to the real part on the right and ditto for the imaginary parts, and solving simultaneous equations.

⁴This content is available online at <http://cnx.org/content/m19425/1.2/>.

Explain the problem we're going to solve, hand it out, and let them go. Hopefully, by the end of class, they have all reached the point where they know that $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ are the two answers, and have tested them.

Note that right after this in the workbook comes a more advanced version of the same thing, where they find $\sqrt[3]{-1}$ (all three answers: -1 , $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$). I tried using this for the whole class, and it was just a bridge too far. But you could give it to some very advanced students—either as an alternative to the $\sqrt{i}$ exercise, or as an extra credit follow-up to it.

Homework:

They should finish the worksheet if they haven't done so, including #7. Then they should also do the "Homework: Quadratic Equations and Complex Numbers." It's a good opportunity to review quadratic equations, and to bring in something new! (It's also a pretty short homework.)

When going over the homework, make sure they did #3 by completing the square—again, it's just a good review, and they can see how the complex answers emerge either way you do it. #4 is back to the discriminant, of course: if $b^2 - 4ac < 0$ then you will have two complex roots. The answer to #6 is no. The only way to have only one root is if that root is 0. (OK, 0 is technically complex...but that's obviously not what the question meant, right?)

The fun is seeing if anyone got #5. The answer, of course, is that the two roots are complex conjugates of each other—real part the same, imaginary part different sign. This is obvious if you rewrite the quadratic formula like this:

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

and realize that the part on the left is always real, and the part on the right is where you get your i from.

9.4.1 Time for another test!

Not much to say here, except that you may want to reuse this extra credit on your own test—if they learn it from the sample (by asking you) and then get it right on your test, they learned something valuable.

Chapter 10

Matrices

10.1 Introduction¹

This is a “double” unit—that is, it is so long that I have a major test right in the middle of it.

The EOC spends an inordinate amount of time on problems like this:

The Kind of Problem I Don't Bother Too Much With

This matrix shows McDonald's sales for a three-day period.

	Big Macs	Fries	Coke
Monday	\$1,000	\$500	\$2,000
Tuesday	\$1,500	\$700	\$2,700
Wednesday	\$800	\$800	\$1,500

Table 10.1

What were their total sales on Monday? What were their total sales of Big Macs? On which day did they make the most profit? etc etc...

I guess the object is to make it appear that “matrices are useful” but it is really deceptive. Of course, matrices are useful, but not because they give you a convenient way to organize tabular data and then add columns or look things up.

So I don't spend much time on this kind of thing. I start with what a matrix is (which is sort of like that). I develop the rules for adding matrices, subtracting them, multiplying a matrix by a constant, and setting two matrices equal to each other—all of which are very obvious, and should not be presented as a mystery, but rather just as something obvious.

Then comes the big two days of magic, in which we learn to multiply matrices. I use a “gradebook” application which gives an example of why you would want to do this strange operation—it makes a lot of sense up to the point where you are multiplying an arbitrary-dimensions matrix by a column matrix, although it gets a bit strained when you expand the second matrix. No matter. They need to get the mechanics of how you multiply matrices, and just practice them.

After that bit of magic, the rest should follow logically. The definition of $[I]$, the definition of an inverse matrix and how you find one, and (the final hoorah) the way you use matrices to solve linear equations, should all be logical and consistent, based on the one magic trick, which is multiplying them. Oh, also there is a magic trick where you find determinants, which doesn't have much to do with anything else.

There is one other thing I need to address, which is calculators. There is a day that I set aside to teach them explicitly how to do matrices on the calculator. But that day is after the first test. Before that day, I

¹This content is available online at <<http://cnx.org/content/m19445/1.1/>>.

don't mention it at all. And even after that day, I stress doing things by hand, and give them problems that will force them to do so (by using variables). But I do love showing them that you can solve five equations with five unknowns quickly and easily by using matrices and a calculator!

10.1.1 Introduction to Matrices

Tell them to get into groups and work on “Introduction to Matrices.” I think it is very self-explanatory. You may want to make the analogy at some point that setting two matrices equal to each other is kind of like setting two complex numbers equal to each other: for “this” to equal “that,” all their respective parts must be equal.

Homework:

“Homework—Introduction to Matrices”

10.2 Multiplying Matrices I (row $\times$ column)²

Once again, you may just want to get them started on the assignment, and let it speak for itself.

However, toward the end of class—after they have all struggled through, or gotten stuck—pull them back and do some blackboard talk. Start by pointing out that we are doing two different things here. Multiplying a matrix times a constant is both easy and intuitive. If you can add two matrices, then you can add $[A] + [A]$ and thus figure out what $2[A]$ has to be.

On the other hand, multiplying two matrices (a row times a column) is not easy or intuitive. Show how to multiply a row matrix by a column matrix, do a few examples, and talk about how it applies to the gradebook example. In other words, make sure they get it.

What I do, about a hundred times, is try to get them to visualize the row floating up in the air and twisting around so that it lines up with the column. I use my hands, I use sticks, anything to get them to visually see this row floating up and twisting to line up with that column. This visualization is not essential today, but if they get it today, it will really help them tomorrow, when we start multiplying full matrices! As I mentioned earlier—this is a radical departure from my normal philosophy—I am more concerned that they get the mechanics here (how to do the multiplication) than any sort of logic behind it. They should be able to see, for instance, that if the row and column do not have the same number of elements, then the matrix multiplication is illegal.

Oh yeah, one more thing—I always stress that when you multiply two matrices, the product is a matrix. In the case of a row times a column, it is a 1×1 matrix, but it is still a matrix, not a number.

Homework

“Homework—Multiplying Matrices I”

10.3 Multiplying Matrices II (the full monty)³

This time, you're going to have to lecture. You are going to have to explain, on the board, how to multiply matrices. Probably a good 20 minutes (half the class) dedicated to showing them that this row goes over here to this column, and then we go down to the next row, and so on. Get them to work problems at their desks, make sure they are cool with it. You can also refer them to the “Conceptual Explanations” to see a problem worked out in a whole lot of detail.

Two things to stress:

1. Keep doing the visualization of a row (in the first matrix) floating up and twisting to get next to a column (in the second matrix). If the two do not line up—that is, they have different numbers of elements—then the multiplication is illegal.

²This content is available online at <http://cnx.org/content/m19448/1.1/>.

³This content is available online at <http://cnx.org/content/m19449/1.2/>.

2. Matrix multiplication does not commute. If you switch the order, you may turn a legal multiplication into an illegal one. Or, you may still have a legal multiplication, but with a different answer. AB and BA are completely different things with matrices.

You may never get to the in-class assignment at all. If you don't, that's OK, just skip it! However, note that the in-class assignment is built on one particular application, which is showing how Professor Snape can do just one matrix multiplication to get the final grades for all his students. This exercise is one of the few applications I have for matrix multiplication.

Homework:

“Homework—Multiplying Matrices II”

#4 is important for a couple of reasons. First, of course, by using variables, it forces them to do the work manually even if they have figured out how to do it on a calculator. More importantly, it continues to hammer home that message about what variables are—you can solve this leaving x , y , and z generic, and then you can plug in numbers for them if you want.

#5 and #7 set up the identity matrix; #6 sets up using matrices to solve linear equations. You don't need to mention any of that now, but you may want to refer back to them later. I don't want them to think of $[I]$ as being defined as “a diagonal row of 1s.” I want them to know that it is defined by the property $AI = IA = A$, and to see how that definition leads to the diagonal row of 1s. #7 is the key to that.

10.4 Identity and Inverse Matrices⁴

This may, in fact, be two days masquerading as one—it depends on the class. They can work through the sheet on their own, but as you are circulating and helping, make sure they are really reading it, and getting the point! As I said earlier, they need to know that $[I]$ is defined by the property $AI = IA = A$, and to see how that definition leads to the diagonal row of 1s. They need to know that A^{-1} is defined by the property $AA^{-1} = A^{-1}A = I$, and to see how they can find the inverse of a matrix directly from this definition. That may all be too much for one day.

I also always mention that only a square matrix can have an $[I]$. The reason is that the definition requires I to work commutatively: AI and IA both have to give A . You can play around very quickly to find that a 2×3 matrix cannot possibly have an $[I]$ with this requirement. And of course, a non-square matrix has no inverse, since it has no $[I]$ and the inverse is defined in terms of $[I]$!

Homework:

“Homework—The Identity and Inverse Matrices”

10.5 Inverse of the Generic 2x2 Matrix⁵

This is one of those things that should be easy, but it isn't. It should be easy because they have already been doing it, with numbers, and it's just the same with letters. But hey, that's what Algebra II is about, right?

They should definitely work in groups here. Make sure they understand what they are doing. A clear sign that they don't understand what they are doing, even a little, is that they wind up solving for a , or solving for w in terms of x , or something like that. They need to understand that the object is to solve for w , x , y , and z in terms of a , b , c , and d . Only by doing this can they come up with a generic solution to the inverse of a 2×2 matrix, which can then be used quickly and easily to find the inverse of any 2×2 matrix. If they don't understand that, they just don't have any idea what we're doing—it's important to get them to understand the problem instead of just focusing on solving it.

⁴This content is available online at <http://cnx.org/content/m19443/1.1/>.

⁵This content is available online at <http://cnx.org/content/m19446/1.1/>.

The answer, by the way, is $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Once they have it, in this form, help them understand how to use it—the numbers in this diagonal switch places, the number in that diagonal change signs. This also helps set up the determinant $(ad-bc)$, by the way.

10.6 Use Matrices for Transformation⁶

This is a fun day. No new math, just a cool application of the math we've seen.

The in-class assignment pretty well speaks for itself. It's worth mentioning that, despite the very simplified and silly nature of this specific assignment, the underlying message—that matrices are used to transform images in computer graphics—is absolutely true.

Some students (good students) may question why the last column $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is necessary in specifying Harpoona's initial condition. The reason is this representation is not simply a list of all the corner points in a shape. Yes, each column represents a point. But the matrix is a set of instructions to the computer, to draw lines from this point to that point. Without the last column, Harpoona would be missing her hypotenuse.

Homework

Homework: "Homework: Using Matrices for Transformation." Here we see a matrix that rotates any object by 300, counter-clockwise. The inverse matrix, of course, rotates clockwise. Hopefully they will discover all that for themselves. And hopefully most of them will realize why an inverse matrix always does the exact opposite of the original matrix, since if you do them one after the other, you end up back where you started.

10.6.1 Time for Another Test!

Our first test on matrices.

10.7 Matrices on the Calculator⁷

This starts with a lecture. You have to show them how to do matrices on the calculator. They should be able to...

- Enter several matrices at once
- Add several matrices
- Subtract
- Multiply
- Find inverses

All of this is explained step-by-step in the "Conceptual Explanations."

There are two things I stress. First, whenever I enter a matrix, I always check it. For instance, after I enter matrix [A], I go back to the home screen and go [A][Enter], and the calculator displays matrix [A] to me, so I can make sure I typed it right. One small mistype will ruin a whole problem, and it's really easy to do!

Second, the calculator is very smart about interpreting equations. After you enter three matrices, you can just type [A][B]⁻¹[C] and it will multiply [A] by the inverse of [B] and then subtract [C].

Homework

"Homework—Calculators." Depending on how things go, they may be able to finish this in class and have no homework.

⁶This content is available online at <http://cnx.org/content/m19451/1.1/>.

⁷This content is available online at <http://cnx.org/content/m19447/1.1/>.

10.8 Determinants⁸

Another very lecture-heavy topic, I'm afraid. Like multiplying matrices, finding the determinant is something you just have to show on the board. And once again, you can refer them in the end to the "Conceptual Explanations" to see an example worked out in detail.

Start by talking about the ad-bc that played such a prominent role in the inverse of a 2×2 matrix. This is, in fact, the determinant of a 2×2 matrix.

Then show them how to find the determinant of a three-by-three matrix, using either the "diagonals" or "expansion by minors" method, whichever you prefer. (I would not do both. Personally, I prefer "expansion by minors," and that is the one I demonstrate in the "Conceptual Explanations.")

Hot points to mention:

- Brackets like this $[A]$ mean a matrix; brackets like this $|A|$ mean a determinant. A determinant is a number associated with a matrix: it is not, itself, a matrix.
- Only square matrices have a determinant.
- Also show them how to find a determinant on the calculator. They need to be able to do this (like everything else) both manually and with a calculator.
- To find the area of a triangle whose vertices are (a,b), (c,d), and (e,f), you can use the formula: Area

$$= \frac{1}{2} \begin{vmatrix} a & c & e \\ b & d & f \\ 1 & 1 & 1 \end{vmatrix}.$$

This is the only use I can really give them for determinants. They will need to know this for the homework. Do an example or two. I like to challenge them to find that area any other way, just to make the point that it is not a trivial problem without matrices. (I don't know any other good way.)

- All we're really going to use "expansion by minors" for is 3×3 matrices. However, I like to point out that it can be obviously extended to 4×4 , 5×5 , etc. It also extends down to a 2×2 —if you "expand minors" on that, you end up with the good old familiar formula ad-bc.
- Finally, mention that any matrix with determinant zero has no inverse. This is analogous to the rule that the number 0 is the only number with no inverse.

Homework

"Homework—Determinants"

10.9 Solving Linear Equations⁹

Just let them start this assignment, it should explain itself.

This is the coolest thing in our whole unit on matrices. It is also the most dangerous.

The cool thing is, you can solve real-world problems very quickly, thanks to matrices—so matrices (and matrix multiplication and the inverse matrix and so on) prove their worth. If you are given:

$$\begin{aligned} 2x - 5y + z &= 1 \\ 6x - y + 2 &= 4 \\ 4x - 10y + 2z &= 2 \end{aligned}$$

you plug into your calculator $[A] = \begin{bmatrix} 2 & -5 & 1 \\ 6 & -1 & 2 \\ 4 & -10 & 2 \end{bmatrix}$, $[B] = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$, ask for $A^{-1}B$, and the answer pops

out! They should be able to do this process quickly and mechanically.

⁸This content is available online at <http://cnx.org/content/m19442/1.1/>.

⁹This content is available online at <http://cnx.org/content/m19450/1.1/>.

But the quick, mechanical nature of the process is also its great danger. I want them to see the logic of it. I want them to see exactly why the equation $\begin{bmatrix} 2 & -5 & 1 \\ 6 & -1 & 2 \\ 4 & -10 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ is exactly like those three separate equations up there. I want them to be able to solve like this:

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

to see why it comes out as it does. If they see that this is all perfectly logical, and they know how to do it, the day is a big success—and in fact, this sort of justifies the whole unit on matrices.

As a final point, mention what happens if the equations were unsolvable: matrix A will have a 0 determinant, and will therefore have no inverse, so the equation won't work. (You get an error on the calculator.)

Homework

“Homework—Solving Linear Equations”

10.9.1 Time for Another Test!

And, time to conclude our unit on matrices.

Chapter 11

Modeling Data with Functions

11.1 Introduction¹

This unit is really three different topics, joined together by a somewhat weak thread.

The first topic is direct and inverse variation. The second topic is finding a parabola that fits any three given points. The third topic is regression on a calculator.

The weak thread that connects them is that they all involve starting with data, and finding a function that models that data. (At least, the latter topics are clearly about that, and the first topic is about that if you approach it the way I do...)

11.2 Direct and Inverse Variation²

This is one of those days where you want to get them working right away (on the “Direct Variation” assignment), let them finish the assignment, and then do 10-15 minutes of talking afterwards. You want to make sure they got the point of what they did.

Direct variation is, of course, just another kind of function—an independent variable, and a dependent variable, and consistency, and so on. But the aspect of direct variation that I always stress is that when the independent number doubles, the dependent number doubles. If one triples, the other triples. If one is cut in half, the other is cut in half...and so on. They are, in a word, proportional.

This is not the same thing as saying “When one goes up, the other goes up.” Of course that is true whenever you have direct variation. But that statement is also true of $\ln(x)$, $\sqrt{x}$, x^2 , $2x$, $x+3$, and a lot of other functions: they do “when one goes up, the other goes up” but not “when one doubles, the other doubles” so they are not direct variation. The only function that has that property is $f(x)=kx$, where k is any constant. (Point out that k could be $\frac{1}{2}$, or it could be -2 , or any other constant—not just a positive integer.)

In #3 they arrived at this point. They should see that the equation $y=kx$ has the property we want, because if you replace x with $2x$ then $y=k(2x)$ which is the same thing as $2kx$ which is twice what it used to be. So if x doubles, y doubles.

They should also see that direct variation always graphs as a line. And not just any line, but a line through the origin.

But the thing I most want them to see is that there are many, many situations where things vary in this way. In other words, #4 is the most important problem on the assignment. You may want to ask them to tell the whole class what they came up with, and then you throw in a few more, just to make the point of how common this is. The amount of time you spend waiting in line varies directly with the number of people

¹This content is available online at <http://cnx.org/content/m19454/1.2/>.

²This content is available online at <http://cnx.org/content/m19452/1.1/>.

in front of you; the amount you pay varies directly with the number of meals you order; the weight of your french fries measured in grams varies directly with the weight of your french fries measured in pounds; and both of these, in turn, vary directly with the number of fries; and so on, and so on, and so on.

11.2.1 Homework

Part I

“Homework: Inverse Variation”

That’s right, no homework on direct variation—it’s time to develop the second one. They should be able to do it pretty well on their own, in analogy to what happened in class. But you will spend a fair amount of the next day debriefing them on inverse variation, just as you did on direct. The defining property is that when the independent variable doubles, the dependent variable chops in half. Again, it is true to say “when one goes up, the other goes down”—but it is not enough. $1/x^2$ has that property, and so does $10-x$, and neither of those is inverse variation.

Examples are a bit harder to think of, off the top of your head. But there is an easy and systematic way to find them. I always warn the students that I will ask for an example of inverse variation on the test, and then (now that I have their attention) I explain to them how to do it. Inverse variation is $y=k/x$ (graphs as a hyperbola). This equation can be rewritten as $xy=k$. This is useful for two reasons. First, it gives you the ability to spot inverse variation—if the product is always roughly the same, it’s inverse. Second, it gives you the ability to generate inverse variation problems, by thinking of any time that two things multiply to give a third thing, and then holding that third thing constant.

For instance: the number of test questions I have to grade is the number of students, times the number of questions on the test. That’s obvious, right?

If I want to turn that into a direct variation problem, I hold one of the two multiplying variables constant. What do I mean “hold it constant?” I mean, pick a number. For instance, suppose there are twenty students in my class. Now the dependent variable (number of questions I have to grade) varies directly with the independent variable (number of questions I put on the test).

On the other hand, if I want to turn that exact same scenario into an inverse variation problem, I hold the big variable constant. For instance, suppose I know that I am only capable of grading 200 problems in a night. So I have to decide how many questions to put on the test, based on how many students I have. You see? Double the number of students, and the number of questions on the test drops in half.

Algebraically, if I call t the number of questions on the test, s the number of students, and g the number of questions I have to grade, then $g=ts$. In the first case, I set $s=20$ so I had the direct variation equation $g=20t$. In the second case I set $g=200$ so I had the inverse variation equation $t=200/s$.

I explain all that to my class, slowly and carefully. They need to know it because the actual test I give them will have a question where they have to make up an inverse variation problem, and I always tell them so. That question, if nothing else, will come as no surprise at all.

Part II

“Homework: Direct and Inverse Variation”

This is on the long side, and introduces a few new ideas: the idea of being proportional to the square (or square root) of a variable, and the idea of dependence on multiple variables. So you should ideally hand it out in the middle of class, so they have time to work on it before the homework—and be prepared to spend a lot of time going over it the next day.

11.3 Calculator Regression³

Lecture time. Talk about the importance of regression again. Walk them, step-by-step, through a few regressions on the calculator. (Detailed instructions for a TI-83 are included in the “Conceptual Explanations” for this section.)

³This content is available online at <http://cnx.org/content/m19453/1.1/>.

My advice right now is to look at the points first (which may require resetting the window!) and then categorize them according to concavity—although I don’t use that word. I talk about three kinds of increasing functions:

- Linear functions increase steadily
- Logs and square roots increase more and more slowly
- Parabolas and exponential functions increase more and more quickly

Similarly, of course, for decreasing functions. Choose an appropriate kind of regression, and let the calculator do the rest.

After a few examples, hand out the homework.

Homework

“Homework: Calculator Regression”

11.3.1 Time for Another Test!

Question 11c is not something I would put on a real test, for a couple of reasons. First, it would be pretty hard to grade; second, and more importantly, it is unlike anything we’ve seen on a homework. But I might use it for an extra credit, and in any case, it can’t hurt them to see it and discuss it on a sample.

Chapter 12

Conics

12.1 Introduction¹

The last topic! And it's a big one. Here is the overall plan.

We start with a day on distance—finding the distance between two points, finding the distance from a point to a line—this will form a basis for the whole unit.

Then we do our shapes: circles, parabolas, ellipses, and hyperbolas. For each shape, there are really two things the students need to learn. One is the **geometric definition** of the shape. The other is what I call the **machinery** associated with the shape—the standard equation and what it represents. I cover these two things separately, and then connect them at the end.

12.2 Distance²

Before I hand this out, I tell them a bit about where we're going, in terms of the whole unit. We're going to do “analytic geometry”—that is, linking geometry with algebra. We're going to graph a bunch of shapes. And everything we're going to do is built upon one simple idea: the idea of **distance**.

Let's start with distance on a number line. (Draw a number line on the board.) Here's 4, and here's 10. What's the distance between them? Right, 6. You don't need to do any math, you can just count—1, 2, 3, 4, 5, 6. There we are.

How about the distance from 4 to 1? Right, 3. 4 to 0? Right, 4. One more: what is the distance from 4 to -5? Again, just count...1,2,3,4,5,6,7,8,9. The distance is nine.

So, what's going on here? In each case, we're counting from 4 to some other number: to 10, to 1, to 0, then to -5. What happened mathematically? We **subtracted**. This is a point that looks incredibly obvious, but really it isn't, so it's worth repeating: **if you subtract two numbers, you get the distance between them**. $10-4$ is 6. $4-1$ is 3. And the last one? $4 - (-5) = 4 + 5 = 9$. So even that works. Remember that we saw, by counting, that the distance from 4 to -5 is 9. Now we see that it works mathematically because subtracting a **negative** is like adding a **positive**.

Oh yeah...what if we had subtracted the other way? You know, $4-9$ or $-5-4$. We would have gotten the right answers, only negative. But the distance would still be positive, because distance is always positive.

So, based on all that, at your seats, write down a **formula** for the distance from a to b on a number line: go! (**give them thirty seconds**) What did you get? $|a - b|$? Good! $|b - a|$? Also good! They are the same thing. $a-b$ and $b-a$ are **not** the same thing, but when you take the absolute value, then they are.

Now, let's get two-dimensional here. We'll start with the easy case, which is when the points line up. In that case, we can use the same rule, right? For instance, let's look at (4,3) and (10,3). How far apart are

¹This content is available online at <http://cnx.org/content/m19307/1.1/>.

²This content is available online at <http://cnx.org/content/m19299/1.2/>.

they? Same as before—6. We can just count, or we can just subtract, because the y -coordinates are the same. **(Show them this visually!)** Similarly, suppose we take $(-2,5)$ and $(-2,-8)$. Since the x -coordinates are the same, we can just count again, or just subtract the y -coordinates, and get a distance of 13.

Now, what if **neither** coordinate is the same? Then it's a bit trickier. But we're not going to use any magic "distance formula"—if you ever memorized one, throw it out. All we need is what we've already done. Let's look at $(-2,1)$ and $(4,9)$. **(Draw it!)** To find that distance, we're going to find the distance **across** and the distance **up**. So draw in this other point at $(4,1)$. Now, draw a triangle, with the distance we want over here, and the distance across here, and the distance up here. These two sides are easy, because they are just what we have already been doing, right? So this is 6 and this is 8. So how do we find this third side, which is the distance we wanted? Right, the Pythagorean Theorem! So it comes out as 10.

The moral of the story is—**whenever you need to find a distance, use the Pythagorean Theorem.**

OK, one more thing before you start the assignment. That was the distance between two **points**. How about the distance from a **point to a line**? For instance, what is the distance from you to the nearest street? The answer, of course, is—it depends on where on the street you want to get. But when we say, distance from you to the street, we mean the **shortest** distance. **(Do a few drawings to make sure they get the idea of shortest distance from a point to a line. If the line is vertical or horizontal, then we are back to just counting. If it's diagonal, life gets much more complicated, and we're not going to get into it. Except I sometimes assign, as an extra credit assignment, "find the distance from the arbitrary point (x,y) to the arbitrary line $y = mx + b$. It's ugly and difficult, but I usually have one or two kids take me up on it. For the rest of the class, just promise to stick with horizontal and vertical lines, and counting.)**

After all that is said, they are ready to start on the assignment.

Homework:

"Homework: Distance"

12.3 Circles³

Our first shape. (Sometimes I have started with parabolas first, but I think this is simpler.)

Here's how you're going to start—and this will be the same for every shape. **Don't tell them what the shape is.** Instead, tell them this. A bunch of points are getting together to form a very exclusive club. The membership requirement for the club is this: **you must be exactly 5 units away from the origin.** Any point that fulfills this requirement is in the club; any point that is either too close or too far, is not in the club. Give them a piece of graph paper, and have them draw all the points in the club. You come around and look at their work. If they are stuck, point out that there are a few very obvious points on the x -axis. The point is that, before any math happens at all, every individual group should have convinced themselves that this club forms a circle.

Then you step back and say—in Geometry class you used circles all the time, but you may never have formally defined what a circle is. We now have a formal definition: **a circle is all the points in a plane that are the same distance from a given point.** That distance is called what? (Someone will come up with "radius.") And that given point is called what? (Someone may or may not come up with "center.") The center plays a very interesting role in this story. It is the most important part, the only point that is key to the definition of the circle—but it is **not itself part of the circle**, not itself a member of the club. (The origin is not 5 units away from the origin.) This is worth stressing even though it's obvious, because it will help set up less obvious ideas later (such as the focus of a parabola).

Point out, also, that—as predicted—the definition of a circle is based entirely on the idea of **distance**. So in order to take our general geometric definition and turn it into math, we will need to mathematically understand distance. Which we do.

At this point, they should be ready for the in-class assignment. They may need some help here, but with just a little nudging, they should be able to see how we can take the **geometric definition** of a circle leads

³This content is available online at <http://cnx.org/content/m19298/1.2/>.

very directly to the **equation** of a circle, $(x - h)^2 + (y - k)^2 = r^2$ where (h, k) is the center and r is the radius. This formula should be in their notes, **etc.** Once we have this formula, we can use it to immediately graph things like $(x - 2)^2 + (y + 3)^2 = 25$ —to go from the equation to the graph, and vice-versa.

Once you have explained this and they get it, they are ready for the homework. They now know how to graph a circle in standard form, but what if the circle **doesn't** come in standard form? The answer is to complete the square—twice, once for x and once for y . If it works out that the homework gets done in class (this day or the next day), you may want to do a brief TAPPS exercise with my little completing-the-square demo in the middle of the homework. However, they should also be able to follow it on their own at home.

Homework:

“Homework: Circles”

It may take a fair amount of debriefing afterward, and even a few more practice problems, before you are confident that they “get” the circle thing. They need to be able to take an equation for a circle in non-standard form, **put** it in standard form, and then graph it. And they need to still see how that form comes directly from the Pythagorean Theorem and the definition of a circle.

There is one other fact that I always slip into the conversation somewhere, which is: how can you look at an equation, such as $2x^2 + 3x + 2y^2 + 5y + 7 = 0$, and even tell if it is a circle? Answer: it has both an x^2 and a y^2 term, and they have the **same coefficient**. If there is no x^2 or y^2 term, you have a line. If one exists but not the other, you have a parabola. By the time we're done with this unit, we will have filled out all the other possible cases, and I expect them to be able to look at any equation and recognize immediately what shape it will be.

12.4 Parabolas, Day 1⁴

Once again, when you start, **don't tell them** we're doing parabolas! Tell them we're going to create another club. This time the requirement for membership is: you must be exactly the same distance from the point $(0, 3)$ that you are from the line $y = -3$. For instance, the point $(3, 3)$ is not part of our club—it is 3 units away from $(0, 3)$ and **six** units away from $y = -3$.

Now, let them work in groups on “All the Points Equidistant from a Point and a Line” to see if they can find the shape from just that. If they need a hint, tell them there is one **extremely** obvious point, and two **somewhat** obvious points. After that they have to dink around.

When all or most groups have it, go through it on the blackboard, something like this. The extremely obvious point is the origin. The “somewhat” obvious points are $(-6, 3)$ and $(6, 3)$. Show why all those work.

Now, can any point **below the x-axis** work? Clearly not. Any point below the x-axis is “obviously” (meaning, after you show them for a minute) closer to the line, than to the point.

So, let's start working up from the origin. The origin was in the club. As we move up, we are getting **closer** to the point, and **farther away** from the line. So how can we maintain equality? The only way is to move **farther away** from the point, by moving out. In this way, you sketch in the parabola.

Now, you introduce the terminology. We're already old friends with the **vertex** of a parabola. This point up here is called the **focus**. This line down here is the **directrix**. The focus and directrix are kind of like the center of a circle, in the sense that they are central to the **definition** of what a parabola is, but they are not themselves **part** of the parabola. The vertex, on the other hand, **is** a part of the parabola, but is not a part of the definition.

The directrix, of course, is a horizontal line: but what if it isn't? What if the directrix is vertical? Then we have a **horizontal parabola**. Of course it isn't a function, but it's still a shape we can graph and talk about, and we have seen them a few times before. If you have time, work through $x = 4y^2 - 8y$.

Homework:

“Homework: Vertical and Horizontal Parabolas”

⁴This content is available online at <<http://cnx.org/content/m19315/1.2/>>.

12.5 Parabolas, Day 2⁵

What good are parabolas? We've already seen some use for graphing parabolas, in terms of modeling certain kinds of behavior. If I throw a ball into the air, **not** straight up, its path through the air is a parabola. But here is another cool thing: if parallel lines come into a parabola, they all bounce to the focus.

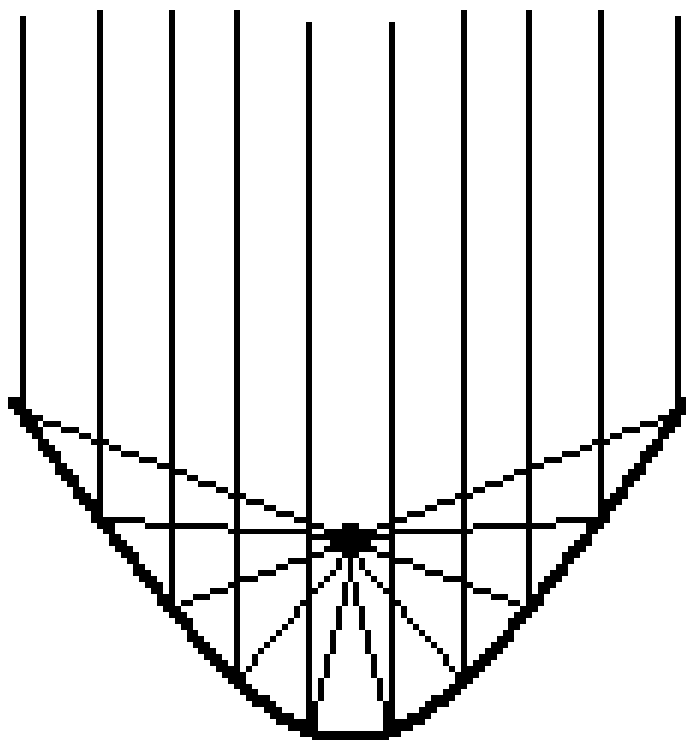


Figure 12.1: Draw this on the board:

So **telescopes** are made by creating **parabolic mirrors**—all the incoming light is concentrated at the focus. Pretty cool, huh?

We are already somewhat familiar with parabola machinery. We recall that the equation for a vertical parabola is $y = a(x - h)^2 + k$ and the equation for a horizontal parabola is $x = a(y - k)^2 + h$. The vertex in either case is at (h, k) . We also recall that if a is positive, it opens up (vertical) or to the right (horizontal); if a is negative, it opens down or to the left. This is all old news.

Thus far, whenever we have graphed parabolas, we have found the vertex, determined whether they open up or down or right or left, and then gone “swoosh.” If we were a little more sophisticated, we remembered that the width of the parabola was determined by a ; so $y = 2x^2$ is narrower than $y = x^2$ which is narrower than $y = \frac{1}{2}x^2$. But how can we use a to actually draw the width accurately? That is the question for today. And the answer starts out with one fact which may seem quite unrelated: the distance from the vertex to the focus is $\frac{1}{4a}$. I'm going to present this as a magical fact for the moment—we will never prove it, though

⁵This content is available online at <http://cnx.org/content/m19313/1.2/>.

we will demonstrate it for a few examples. But that one little fact is the only thing I am going to ask you to “take my word for”—everything else is going to flow logically from that.

(Go back to your drawing of our parabola with focus $(0,3)$ and directrix $y = -3$, and point out the distance from vertex to focus—label it $\frac{1}{4a}$.) Now, what is the distance from the vertex to the **directrix**? Someone should get this: it must also be $\frac{1}{4a}$. Why? Because the vertex is part of the parabola, so by definition, it must be the same distance from the directrix that it is from the focus. Label that.

Now, let’s start at this point here (point to $(-6,3)$) and go down to the directrix. How long is that? Again, someone should get it: $\frac{1}{4a} + \frac{1}{4a}$, which is $\frac{2}{4a}$ or $\frac{1}{2a}$. Now, how far **over** is it, from this same point to the focus? Well, again, this is on the parabola, so it must be the same distance to the focus that it is to the directrix: $\frac{1}{2a}$.

If you extend that line all the way to the other side, you have two of those: $\frac{1}{2a} + \frac{1}{2a} = \frac{2}{2a}$, or $\frac{1}{a}$, running between $(6,3)$ and $(-6,3)$. This line is called the **latus rectum**—Latin for “straight line”—it is always a line that touches the parabola at two points, runs **parallel** to the directrix, and goes **through** the focus.

At this point, the blackboard looks something like this:

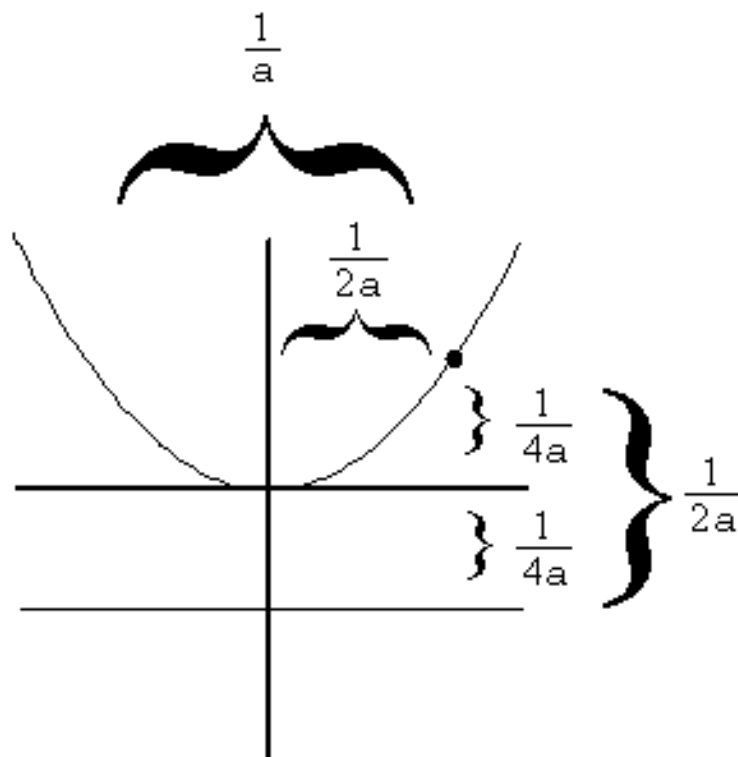


Figure 12.2

The point should be clear. Starting with the one magical fact that the distance from the focus to the vertex is $\frac{1}{4a}$, we can get everything else, including the length of the latus rectum, $\frac{1}{a}$. And that gives us a way of doing exactly what we wanted, which is using a to more accurately draw a parabola.

At this point, you do a sample problem on the board, soup to nuts. Say, $x = \frac{1}{2}(y+3)^2 - 4$. They should be able to see quickly that it is horizontal, opens to the right, with vertex at $(-4,-3)$ —that’s all review. But

now we can also say that the distance from the focus to the vertex is $\frac{1}{4a}$ which in this case (since $a = \frac{1}{2}$) is $\frac{1}{2}$. So where is the focus? Draw the parabola quickly on the board. They should be able to see that the focus is to the **right** of the vertex, so that puts us at $(-3\frac{1}{2}, -3)$. The directrix is to the **left**, at $x = -4\frac{1}{2}$. (I always warn them at this point, a very common way to miss points on the test is to say that the directrix is $(x = -4\frac{1}{2}, -3)$. That's a point—the directrix is a line!)

How wide is the parabola? When we graphed them before, we had no way of determining that: but now we have the latus rectum to help us out. The length of the latus rectum is $\frac{1}{a}$ which is 2. So you can draw that in around the focus—going up one and down one—and then draw the parabola more accurately, staring from the focus and touching the latus rectum on both sides.

Whew! Got all that? Good then, you're ready for the homework! (You may want to mention that a different sample problem is worked in the "Conceptual Explanations.")

Homework:

"Homework: Parabolas and the Latus Rectum"

This one will take a lot of debriefing afterwards. Let's talk about #3 for a minute. The first thing they should have done is draw it—that can't be stressed enough—always draw it! Drawing the vertex and focus, they should be able to see that the parabola opens to the right. Since we know the vertex, we can write immediately $x = a(y - 6)^2 + 5$. Many of them will have gotten that far, and then gotten stuck. Show them that the distance from the vertex to the focus is 2, and we know that this distance must be $\frac{1}{4a}$, so $\frac{1}{4a} = 2$. We can solve this to find $a = \frac{1}{8}$ which fills in the final piece of the puzzle.

On #5, again, before doing anything else, they should draw it! They should see that there is one vertical and one horizontal parabola that fits this definition.

But how can they find them? You do the vertical, they can do the horizontal. Knowing the vertex, we know that the vertical one will look like $y = a(x + 2)^2 + 5$. How can we find a this time, by using the information that it "contains the point (0,1)"? Well, we have to go back to our last unit, when we found the equation for a parabola containing certain points! Remember, we said that if it **contains** a point, then that point must make the equation **true**! So we can plug in (0,1) to the equation, and solve for a . $1 = a(0 + 2)^2 + 5$, $4a + 5 = 1$, $a = -1$. The negative a tells us that the parabola will open down—and our drawing already told us that, so that's a good reality check.

12.6 Parabolas: From Definition to Equation⁶

OK, where are we? We started with the geometric definition of a parabola. Then we jumped straight to the machinery, and we never attempted to connect the two. But that is what we're going to do now.

Remember what we did with circles? We started with our geometric definition. We picked an arbitrary point on the circle, called it (x, y) , and wrote an equation that said "you, Mr. (x, y) , are exactly 5 units away from the origin." That equation became the equation for the circle.

Now we're going to do the same thing with a parabola. We're going to write an equation that says "you, Mr. (x, y) , are the same distance from the focus that you are from the directrix." In doing so, we will write the equation for a parabola, **based on** the geometric definition. And we will discover, along the way, that the distance from the focus to the vertex really is $\frac{1}{4a}$.

That's really all the setup this assignment needs. But they will need a lot of help doing it. Let them go at it, in groups, and walk around and give hints when necessary. The answers we are looking for are:

1. $\sqrt{x^2 + y^2}$. As always, hint at this by pushing them toward the Pythagorean triangle.
2. $y + 4$. The way I always hint at this is by saying "Try numbers. Suppose instead of (x, y) this were (3,10). Now, how about (10,3)?" and so on, until they see that they are just adding 4 to the y-coordinate. Then remind them of the rule, from day one of this unit—to find distances, subtract. In this case, subtract -4.
3. $\sqrt{x^2 + y^2} = y + 4$. This is the key step! By asserting that $d_1 = d_2$, we are writing the definition equation for the parabola.

⁶This content is available online at <<http://cnx.org/content/m19311/1.2/>>.

4. Good algebra exercise! Square both sides, the y^2 terms cancel, and you're left with $x^2 = 8y + 16$. Solve for y , and you end up with $y = \frac{1}{8}x^2 - 2$. Some of them will have difficulty seeing that this is the final form—point out that we can rewrite it as $y = \frac{1}{8}(x - 0)^2 - 2$, if that helps. So the vertex is (0,-2) and the distance from focus to vertex is 2, just as they should be.

Warn them that this **will be on the test!**

Which brings me to...

12.6.1 Time for another test!

Our first test on conics. I go back and forth as to whether I should give them a bunch of free information at the top of the test—but it's probably a good idea to give them a chart, sort of like the one on top of my sample.

12.7 Ellipses⁷

Only two shapes left! But these two are doozies. Expect to spend at least a couple of days on each—they get a major test all to themselves.

In terms of teaching order, both shapes are going to follow the same pattern that we set with parabolas. First, the geometry. Then, the machinery. And finally, at the end, the connection between the two.

So, as always, don't start by telling them the shape. Let them do the assignment "Distance to this point plus distance to that point is constant" in groups, and help them out until they get the shape themselves. A good hint is that there are two pretty easy points to find on the x -axis, and two harder points to find on the y -axis. As always, keep wandering and hinting until most groups have drawn something like an ellipse. Then you lecture.

The lecture starts by pointing out what we have. We have two points, called the **foci**. (One "focus," two "foci.") They are the defining points of the ellipse, but they are not part of the ellipse. And we also have a distance, which is part of the definition.

Because the foci were horizontally across from each other, we have a horizontal ellipse. If they were vertically lined up, we would have a vertical ellipse. You can also do diagonal ellipses, but we're not going to do that here.

Let's talk more about the geometry. One way you can draw a circle is to thumbtack a piece of string to a piece of cardboard, and tie the other end of the string to a pen. Keeping the string taut, you pull all the way around, and you end up with a circle. Note how you are using the geometric definition of a circle, to draw one: the thumbtack is the center, and the piece of string is the radius.

Now that we have our geometric definition of an ellipse, can anyone think of a way to draw one of those? **(probably not)** Here's what you do. Take a piece of string, and thumbtack **both** ends down in a piece of cardboard, so that the string is **not** taut. Then, using your pen, pull the string taut.

⁷This content is available online at <<http://cnx.org/content/m19303/1.2/>>.

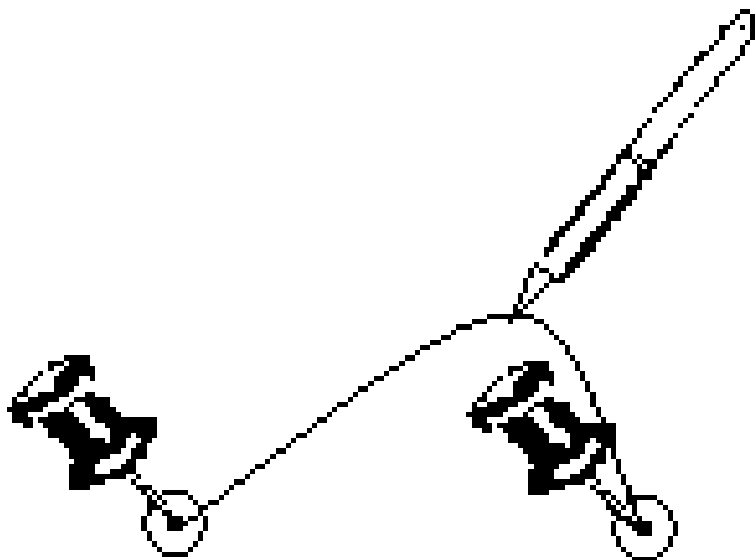


Figure 12.3

Now, pull the pen around, keeping the string taut. You see what this does? While the string is taut, the **distance from the pen to the left thumbtack, plus the distance from the pen to the right thumbtack, is always a constant**—namely, the length of the string. So this gives you an ellipse. I think most people can picture this if they close their eyes. Sometimes I assign them to do this at home.

OK, so, what good are ellipses? The best example I have is orbits. The Earth, for instance, is traveling in an **ellipse**, with the sun at one of the two foci. The moon's orbit around the Earth, or even a satellite's orbit around the Earth, are all ellipses.

Another cool ellipse thing, which a lot of people have seen in a museum, is that if you are in an elliptical room, and one person stands at each focus, you can hear each other whisper. Just as a parabola collects all incoming parallel lines at the focus, an ellipse bounces everything from one focus straight to the other focus.

OK, on to the machinery. Here is the equation for a horizontal ellipse, centered at the origin.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (12.1)$$

Here is a drawing of a horizontal ellipse.

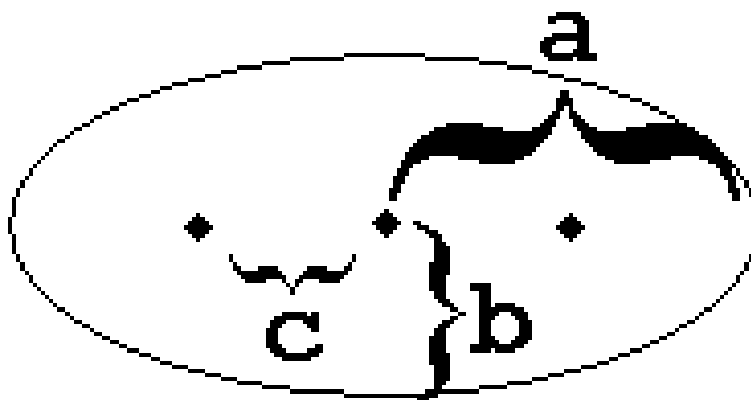


Figure 12.4

There are three numbers in this drawing. a is the distance from the center to the far edge (right or left). If you double this, you get the horizontal length of the entire ellipse—this length, $2a$ of course, is called the **major axis**.

b is the distance from the center to the top or bottom. If you double this, you get the vertical length of the entire ellipse—this length, $2b$ of course, is called the **minor axis**.

c is the distance from the center to either focus.

a and b appear in our equation. c does not. However, the three numbers bear the following relationship to each other: $a^2 = b^2 + c^2$. Note also that a is always the biggest of the three!

A few more points. If the center is **not** at the origin—if it is at, say, (h, k) , what do you think that does to our equation? They should all be able to guess that we replace x^2 with $(x - h)^2$ and y^2 with $(y - k)^2$.

Second, a vertical ellipse looks the same, but with a and b reversed:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (12.2)$$

Let's see how all that looks in an actual problem—walk through this on the blackboard to demonstrate. Suppose we want to graph this:

$$x^2 + 9y^2 - 4x + 54y + 49 = 0$$

First, how can we recognize it as an ellipse? Because it has both an x^2 and a y^2 term, and the coefficient are **different**. So we complete the square twice, sort of like we did with circles. But with circles, we always divided by the coefficient right away—this time, we pull it out from the x and y parts of the equation **separately**.

$x^2 + 9y^2 - 4x + 54y + 49 = 0$		The original problem
$x^2 - 4x + 9y^2 + 54y = -49$		Group the x and y parts
<i>continued on next page</i>		

$(x^2 - 4x) + 9(y^2 + 6y) = -49$		Factor out the coefficients. In this case, there is no x coefficient, so we just have to do y. In general, we must do both.
$(x^2 - 4x + 4) + 9(y^2 + 6y + 9) = -49 + 4 + 81$		Complete the square, twice
$(x - 2)^2 + 9(y + 3)^2 = 36$		Finish completing that square
$\frac{(x-2)^2}{36} + \frac{(y+3)^2}{4} = 1$		Divide by 36. This is because we need a 1 on the right, to be in our standard form!

Table 12.1

OK, get all that? Now, what do we have?

First of all, what is the center? That's easy: (2,-3).

Now, here is a harder question: does it open vertically, or horizontally? That is, does this look like $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or like $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$? The way to tell is by remembering that a is always bigger than b . So in this case, the 36 must be a^2 and the 4 must be b^2 , so it is horizontal. We can see that the major axis ($2a$) will be 12 long, stretching from (-4,-3) to (8,-3). (Draw all this as you're doing it.) And the minor axis will be 4 long, stretching from (2,-5) to (2,-1).

And where are the foci? Since this is a horizontal ellipse, they are to the left and right of the center. By how much? By c . What is c ? Well, $a^2 = b^2 + c^2$. So $c^2 = 32$, and $c = \sqrt{32} = 4\sqrt{2}$, or somewhere around $5\frac{1}{2}$. (They should be able to do that without a calculator: 32 is somewhere between 25 and 36, so $\sqrt{32}$ is around $5\frac{1}{2}$.) So the foci are at more or less $(-3\frac{1}{2}, -3)$ and $(7\frac{1}{2}, -3)$. We're done!

Homework:

"Homework: Ellipses"

There are two things here that may throw them for a loop.

One is the fractions in the denominator, and the necessity of (for instance) turning $25y^2$ into $\frac{y^2}{1/25}$. You may have to explain very carefully **why** we do that (standard form allows for a number on the bottom but not on the top), and **how** we do that.

The other is number 7. Some students will quickly and carelessly assume that 94.5 is a and 91.4 is b . So you want to draw this very carefully on the board when going over the homework. Show them where the 94.5 and 91.4 are, and remind them of where a , b , and c are. Get them to see from the drawing that $94.5 + 91.4$ is the major axis, and is therefore $2a$. And that $a - 91.4$ is c , so we can find c , and finally, we can use $a^2 = b^2 + c^2$ to find b . This is a really hard problem, but it's worth taking a lot of time on, because it really drives home the importance of visually being able to see an ellipse in your head, and knowing where a , b , and c are in that picture.

Oh, yeah...number 6 may confuse some of them too. Remind them again to **draw it first**, and that they can **plug in** (0,0) **and get a true equation**.

OK, now we're at a bit of a fork in the road. The next step is to connect the geometry of the ellipse, with the machinery. This is a really great problem, because it brings together a lot of ideas, including some of the work we did forever ago in radical equations. It is also really hard. So you can decide what to do based on how much time you have left, and how well you think they are following you. You may want to have them go through the exercise in class (expect to take a day). Or, you may want to make photocopies of the completely-worked-out version (which I have thoughtfully included here in the teacher's guide), and have them look it over as a TAPPS exercise, or just ask them to look it over. Or you could skip this entirely, or make it an extra credit.

12.8 Ellipses: From Definition to Equation⁸

⁸This content is available online at <http://cnx.org/content/m19305/1.1/>.

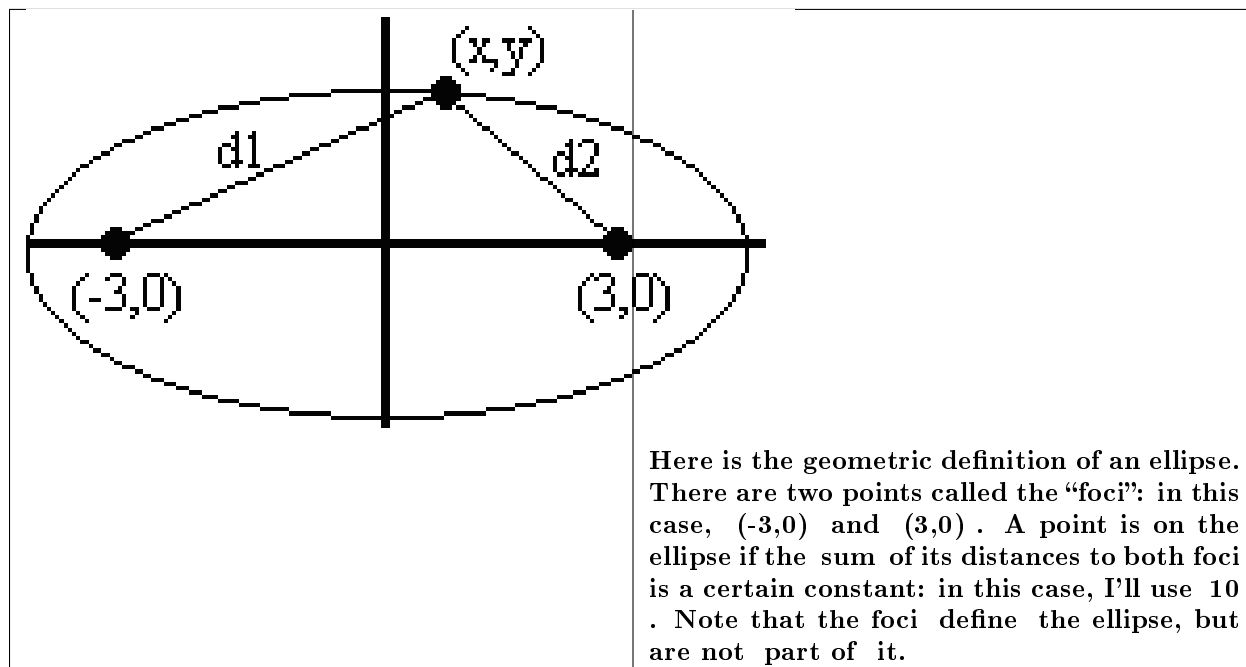


Table 12.2

The point (x, y) represents any point on the ellipse. $d1$ is its distance from the first focus, and $d2$ to the second. So the ellipse is **defined geometrically** by the relationship: $d1 + d2 = 10$.

To calculate $d1$ and $d2$, we use the Pythagorean Theorem as always: drop a straight line down from (x, y) to create the right triangles. Please verify this result for yourself! You should find that $d1 = \sqrt{(x+3)^2 + y^2}$ and $d2 = \sqrt{(x-3)^2 + y^2}$. So the equation becomes:

$$\sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2} = 10. \quad \text{This defines our ellipse}$$

The goal now is to simplify it. We did problems like this earlier in the year (radical equations, the “harder” variety that have two radicals). The way you do it is by isolating the square root, and then squaring both sides. In this case, there are two square roots, so we will need to go through that process twice.

$\sqrt{(x+3)^2 + y^2} = 10 - \sqrt{(x-3)^2 + y^2}$	Isolate a radical
$(x+3)^2 + y^2 = 100 - 20\sqrt{(x-3)^2 + y^2} + (x-3)^2 + y^2$	Square both sides
$(x^2 + 6x + 9) + y^2 = 100 - 20\sqrt{(x-3)^2 + y^2} + (x^2 - 6x + 9) + y^2$	Multiply out the squares
$12x = 100 - 20\sqrt{(x-3)^2 + y^2}$	Cancel & combine like terms
$\sqrt{(x-3)^2 + y^2} = 5 - \frac{3}{5}x$	Rearrange, divide by 20
continued on next page	

$(x - 3)^2 + y^2 = 25 - 6x + \frac{9}{25}x^2$	Square both sides again
$(x^2 - 6x + 9) + y^2 = 25 - 6x + \frac{9}{25}x^2$	Multiply out the square
$\frac{16}{25}x^2 + y^2 = 16$	Combine like terms
$\frac{x^2}{25} + \frac{y^2}{16} = 1$	Divide by 16

Table 12.3

...and we're done! Now, according to the “machinery” of ellipses, what should that equation look like? Horizontal or vertical? Where should the center be? What are a , b , and c ? Does all that match the picture we started with?

12.9 Hyperbolas⁹

The good news about hyperbolas is, they are a lot like ellipses—a lot of what has already been learned, will come in handy here. The bad news about hyperbolas is, they are a lot like ellipses—so all the little differences can be very confusing.

We will start as always with the geometric definition. Let them do the assignment “Distance to this point **minus** distance to that point is constant” in groups, and help them out until they get the shape themselves. There are two pretty easy points to find on the x -axis, but from there they just sort of have to noodle around like we did with parabolas, asking...what happens as I move inside? What happens as I move outside? As always, keep wandering and hinting until most groups have drawn something like a hyperbola. Then you lecture.

The lecture starts by pointing out what we have. We have two points, once again called the **foci**. They are the defining points of the hyperbola, but they are not part of the hyperbola. And we also once again have a distance which is part of the definition.

Because the foci were horizontally across from each other, we have a horizontal hyperbola. If they were vertically lined up, we would have a vertical hyperbola. You can also do diagonal hyperbolas—anyone remember where we have seen one of those? That's right, inverse variation! That was a hyperbola, just like these. But we're not going to talk about those in this unit, just the horizontal and vertical ones.

Incidentally, a hyperbola is **not** two back-to-back parabolas. It looks like it, but these shapes are actually different from parabolic shapes, as we will see.

OK, so, what good are hyperbolas? The analogy continues...orbits! Suppose a comet is heading toward the sun. (Draw.) If it has a low energy—that is, a low velocity—it gets trapped by the sun, and winds up orbiting around the sun in an elliptical orbit. But if it has high energy (high velocity) it zooms around the sun and then zooms away forever. Its path in this case is half a hyperbola.

Another cool use is in submarine detection. A submarine sends out a pulse. Two receiving stations get the pulse. They don't know what direction it came from or when it was sent, but they do know that one station received it exactly two seconds before the other one. This enables them to say that the distance from the sub to this station, **minus** the distance to this other station, is such-and-such. And this, in turn, locates the sub on a hyperbola.

OK, on to the machinery. Here is the equation for a horizontal hyperbola, centered at the origin.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (12.3)$$

Looks familiar, doesn't it? But the plus has changed to a minus, and that makes all the difference in the world.

Here is a drawing of a horizontal hyperbola.

⁹This content is available online at <<http://cnx.org/content/m19306/1.2/>>.

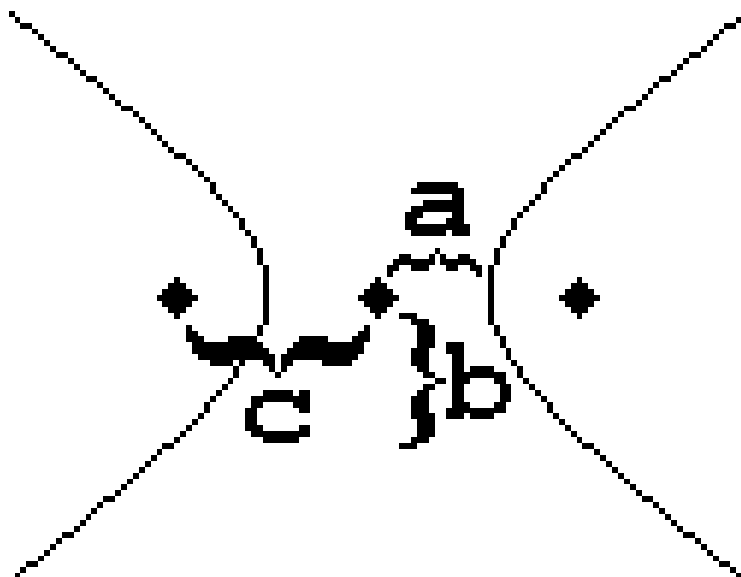


Figure 12.5

Let's be very careful in seeing how this **is**, and **is not**, like an ellipse.

a is defined in a very similar way. It goes from the center, to the edges. In this case, the edges are called the "vertices" (in analogy to parabolas). The distance from one vertex to the other ($2a$ of course) is called the **transverse axis**.

c is defined in a very similar way: it goes from the center to either focus.

b is perpendicular to the other two, just as before. But it goes from the center to...well, to a sort of strange point in the middle of nowhere. We're going to use this point. The distance from the top point to the bottom point ($2b$) is called the **conjugate axis**.

But here is one major difference. In an ellipse, the foci are **inside**; in a hyperbola, they are **outside**. So you can see, just looking at an ellipse, that $a > c$; and you can see, just looking at a hyperbola, that $c > a$. Hence, our equation relating the three shapes is going to be different. Instead of $a^2 = b^2 + c^2$, we have $c^2 = a^2 + b^2$. This reflects the fact that c is the biggest one in this case.

Once again, the class should be able to see that if the center is (h, k) instead of the origin, we replace x^2 with $(x - h)^2$ and y^2 with $(y - k)^2$.

How about a vertical hyperbola? That looks like this:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad (12.4)$$

The way we tell vertical from horizontal is **completely different**. In an ellipse, we told by assuming that $a > b$. In a hyperbola, we have no such guarantee; either a or b could be the greater, or they could even be the same. Instead, we look at which one comes **first**. If we are doing an $x^2 - y^2$ thing, it's horizontal; if we are doing a $y^2 - x^2$ thing, it's vertical. **This is a very common source of errors...be careful!**

There should be no need to go through the whole completing-the-square rigmarole on the board—just tell them it is exactly like with ellipses, including making sure you have a 1 on the right, and there is nothing

multiplied by x^2 or y^2 . But the really different part is the graphing. So let's just pick it up there. Suppose you want to graph:

$$\frac{(x-2)^2}{36} - \frac{(y+3)^2}{4} = 1$$

First of all, what is the center? That's easy: (2,-3).

Now, here is a harder question: does it open vertically, or horizontally? We can answer this question without even looking at the numbers on the bottom! The x^2 is positive and the y^2 is negative, so this is horizontal.

As we did with ellipses, we will then find a , b , and c . $a = 6$ and $b = 2$. We will find c with the hyperbola equation $c^2 = a^2 + b^2$ (different from the ellipse equation!) and get $c = \sqrt{40}$ which is $2\sqrt{10}$ or somewhere just above 6 (again, because 40 is just above 36).

Now, it's drawing time. We start at the center. We go out horizontally by 6 to find the vertices, and by a little more than 6 to find the foci. We go out vertically by 2 to find the endpoints of the conjugate axis, those weird little points in space. Then what?

Here's what you do. Draw a rectangle, going through the vertices, and the endpoints of the conjugate axis. Then, draw diagonal lines through the corners of that rectangle. Those diagonal lines are going to serve as **asymptotes**, or guides: they are not part of the hyperbola, but they help us draw it. Why? Because as it moves out, the hyperbola gets closer and closer to the asymptotes, but never quite reaches them. So once you have drawn your asymptotes, you have a guide for drawing in your hyperbola.

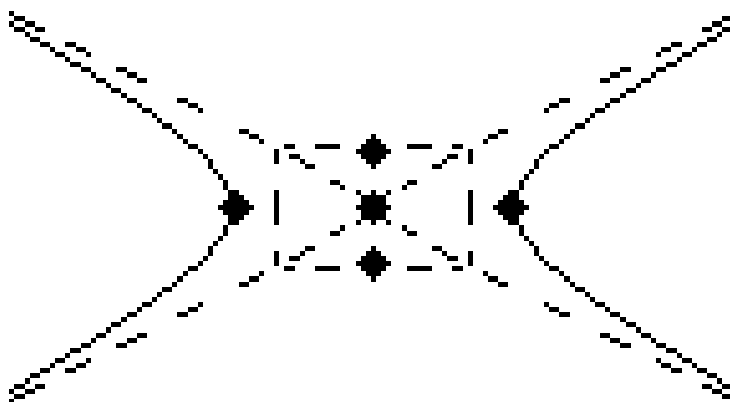


Figure 12.6

Note that I draw the box and the asymptotes in dotted lines, indicating that they are not really part of the hyperbola.

It's worth talking for a while about what an asymptote is, since it is such an important concept in Calculus—the line that the curve gets closer and closer and closer to, without ever quite reaching. It's also worth pointing out that this shows that a hyperbola is not two back-to-back parabolas, since parabolas do not display asymptotic behavior.

Finally, I always mention the comet again. Remember that if a comet comes in with high energy, it swoops around the sun and then flies away again. Now, if there were no sun—if there were nothing in the universe but our comet—the comet would travel in a straight line. And clearly, when the comet is very, very far away from the sun (either **before** or **after** its journey through our solar system), the effect of the sun is very small, so the comet travels almost in a straight line. That straight line is the asymptote. The farther away from the sun the comet gets, the closer it gets to that straight line.

Homework:

“Homework: Hyperbolas”

The only really unusual thing here is that I ask for the equation of one of the asymptotes. This is just a quick review of the skill of finding the equation for a line, given that you already know two points on the line.

You will note that I have absolutely nothing here about going from the geometry of the hyperbola, to the equation. The reason is that it is exactly like the ellipse. You may want to do it, or you may not. If you do it, you shouldn't need to hand them anything—just say “By analogy to what we did with the ellipse, do this.”

Now that you have done all the shapes, the one vital skill that cuts across all of it is looking at an equation ($ax^2 + by^2 + cx + dy + e = 0$) and telling what shape it is. This is done entirely by looking at the coefficients of the squared terms (a and b), and you should refer them to the chart at the end of the “Conceptual Explanations.”

12.9.1 Time for our very last test!

With luck, you have two or three weeks left for review after this, before the final test. Congratulations, you made it through!

Chapter 13

Sequences and Series

13.1 Prerequisites¹

The major “prerequisite” for this unit is the introductory unit on **functions**. However, the introduction to geometric sequences will work best if the unit on **exponents** has already been covered; and the inductive proofs often require skills covered in the unit on **rational expressions**.

13.2 Arithmetic and Geometric Sequences²

The in-class assignment does not need any introduction. Most of them will get the numbers, but they may need help with the last row, with the letters.

After this assignment, however, there is a fair bit of talking to do. They have all the concepts; now we have to dump a lot of words on them.

A “sequence” is a list of numbers. In principal, it could be anything: the phone number 8,6,7,5,3,0,9 is a sequence.

Of course, we will not be focusing on random sequences like that one. Our sequences will usually be expressed by a formula: for instance, “the xxx th terms of this sequence is given by the formula $100+3(n-1)$ ” (or $3n+97$ in the case of the first problem on the worksheet. This is a lot like expressing the **function** $y = 100+3(x-1)$, but it is not exactly the same. In the function $y = 3x+97$, the variable x can be literally any number. But in a **sequence**, xxx n must be a positive integer; you do not have a “minus third term” or a “two-and-a-halfth term.”

The first term in the sequence is referred to as t_1 and so on. So in our first example, $t_5 = 112$.

The number of terms in a sequence, or the particular term you want, is often designated by the letter n .

Our first sequence adds the same amount every time. This is called an **arithmetic sequence**. The amount it goes up by is called the **common difference** d (since it is the difference between any two adjacent terms). Note the relationship to linear functions, and slope.

Exercise 13.1

If I want to know all about a given arithmetic sequence, what do I need to know? Answer: I need to know t_1 and d .

Exercise 13.2

OK, so if I **have** t_1 and d for the arithmetic sequence, give me a formula for the n^{th} term in the sequence. (Answer: $t_n = t_1 + d(n-1)$. Talk through this carefully before proceeding.)

Time for some more words. A **recursive definition** of a sequence defines each term in terms of the previous. For an arithmetic sequence, the recursive definition is $t_{n+1} = t_n + d$. (For instance, in our

¹This content is available online at <<http://cnx.org/content/m19495/1.1/>>.

²This content is available online at <<http://cnx.org/content/m19490/1.1/>>.

example, $t_{n+1} = t_n + 3$. An **explicit definition** defines each term as an absolute formula, like the $3n + 97$ or the more general $t_n = t_1 + d(n - 1)$ we came up with.

Our second sequence multiplies by the same amount every time. This is called a **geometric sequence**. The amount it multiplies by is called the **common ratio** r (since it is the ratio of any two adjacent terms).

Exercise 13.3

Find the recursive definition of a geometric sequence. (Answer: $t_{n+1} = rt_n$. They will do the explicit definition in the homework.)

Exercise 13.4

Question: How do you make an arithmetic sequence go **down**? Answer: $d < 0$

Exercise 13.5

Question: How do you make a geometric series go down? Answer: $0 < r < 1$. (Negative r values get weird and interesting in their own way...why?)

Homework

“Homework: Arithmetic and Geometric Sequences”

13.3 Series and Series Notation³

Begin by defining a series: it’s like a sequence, but with plusses instead of commas. So our phone number example of a sequence, “8,6,7,5,3,0,9” becomes the series “ $8 + 6 + 7 + 5 + 3 + 0 + 9$ ” which is 38.

Many of the other words stay the same. The first term is t_1 , the n^{th} term is t_n , and so on. If you add up all the terms of an arithmetic sequence, that’s called an arithmetic series; and similarly for geometric.

The hardest part about this introduction is the notation. Explain about series notation, using weird examples like $\sum_{n=3}^7 \frac{n^2-2}{5}$ just to make the point that even when the function looks complicated, it is not hard to **write out the terms**.

Note that the “counter” always goes by ones. Does this mean you can’t have a series that goes up by 2s? Ask them how to use series notation for the series $10 + 12 + 14 + 16$.

Homework:

“Homework—Series and Series Notation”

13.4 Arithmetic and Geometric Series⁴

Going over the homework, make sure to mention #3(e), an **alternating series**. You get that kind of alternation by throwing in a $(-1)^n$ or, in this case, $(-1)^{n-1}$.

Last night’s homework ended with the series “all the even numbers between 50 and 100.” Some students may have written $\sum_{n=1}^{26} (48 + 2n)$. Others may have written the answer differently. But one thing they probably all agree on is that adding it up would be a pain. If only there were...a shortcut!

Let’s consider the series $3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$. (Write that on the board.)

Exercise 13.6

What do we get if we add the first term to the last? Answer: 20. Modify your drawing on the board to look like this:

³This content is available online at <http://cnx.org/content/m19491/1.1/>.

⁴This content is available online at <http://cnx.org/content/m19494/1.1/>.

$$3+5+7+9+11+13+15+17$$


Figure 13.1

OK, what about the second term to the second-to-last? Hmm....20 again. Add to the drawing, and then keep adding until it looks like this:

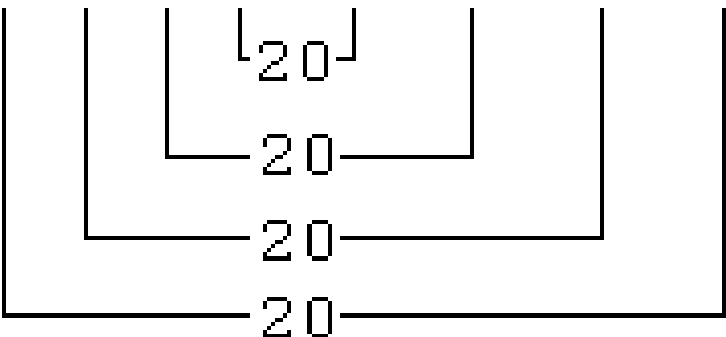
$$3+5+7+9+11+13+15+17$$


Figure 13.2

Exercise 13.7

So, looking at that drawing, what does $3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$ add up to? Hopefully everyone can see that it adds up to four 20s, or 80.

Exercise 13.8

And this is the big one—will that trick work for all series? If so, why? If not, which series will it work for? Answer: It will work for all **arithmetic series**. The reason that the second pair added up the same as the first pair was that we went up by two on the left, and down by two on the right. As long as you go up by the same as you go down, the sum will stay the same—and this is just what happens for arithmetic series.

OK, what about geometric series? Write the following on the board:

$$2 + 6 + 18 + 54 + 162 + 486 + 1458$$

Clearly the “arithmetic series trick” will not work here: $2 + 1458$ is not $6 + 486$. We need a whole new trick. Here it comes. First, to the left of your equation, write $S =$ so the board looks like:

$$S = 2 + 6 + 18 + 54 + 162 + 486 + 1458$$

where S is the mystery sum we’re looking for. Now, **above** that, write:

$$3S =$$

ask the class what comes next. Can we just multiply each term by 3? (Yes, distributive property.) When you write this line, line up the numbers like this:

$$3S = 6 + 18 + 54 + 162 + 486 + 1458 + 437$$

$$S = 2 + 6 + 18 + 54 + 162 + 486 + 1458$$

But don’t go too fast on that step—make sure they see why, if S is what we said, then $3S$ must be that!

Now, underline the second equation (as I did above), and then **subtract the two equations**. What do we get on the left of the equal sign? What do we get on the right? See how things cancel? See if you can get the class to tell you that...

$$2S = 4374 - 2$$

So then S is just 2186. They may want to verify this one on their calculators. Once again, however, the key is to understand why this trick **always works** for any Geometric series.

13.4.1 Homework:

“Homework: Arithmetic and Geometric Series”

13.5 Proof by Induction⁵

13.5.1 Proof by Induction

Going over last night’s homework, make sure they got the right formulas. For an arithmetic series, $S_n = \frac{n}{2}(t_1 + t_n)$. For a geometric, $S_n = \frac{t_1 r^n - t_1}{r - 1}$, in some form or another.

Example 13.1

Students sometimes ask if that formula will still work for an arithmetic series with an odd number of terms. Obviously, you can’t still pair them up in the way we have been doing. The answer is, it does still work. One proof—which I usually don’t mention unless the right questions are asked—is that, for an arithmetic series, the average of all the terms is right in the middle of the first and last terms. (It can take a minute to convince yourself that this is not always true for any series, but it is for any **arithmetic** series.) So the average is $\left(\frac{t_1 + t_n}{2}\right)$, and there are n terms. This leads us to the total sum being $n\left(\frac{t_1 + t_n}{2}\right)$, which is the old formula written in a new way. This lacks some of the elegance of the original proof, but it has the advantage that it doesn’t matter if n is even or odd.

Anyway, on to today’s topic.

NOTE: Important note. The following lecture can be done in 5-10 minutes—I’ve done it many times—and if you do it that way, it doesn’t work. This can be one of the most confusing topics in the whole unit. It must be taken **very slowly and carefully!**

Today, we’re going to learn a new way of proving things. This method, called “proof by induction,” is a very powerful and general technique that turns up in many different areas of mathematics: we are going to be applying it to series, but the real point is to learn the technique itself.

So...to begin with, we are going to prove something we already know to be true:

$$1 + 2 + 3 + 4 \dots n = \frac{n}{2}(n + 1)$$

Of course we know how to prove that using the arithmetic series trick, but we’re going to prove it a **different** way.

⁵This content is available online at <<http://cnx.org/content/m19492/1.1/>>.

Let's start by seeing if that formula works when $n = 1$: in other words, for a 1-term series. In that case, what is the left side of the equation? (Even this seemingly innocuous question can baffle good students sometimes. Give them a minute. Point to the equation. Remind them that the equal sign divides any equation into a left side, and a right side. So, what is the left side of this equation, when there is only one term?) Yes, it is just...1.

How about the right side? Well, that's... $\frac{1}{2}(1 + 1) = 1$. So at least, for this particular case, it works.

To build up to the next step, ask this hypothetical question. Suppose we had not yet proven that this equation **always** works. But suppose that I had proven that it works **when** $n = 200$. Just say, I had sat down with my calculator and added up all the numbers from 1 to 200, which took a very long time, but in the end, I did indeed get what the formula predicts (which is, of course, $100 \times 201 = 20,100$. And now I ask you to confirm that the formula works when $n = 201$.

Well, you can do the right side easily enough: $\frac{201}{2}(202) = 20,301$. But what about the left side? Do you have to add up all those numbers on **your** calculator? No, you don't, if you're clever. (See if they can figure this next part out—this is the key.) I already told you what the first 200 numbers add up to. So you can simply add 201 to my total. $20,100 + 201 = 20,301$.

The point here is not just "it works." The point is that you can confirm that it works, **without adding up all 200 numbers again**, because I already did that part—all you have to add is the last number.

Now...suppose I had already proven that it works for $n = 325$. How would we show that it works for $n = 326$? Good—we would add 326 to the old answer (for $n = 325$), and see if we got what the formula predicted we **should** get for $n = 326$. Let's try it...

Now...suppose I had already proven that it works for $n = 1000$. How would we show that it works for $n = 1001$? Good—we would add 1001 to the old answer (for $n = 1000$), and see if we got what the formula predicted we **should** get for $n = 1001$. Let's try it... *Repeat this exercise until they are sick of it, but boy, do they get it. Then hit them with the big one: what is the general form of this question? See if they can figure out that it is:*

Suppose I had already proven that it works for some n . How would we show that it works for $n + 1$?

Give them time here...see if they can find the answer...

We would add $(n + 1)$ to the old answer (for n , and see if we got what the formula predicted we **should** get for $(n + 1)$.

What does that look like? Well, for the old n , the formula predicted we would get $\frac{n}{2}(n + 1)$. So if we add $(n + 1)$ to that, we get $\frac{n}{2}(n + 1) + (n + 1)$. And what **should** we get? Well, for $(n + 1)$, the formula predicts we should get $\frac{n+1}{2}(n + 1 + 1)$.

Do the algebra to show that they are equal. Then, step back and say...so, what have we done? Well, first we proved that the formula works for $n = 1$. Then we proved—not for one specific case, but quite generally—that **if it works for any number, it must also work for the next number**. If it works for $n = 1$, then it must work for $n = 2$. If it works for $n = 2$, then it must work for $n = 3$...and so on. It must always work.

At this point, I think it's helpful to work through one more example. I recommend going through $\sum \frac{1}{n(n+1)}$, just as it is done in the Conceptual Explanations. This time, you're using a little less explanation and focusing more on the process, so it makes a better model for their homework.

Homework:

"Homework—Proof by Induction"

At this point, you're ready for the test. Unlike most of my "Sample Tests," this one is probably too **short**, but it serves to illustrate the sorts of problems you will want to ask, and to remind the students of what we've covered.

13.6 Extra Credit⁶

13.6.1 An extra cool problem you may want to use as an extra credit or something

Exercise 13.9

(Solution on p. 85.)

A bank gives $i\%$ interest, compounded annually. (For instance, if $i = 6$, that means 6% interest.) You put A dollars in the bank **every year for n years**. At the end of that time, how much money do you have?

NOTE: (The fine print: Let's say you make your deposit on January 1 every year, and then you check your account on December 31 of the last year. So if $n = 1$, you put money in exactly once, and it grows for exactly one year.)

The **previous** year's money receives interest twice, so it is worth $A(1 + \frac{i}{100})^2$ at the end. And so on, back to the first year, which is worth $A(1 + \frac{i}{100})^n$ (since that initial contribution has received interest n times).

So we have a Geometric series:

$$S = A(1 + \frac{i}{100}) + A(1 + \frac{i}{100})^2 + \dots + A(1 + \frac{i}{100})^n$$

We resolve it using the standard trick for such series: multiply the equation by the common ratio, and then subtract the two equations.

$$(1 + \frac{i}{100})S = A(1 + \frac{i}{100})^2 + \dots + A(1 + \frac{i}{100})^n + A(1 + \frac{i}{100})^{n+1}$$

$$S = A(1 + \frac{i}{100}) + A(1 + \frac{i}{100})^2 + \dots + A(1 + \frac{i}{100})^n$$

$$(\frac{i}{100})S = A(1 + \frac{i}{100})^{n+1} - A(1 + \frac{i}{100})$$

$$S = \frac{100A}{i} \left[(1 + \frac{i}{100})^{n+1} - (1 + \frac{i}{100}) \right]$$

Example 13.2

Example: If you invest \$5,000 per year at 6% interest for 30 years, you end up with:

$$\frac{100(5000)}{6} [1.0631 - 1.06] = \$419,008.39$$

Not bad for a total investment of \$150,000!

⁶This content is available online at <<http://cnx.org/content/m19493/1.1/>>.

Solutions to Exercises in Chapter 13

Solution to Exercise 13.9 (p. 84)

The money you put in the very **last** year receives interest exactly once. “Receiving interest” in a year always means being multiplied by $(1 + \frac{i}{100})$. (For instance, if you make 6% interest, your money multiplies by 1.06.) So the A dollars that you put in the last year is worth, in the end, $A(1 + \frac{i}{100})$.

Chapter 14

Probability

14.1 Tree Diagrams¹

You can start them off here with the in-class assignment “How Many Groups?” or even hand it out after the previous test: it isn’t long or difficult, and it does not require any introduction.

It does, however, require a lot of follow-up. The worksheet leads to a lecture, and the lecture goes something like this.

The big lesson for the first day is how to make a chart of all the possibilities for these kinds of scenarios. For the die-and-coin problem, the tree diagram looks like (draw this on the board):

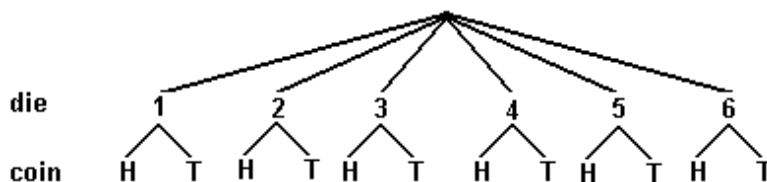


Figure 14.1

It may seem silly to repeat “Heads-Tails, Heads-Tails,” and so on, six times. But if you do so, each “leaf” of this “tree” represents exactly one possibility. For instance, the third leaf represents “The die rolls a 2, and the coin gets heads.” That’s a completely different outcome from the fifth leaf, “The die rolls a 3 and the coin gets heads.”

Using a tree like this, we can answer probability questions.

Exercise 14.1

What is the probability of the outcome “Die rolls 2, coin gets heads?” Just ask this question, give them 15 seconds or so to think about it, then call on someone for the answer. But then, talk through the following process for getting the answer.

1. Count the number of leaves that have this particular outcome. In this case, only one leaf.
2. Count the total number of leaves. In this case, twelve.
3. Divide. The probability is $\frac{1}{12}$.

¹This content is available online at <<http://cnx.org/content/m19463/1.1/>>.

Exercise 14.2

What is the probability of the outcome “Die rolls a prime number, coin gets tails?” Give them 30 seconds or so, then go through it carefully on the chart. There are three such leaves. (*Trivia fact: 1 is not considered a prime number.) So the probability is $\frac{3}{12}$, or $\frac{1}{4}$.

Exercise 14.3

What does that really mean? I mean, either you’re going to get that outcome, or you’re not. After you roll-and-flip, does it really mean anything to say “The probability of that outcome was $\frac{1}{4}$?”

Give the class a minute, in pairs, to come up with the best possible explanation they can of what that statement, “The probability of this event is $\frac{1}{4}$,” really means. Call on a few. Ultimately, you want to get to this: it doesn’t really mean much, for one particular experiment. But if you repeat the experiment 1,000 times, you should expect to get this result about 250 of them.

Exercise 14.4

(Solution on p. 96.)

Why are there twelve leaves? (Or: How could you figure out that there are twelve leaves, without counting them?)

Get the idea? OK, let’s try a new one.

A Ford dealer has three kinds of sedans: Ford Focus, Taurus, or Fusion. The Focus sedan comes in three types: S, SE, or SES. The Fusion sedan also comes in three types: S, SE, or SEL. The Taurus comes in only two models: SEL, and Limited. (All this is more or less true, as far as I can make out from their Web site.)

Ask everyone in the class, in pairs, to draw the appropriate tree and use it to answer the following questions.

- If you choose a car at random from the dealer lot, what are the odds that it is a Fusion? (Answer: $\frac{3}{8}$.)
- What are the odds that it is a Fusion SE? (Answer: $\frac{1}{8}$.)
- What are the odds that it is any kind of SEL? (Answer: $\frac{1}{4}$.)
- Finally—and most important—what assumption must you make in answering all of the above questions, that was not stated in the original description of the situation?

Hopefully someone will come up with the answer I’m looking for to that last question: you’re assuming that the dealer’s lot has exactly the same number of each possible kind of car. In real life, of course, that assumption is very likely wrong.

This ties in, of course, to the last question on the assignment they did. Red-haired people are considerably more rare than the other types. So this business of “counting leaves” only works when each leaf is exactly as common, or probable, as each other leaf. This does not mean we cannot do probability in more complicated situations, but it means we will need to develop a more sophisticated rule.

14.1.1 Homework

“Homework: Tree Diagrams”

When going over this homework the next day, there are two things you want to emphasize about problem #2 (stars). First: because the actual tree diagram would have 70 leaves, you don’t want to physically draw it. You sort of have to imagine it. They have seen three diagrams now: the coin-and-die, the cars, and the three-coins. That should be enough for them to start to imagine them without always having to draw them.

Second: the answer to question 2(c) is not “one in 70, so maybe 14 or so.” That logic worked fine with 1(e), but in this case, it is not reasonable to assume that all types of stars are equally common. The right answer is, “I can’t answer this question without knowing more about the distribution of types.”

Problem #3(e) is subtler. The fact that $\frac{1}{4}$ of the population is children does not mean that $\frac{1}{4}$ of the white population is children. It is quite possible that different ethnic groups have different age breakdowns. But ignoring that for the moment, problem #3 really brings out a lot of the main points that you want to make the next day:

14.2 Introduction to Probability²

OK, let's really talk about problem #3 from last night's homework. And for the moment, let's ignore part (e), and go ahead and assume that $\frac{1}{4}$ of all white people are children. Based on that assumption, you would expect roughly 75 white people, about 19 of whom are children, and about 9 of whom are boys. If you answered exactly $\frac{75}{8}$, or 9.375, that isn't a crazy answer. Of course, you can't actually have 9.375 white boys in a room. But that is actually the "expectation value" for such an experiment. If you have a thousand rooms with a hundred people each, the average number of white boys in each room will probably be 9.375.

In the formal language of probability, we would say that for any randomly chosen person in the U.S. in 2006, there is a 9.375% chance that this person will be a white boy. That's what "percent" means: out of a hundred.

Exercise 14.5

What percent of the people in this class, right now, are girls?

Exercise 14.6

If you roll a die, what is the percent chance that you will get an even number?

OK, that's easy enough. But we're going to tweak it a bit. Obviously, there is nothing magical about the number 100. We could just as easily ask "How many out of a thousand?" or "How many out of 365?" But what turns out to be most convenient, mathematically, is to ask the question "How many out of 1?"

This is how we are going to work with probability numbers from here on out, so it is very important to understand this numbering system!

- The probability of any event whatsoever, under any and all circumstances, is always between 0 and 1. A probability of -2 , or a probability of 2 , is meaningless.
- A probability of 0 means "It cannot possibly happen."
- A probability of 1 means "It is guaranteed to happen."
- A probability of $\frac{1}{4}$ means "It has a one in four chance of happening," or "If you try this 100 times, it will probably happen 25 of them," so it is the same as a 25% chance.

After you have said all that, you're ready to hit them with the worksheet "Introduction to Probability." It should only take 10 minutes (half of which is spent on #2a).

Then come back. Let's go over #2 carefully.

2b. The probability is $\frac{1}{16}$. You can see this from the tree diagram, but how could we have figured it out without a drawing? The answer—we've discussed this before, and it is absolutely central—is by multiplying. 4 possibilities for the first die, times 4 possibilities for the second die, makes 16 possibilities for the combination.

But here's another way we can look at that same multiplication. The probability of "3 on the first die" is $\frac{1}{4}$. The probability of "2 on the second die" is also $\frac{1}{4}$. So the probability of both these events happening is $\frac{1}{4} \times \frac{1}{4}$, or $\frac{1}{16}$.

When you have two different independent events—that is, neither one has an effect on the other—the probability of both happening is the probability of the first one, times the probability of the second one.

The idea of "independent" events is crucial here, of course, and you have to stress it. But it's also a fairly obvious point, and there is a real danger of making it sound more esoteric than it is. If you spend ten minutes discussing the word "independent" you may do more harm than good. Consider trying this instead. Tell that class that you're looking into a big box full of bananas. One out of every four bananas in the box is green; the rest are yellow. Also, one out of every three bananas is stamped "Ship to California"; the rest say "ship to New York." Finally, half the bananas are over four days old.

- What is the probability that a given banana is green, and destined for New York? $\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$. One out of every six bananas have both of these attributes. Or, to put it another way, a given randomly chosen banana has a $\frac{1}{6}$ chance of having both attributes.

²This content is available online at <http://cnx.org/content/m19461/1.1/>.

- What is the probability that a given banana is green, and over four days old? Well, not much. Not $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$. Because in general, as a banana gets older, it turns from green to yellow. So being green, and being old, are not independent: one makes the other less likely.

Now, ask the class, in pairs, to come up with a similar scenario. (It should not involve fruit!) They should think of two events that are independent, and calculate the probability of both of them happening. Then they should think of two events that are not independent, and explain why the probability of both of them happening is not the product of their individual probabilities.

Homework

“Homework: The Multiplication Rule”

Going over this homework, of course you want to make sure that the last problem gets answered. With a little thought, it should be obvious to anyone that if P is the probability that something will occur, $1 - P$ is the probability that it will not occur. If it happens 1 time out of 5, then it doesn’t happen 4 times out of 5. This can be memorized as a new rule, along with the multiplication rule, but it is easier to see why it works.

14.3 Trickier Probability Problems³

The thing that makes probability problems the darling of math contest writers everywhere, is also the thing that makes them frustrating for so many students: no two problems are exactly alike. Most probability problems can be solved with the multiplication rule, combined with a lot of good, hard thinking about the problem.

I’m going to present two scenarios with five questions here, in the lesson plan. The idea is for you to talk them through with the class. In each case, explain the scenario and the question clearly. Then give them a minute or two, with no guidance, to think about it. Then take their answers and go over the correct answer very slowly and clearly. None of them should be presented as if it were a symbol of a whole, unique, important class of problems. Each should be presented as simply another example of you can solve a wide variety of problems, if you’re willing to think about them patiently and clearly.

Example 14.1: Scenario 1

You reach your hand into a bag of Scrabble® tiles. The bag has one tile with each letter. You pull out, first one tile, and then another.

1. What is the probability that you will pull out, first the letter A , and then the letter B ? The quick, easy answer is $\frac{1}{26} \times \frac{1}{26}$. Quick, easy...and not quite right. Yes, there is a $\frac{1}{26}$ chance that the first tile will be an A . But once you have that tile, there are only 25 left. So the odds of the second tile being a B are actually $\frac{1}{25}$. The odds of getting an A followed by a B are $\frac{1}{26} \times \frac{1}{25}$.
2. What is the probability that your two tiles are the letters A and B ? It looks like the same question, but there is a subtle difference. You could pull out A followed by B (as in the last example), or you could pull out B followed by A . So there are really two ways to do it, and the probability is $\frac{1}{25} \times \frac{1}{26} \times 2$.

Example 14.2: Scenario 2

You roll two 6-sided dice.

1. What is the probability that the sum of the two dice is 10? Imagine making a tree diagram. It would have 36 leaves. How many of them would have a sum of 10? 6–4, 5–5, and 4–6. (Of course, on the tree diagram, “6 on the first die, 4 on the second” is a different leaf from “4 on the first die, 6 on the second”...just as in the AB problem above.) So the probability is $\frac{3}{36}$, or $\frac{1}{12}$.

³This content is available online at <<http://cnx.org/content/m19464/1.1/>>.

2. What is the probability that neither die rolls a 1? We do not have a “neither” rule, so we have to reframe the question in terms of the rules we do have. We can rephrase the question like this: what is the probability that the first die doesn’t roll a 1, and the second die also doesn’t roll a 1? The first is $\frac{5}{6}$, and the second is also $\frac{5}{6}$. So the odds of both happening is $\frac{25}{36}$. It’s an easy question to answer, once you reword it correctly.
3. What is the probability that either die (or “at least one die”) rolls a 1? (Most people think the answer will be $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. By that logic, by the time you roll six dice, you are guaranteed to get at least one 1: obviously not true!)

The right way to think about this problem is as the reverse, the “not,” of the previous problem. We said that 25 out of 36 times, neither die will roll a 1. So the remaining 11 out of 36 times, at least one of them will. This is an example of the “not” rule we got from last night’s homework: the probability of “no ones” is $\frac{25}{36}$, so the probability of NOT “no ones” is $1 - \frac{25}{36} = \frac{11}{36}$. (*It’s interesting to note that the “naïve” guess of $\frac{1}{3}$ is not too far off, and makes a reasonable approximation. If you have a 1 in 10 chance of doing something, and you try three times, there is a roughly $\frac{3}{10}$ chance that you will succeed at least once—but not exactly $\frac{3}{10}$.)

The last thing you need to assure the class, before you hit them with the worksheet, is that no one is born knowing how to do this. Probability problems are just like everything else: they make more sense, and get easier, with practice. It’s OK to get frustrated, but don’t give up!

Then give them the worksheet. Ideally they should be able to make a good (10-15 minute) start in class, and then finish it up for homework. Expect to spend a lot of the next day going over these. It’s worth it.

Homework

“Homework: Trickier Probability Problems”

14.4 Permutations⁴

No in-class worksheet today—a day of lecture.

How many different three-digit numbers can we make using only the digits 1, 2, and 3? Answer: 27. Here they are, listed very systematically. (If possible, project this table onto a screen where everyone can see it and look at it for a moment, to see the pattern and how it is generated.)

First Digit	Second Digit	Third Digit	Resulting Number
1	1	1	111
		2	112
		3	113
	2	1	121
		2	122
		3	123
		1	131
3			<i>continued on next page</i>

⁴This content is available online at <<http://cnx.org/content/m19462/1.2/>>.

2	1	2	132
		3	133
		1	211
	2	2	212
		3	213
		1	221
	3	2	222
		3	223
		1	231
3	1	2	232
		3	233
	2	1	311
		2	312
		3	313
	3	1	321
		2	322
		3	323
	3	1	331
		2	332
		3	333

Table 14.1

Effective, and not particularly difficult...but tedious. How could we have answered without the table? Well, of course, it's the rule of multiplication again. There 3 possibilities for the first digit. For each of these, there are 3 possibilities for the second digit; and for each of these, 3 possibilities for the third digit. $3 \times 3 \times 3 = 27$.

Now, let's ask a different problem: how many possible 3-digit numbers can be made using the digits 1, 2, and 3, if every digit is used only once? Once again, we can list them systematically—and it's a lot easier this time. Once you have chosen the first two digits, the third digit is forced. There are only six possibilities.

First Digit	Second Digit	Third Digit	Number
1	2	3	123
	3	2	132
2	1	3	213

continued on next page

	3	1	231
3	1	2	312
	2	1	321

Table 14.2

I really do believe it is important to show them these tables before doing any calculations!!! There is no substitute for seeing everything laid out in an organized manner to get a feeling for the space.

Once again, however, once they have seen the table, we can ask the question: why 6? And once again, we can answer that question using the rule of multiplication. There are three possible numbers that can go in the first digit. Once you have chosen that digit, there are only two possible numbers that can go in the second digit. And once you have chosen that, there is only one number that can possibly go in the third. $3 \times 2 \times 1 = 6$.

Exercise 14.7

Repeat the above problems, only with nine digits instead of three. First, how many different nine-digit numbers can be made using the digits 1–9? Second, how many different nine-digit numbers can be made if you use the digits 1–9, but use each digit only once? Obviously we don’t want to make these tables (even the second one is prohibitive!) but with the rule of multiplication, and our calculators, we can figure out how big the tables would be. Give them a couple of minutes on this.

The first is $9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9$. (Nine possibilities for the first digit; for each of those, nine for the second; and so on.) Even that is tedious to write. Let’s write it like this instead: 9^9 . We can punch it into the calculator just like that.

The second is $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$. (Nine possibilities for the first digit; for each of those, only eight for the second, because one of them is used up; and so on.) Is there any easy way to write that? In fact, there is. It is called 9 factorial, and it is written $9!$. You may want to show them how to get factorials on their calculators. On some (such as the TI-83) the factorial option is actually listed under probability, reflecting the fact that factorials are used so often in probability problems, for this very reason.

Incidentally, $9^9 = 387,420,489$ possibilities for the first scenario. $9! = 362,880$ for the second: still a pretty big number, but only about a thousandth as big as the first one. This should come as no surprise: almost all of the nine-digit numbers use the same digit twice somewhere or other!

Exercise 14.8

Question: How many different ways can five books be arranged on a shelf? Give them a minute, and see if they can figure out that it is the same problem we just did. 5 books can go in the first position; for each of these, 4 in the second position; and so on. $5! = 120$ possibilities.

Exercise 14.9

Question: How many three-digit numbers can be made using the digits 1–9?

If we are allowed to repeat digits, this is hopefully pretty easy by this point: $9 \times 9 \times 9$. Written more concisely, 9^3 .

But what if we’re not? Is there any way we can write $9 \times 8 \times 7$ more concisely? There is, and it’s a bit sneaky: it is $\frac{9!}{6!}$. Explain why this works. Point out that, while they may not particularly need it for $9 \times 8 \times 7$, it’s really nice as a shortcut for $20 \times 19 \times 18 \dots 8$.

If you have extra time, ask everyone in class to come up with two scenarios: one of the “the same thing can be used twice” (exponential) variety, and one of the “the same thing cannot be used twice” (factorial) variety.

Homework

“Homework: Permutations”

14.5 Combinations⁵

Once again, this one is lecture, without an in-class worksheet, walking through a series of questions.

Suppose you have a Daisy, an Iris, a Lily, a Rose, and a Violet. You are going to make a floral arrangement with three of them. How many possible arrangements can you make?

Based on yesterday, it's tempting to answer $5 \times 4 \times 3$, or $\frac{5!}{2!}$. But here's why this is different from yesterday's problems: order doesn't matter. "A rose, a daisy, and an iris" is the same arrangement as "A daisy, an iris, and a rose": you don't want to count it twice.

So, let's start the way we did yesterday: list them all. Give the class a minute to do this. Remind them, as always, that it's best to be systematic to make sure you list every possibility exactly once. Here's my list.

First Flower	Second Flower	Third Flower	Arrangement
Daisy	Iris	Lily	Daisy, Iris, Lily
		Rose	Daisy, Iris, Rose
		Violet	Daisy, Iris, Violet
	Lily	Rose	Daisy, Lily, Rose
		Violet	Daisy, Lily, Violet
	Rose	Violet	Daisy, Rose, Violet
Iris	Lily	Rose	Iris, Lily, Rose
		Violet	Iris, Lily, Violet
	Rose	Violet	Iris, Rose, Violet
Lily	Rose	Violet	Lily, Rose, Violet

Table 14.3

10 items in all. I generated the list by using the same kind of systematic approach I used for the permutations, but always moving forward in the list: so after "Lily" I'm allowed to list "Rose" and "Violet," but not "Daisy" or "Iris."

This turns out to be such a common and important operation that it gets its own name: "choose." We would say "5 choose 3 is 10," sometimes written $\binom{5}{3} = 10$. It means, if you have five items, and want to choose 3 of them, there are ten ways to do so.

Exercise 14.10

The four Beatles are John, Paul, Georg, and Ringo. Suppose you are going to put photographs of three of them on your door. List all the possible combinations. How many are there?

Give the class a minute to list—they should make a diagram like the one I made above!—and count. When they are done, you can point out that the answer (four combinations) is very obvious if you look at it backward: each combination leaves exactly one Beatle out. There is a very important insight here, which can also be applied to the flower example: each arrangement listed leaves exactly two flowers out. So 5 choose 3 ("which flowers should I include?") is the same number as 5 choose 2 ("which flowers should I leave out?").

Now we're going to try something with bigger numbers. A drama teacher looks out at a class of 30 students, and wants to choose 2 of them to run a scene. How many possible pairs of students are there? As before, when the numbers get this large, we can't reasonably list all the combinations: we need an algebraic way of figuring out how many there are, without actually counting them all.

⁵This content is available online at <http://cnx.org/content/m19460/1.1/>.

To start off, let's turn this combinations ("order doesn't matter") problem into a permutations ("order does matter") problem. The teacher wants to choose one student to play the Child and one to play the Dog. In this version of the problem, "John plays the Child and Susan plays the Dog" is different from "Susan plays the Child and John plays the Dog," and should be counted separately. So it is a straightforward permutations problem. There are 30 possible actors for the Child, and for each of those, 29 for the Dog. $30 \times 29 = 870$ possible scenes.

Now let's return to the original problem, how many pairs of students are there? In this problem, "John-Susan" and "Susan-John" are the same pair. The key insight here is that, when we ran the permutations problem, we counted each pair twice. So the answer is $870/2 = 435$ pairs.

Take this slow and easy. This is a very general approach to combinations problem. First, you solve the (easier) permutations problem. Then you ask, "How many times did I count every group?" and divide by that.

Ask the class to try this approach on the original (flower) problem. They should answer three questions.

1. How many permutations are there? Remember that in this question, "rose-daisy-iris" and "daisy-iris-rose" are two different arrangements, as if each flower is being placed in a numbered slot.
2. Now, when we counted up the permutations, how many times did they redundantly count each combination?
3. Divide the first answer by the second, and you have the total number of combinations.

Lay out the process, then have them work the problem. Many of them will get stuck on step (2), incorrectly thinking that we counted each permutation three times. In fact, we counted each one six times:

daisy-iris-rose, daisy-rose-iris, iris-daisy-rose, iris-rose-daisy, rose-daisy-iris, rose-iris-daisy

The permutations are $5 \times 4 \times 3$, or $\frac{5!}{2!}$, or 60. But each combination is listed six times, so the total number of combinations is 10, as we counted before.

So...why six times? This is the last question, and it's a hard one. When we were choosing two items, we counted each combination twice (John-Susan, Susan-John). When we were choosing three items, we counted each combination six times. What's the pattern? Give them a minute to think about this. Then make sure they understand that it is, in fact...another permutations problem! The list of six I gave above is simply the number of ways you can arrange three items, or $(3!)$. You would divide by $(4!)$ for 4 items, and so on.

Homework

"Homework: Permutations and Combinations"

A sample test, and real test, and you're done!

Solutions to Exercises in Chapter 14

Solution to Exercise 14.4 (p. 88)

It's six (number of possibilities for the die) times two (number of possibilities for the coin). For the second question on the worksheet, with the frogs, there are 15,000 groups, or $5,000 \times 3$. This multiplication rule is really the heart of all probability work, so it's best to get used to it early.

Index of Keywords and Terms

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