

# Systems of Equations and Inequalities

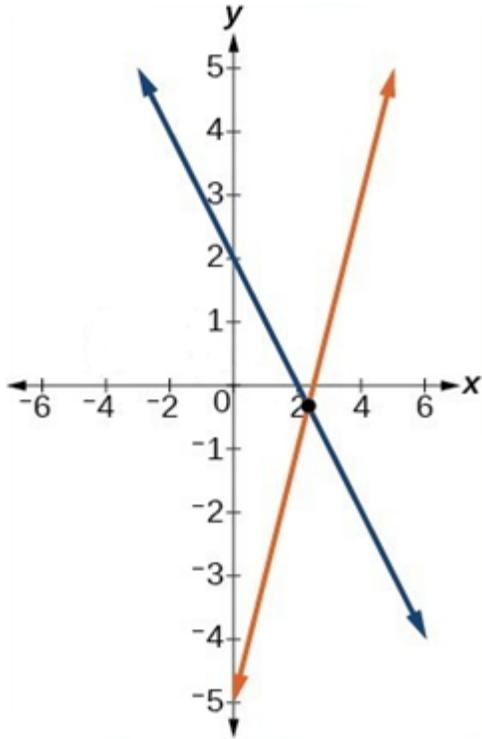
College Algebra

# System of Linear Equations

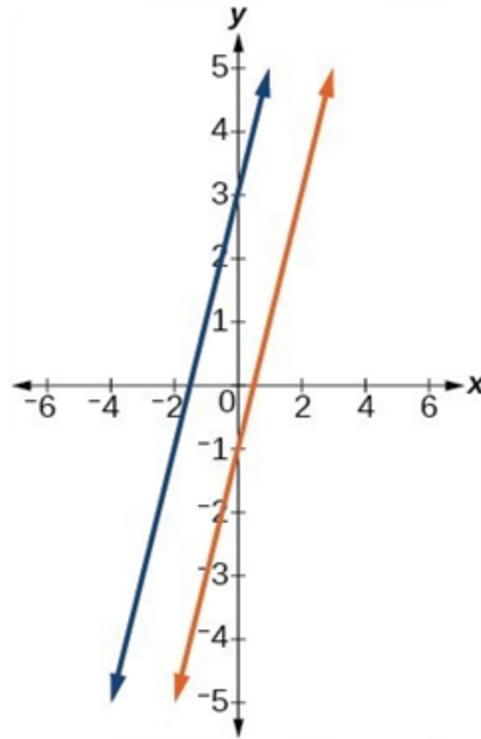
There are three types of systems of linear equations in two variables, and three types of solutions.

1. An **independent system** has exactly one solution pair  $(x, y)$ . The point where the two lines intersect is the only solution.
2. An **inconsistent system** has no solution. The two lines are parallel and will never intersect.
3. A **dependent system** has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

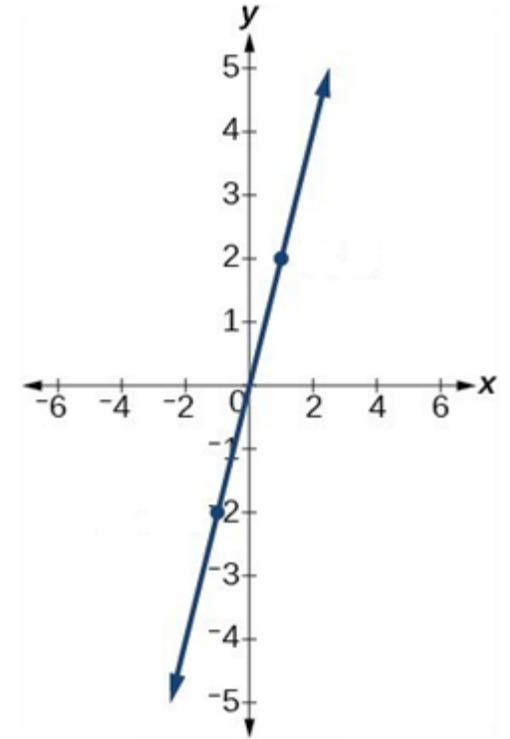
# Graphs of Linear Systems



Independent System



Inconsistent System



Dependent System

# Substitution Method

**Given a system of two equations in two variables, solve using the substitution method.**

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

# Solving Systems of Equations by Substitution

**Example:** Solve the following system of equations:

$$-x + y = -5$$

$$2x - 5y = 1$$

**Solution:**

$$y = x - 5$$

Solve the first equation for  $y$

$$2x - 5(x - 5) = 1$$

Substitute the expression for  $y$  into the second equation

$$x = 8$$

Solve for  $x$

$$-8 + y = -5$$

Substitute the value of  $x$  back into the first equation

$$y = 3$$

Solve for  $y$ . The solution is (8,3)

# Addition Method

**Given a system of two equations in two variables, solve using the addition method.**

1. Write both equations with  $x$ - and  $y$ -variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. Use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.

# Solving A System by the Addition Method

**Example:** Solve the following system of equations:

$$3x + 5y = -11$$

$$x - 2y = 11$$

**Solution:**

$$-3(x - 2y) = -3(11)$$

$$\begin{cases} 3x + 5y = -11 \\ -3x + 6y = -33 \end{cases}$$

$$11y = -44, \text{ or } y = -4$$

$$x - 2(-4) = 11$$

$$x = 3$$

Multiply the second equation by  $-3$

Add the two equations

Solve for  $y$

Substitute  $y$  into either equation

Solve for  $x$ . The solution is  $(3, -4)$

# Inconsistent System of Equations

**Inconsistent systems** consist of parallel lines that have the same slope but different  $y$ -intercepts.

$$\text{Example: } \begin{cases} x = 9 - 2y \\ x + 2y = 13 \end{cases}$$

Solution:

$$(9 - 2y) + 2y = 13$$

$$9 = 13$$

Use the substitution method

Clearly, this is a contradiction, so this system has no solution.

# Dependent System of Equations

**Dependent systems** of equations in two variables is a system in which the two equations represent the same line, with an infinite number of solutions.

**Example:** 
$$\begin{cases} x + 3y = 2 \\ 3x + 9y = 6 \end{cases}$$

**Solution:**

$$\begin{cases} -3(x + 3y) = -3(2) \\ 3x + 9y = 6 \end{cases}$$

$$0 = 0$$

Use the addition method

This is an identity, so there are an infinite set of points  $(x, y)$  that satisfy both equations.

# Writing the General Solution

In the previous example, we presented an analysis of the solution to the following system of equations:

$$x + 3y = 2$$

$$3x + 9y = 6$$

After a little algebra, we found that these two equations were exactly the same. We then wrote the general solution as  $(x, -13x + 23)$ . Why would we write the solution this way? In some ways, this representation tells us a lot. It tells us that  $x$  can be anything,  $x$  is  $x$ . It also tells us that  $y$  is going to depend on  $x$ , just like when we write a function rule. In this case, depending on what you put in for  $x$ ,  $y$  will be defined in terms of  $x$  as  $-13x + 23$

In other words, there are infinitely many  $(x, y)$  pairs that will satisfy this system of equations, and they all fall on the line  $f(x) = -13x + 23$ .

# Models Using a System of Linear Equations

**Given a situation that represents a system of linear equations, write the system of equations and identify the solution.**

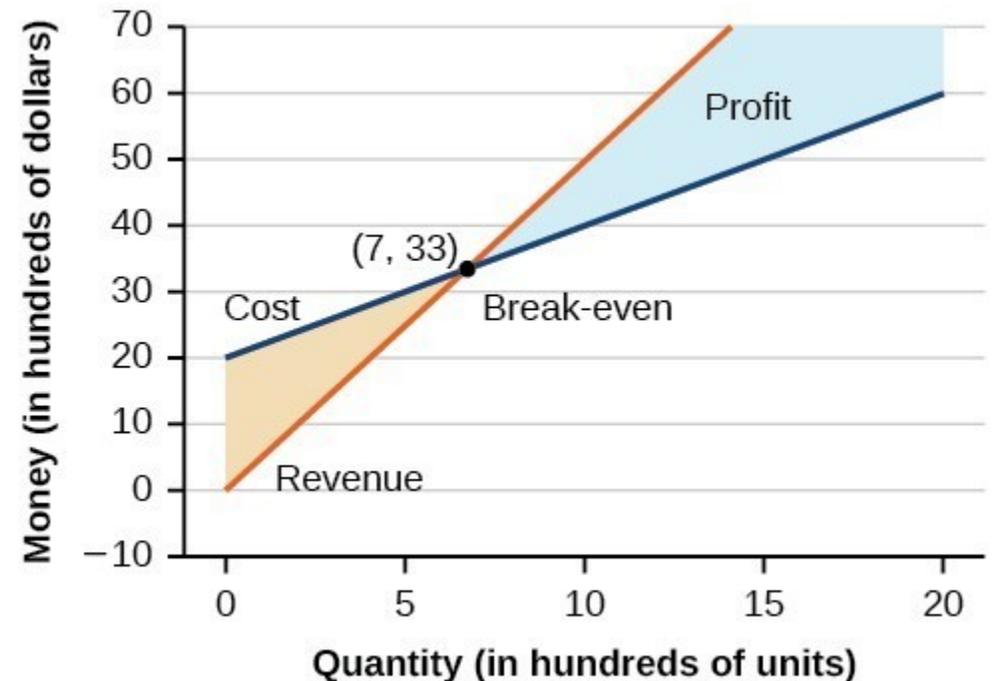
1. Identify the input and output of each linear model.
2. Identify the slope and  $y$ -intercept of each linear model.
3. Find the solution by setting the two linear functions equal to another and solving for  $x$ , or find the point of intersection on a graph.

# Using Linear Systems to Investigate Profits

The **revenue function** of a business can be represented by  $R(x) = px$ , where  $x$  = quantity and  $p$  = price.

The **cost function** includes fixed costs and variable costs and can be represented as  $C(x) = F + vx$ .

The **break-even point** is where the two lines intersect,  $R(x) = C(x)$  for a given quantity  $x$ .

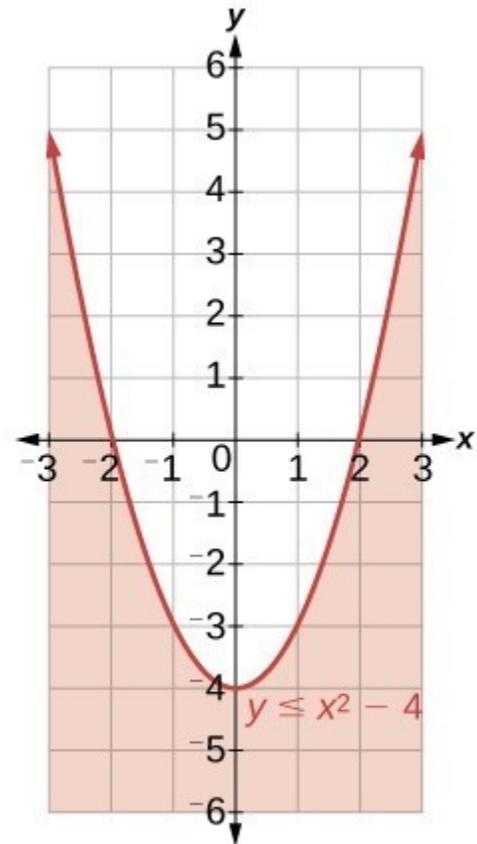
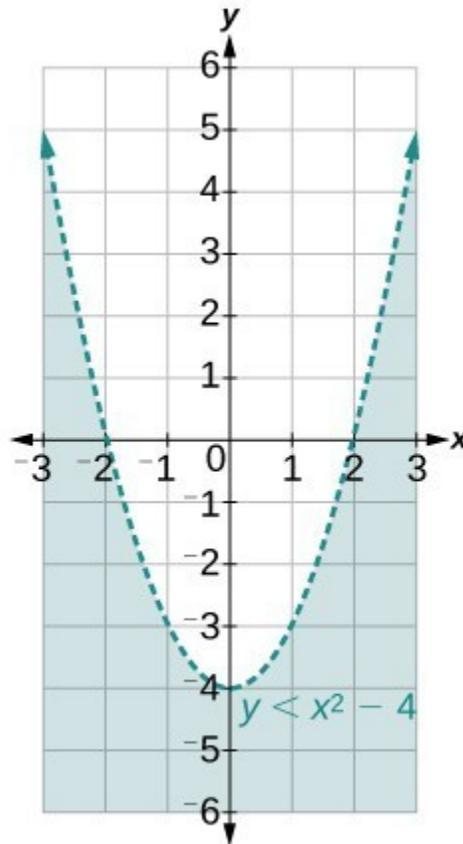
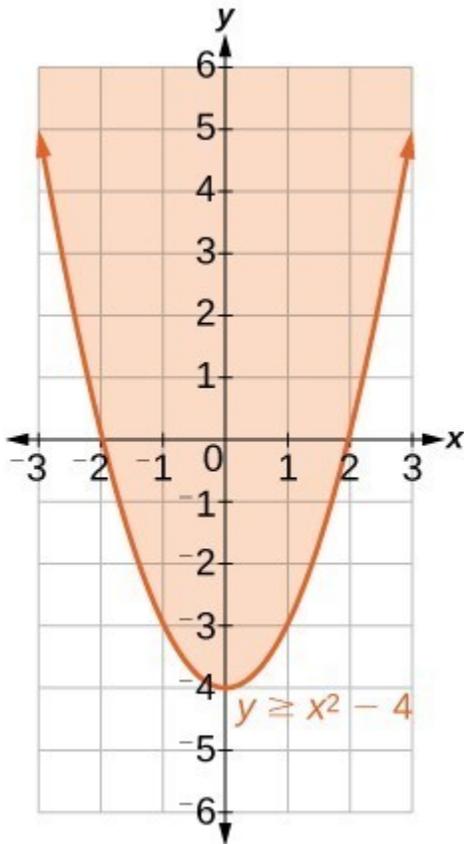
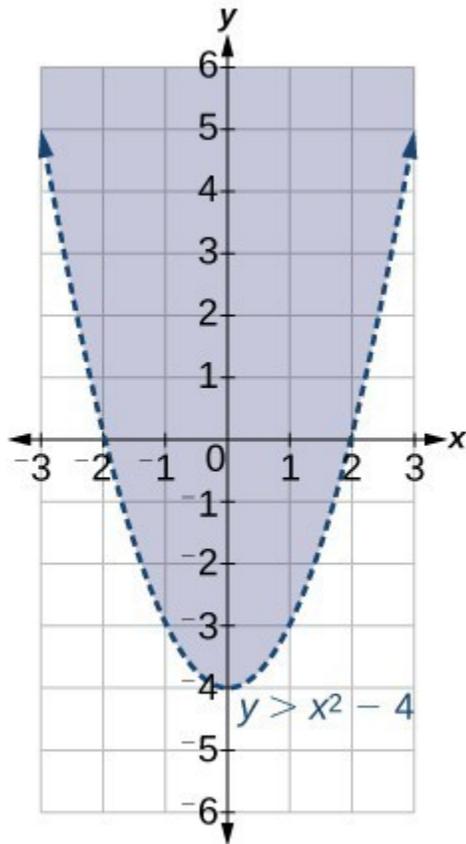


# Graphing Nonlinear Inequalities

**Given an inequality bounded by a parabola, sketch a graph.**

1. Graph the parabola as if it were an equation. This is the boundary for the region that is the solution set.
2. If the boundary is included in the region (the operator is  $\leq$  or  $\geq$ ), the parabola is graphed as a solid line. If the boundary is not included in the region (the operator is  $<$  or  $>$ ), the parabola is graphed as a dashed line.
3. Test a point in one of the regions to determine whether it satisfies the inequality statement. If the statement is true, the solution set is the region including the point. If the statement is false, the solution set is the region on the other side of the boundary line.
4. Shade the region representing the solution set.

# Inequalities Bounded by a Parabola



# Graphing a System of Nonlinear Inequalities

**Given a system of nonlinear inequalities, sketch a graph.**

1. Find the intersection points by solving the corresponding system of nonlinear equations.
2. Graph the nonlinear equations.
3. Find the shaded regions of each inequality.
4. Identify the feasible region as the intersection of the shaded regions of each inequality or the set of points common to each inequality

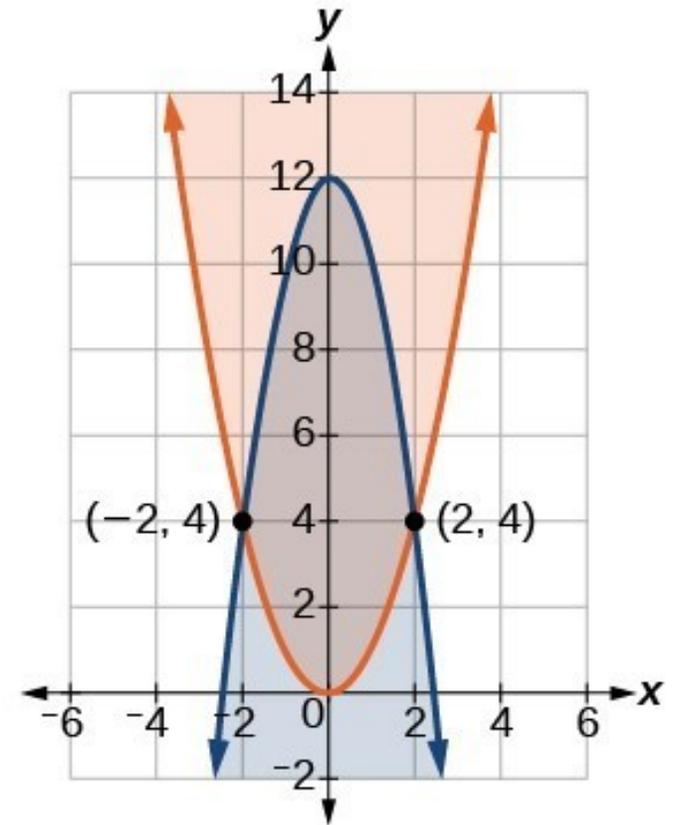
# Graphing a System of Nonlinear Inequalities

**Example:** Graph the given system of inequalities

$$\begin{aligned}x^2 - y &\leq 0 \\ 2x^2 + y &\leq 12\end{aligned}$$

**Solution:**

The feasible region is between the two equations bounded by  $y \leq -2x^2 + 12$  on the top and  $y \geq x^2$  on the bottom.



# Intersection of a Parabola and a Line

There are three possible solution sets for a system of equations involving a parabola and a line.

- No solution. The line will never intersect the parabola.
- One solution. The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions. The line crosses on the inside of the parabola and intersects the parabola at two points.

# Solving a System of Equations Containing a Line and a Parabola

**Given a system of equations containing a line and a parabola, find the solution.**

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the parabola equation.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

# Solving a System of Equations Containing a Line and a Parabola

**Example:** Solve the system of equations.

$$\begin{aligned}x - y &= -1 \\ y &= x^2 + 1\end{aligned}$$

**Solution:**

$$x = y - 1$$

Solve the first equation for  $x$

$$y = (y - 1)^2 + 1$$

Substitute for  $x$  in the second equation

$$y^2 - 3y + 2 = 0$$

Expand and rearrange the polynomial

$$(y - 2)(y - 1) = 0$$

Find the zeros of  $y = 2$  and  $y = 1$

$$x = (2) - 1 = 1$$

Substitute  $y$  into the line for the first solution at (1,2)

$$x = (1) - 1 = 0$$

Use the other value of  $y$  for the second solution at (0,1)

# Desmos Interactive

Topic: explore intersections of a parabola and a line

<https://www.desmos.com/calculator/yq47tetslx>

# Intersection of a Circle and a Line

Just as with a parabola and a line, there are three possible solution sets when solving a system of equations representing a circle and a line.

**Given a system of equations containing a line and a circle, find the solution.**

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the equation for the circle.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

# Find the Intersection of a Circle and a Line by Substitution

## Example:

$$\begin{aligned}x^2 + y^2 &= 5 \\ y &= 3x - 5\end{aligned}$$

## Solution:

$$x^2 + (3x - 5)^2 = 5$$

$$10x^2 - 30x + 20 = 0$$

$$10(x - 2)(x - 1) = 0$$

$$y = 3(2) - 5 = 1$$

$$y = 3(1) - 5 = -2$$

Substitute for  $y$  into the equation of the circle

Expand and rearrange the polynomial

Find the zeros of  $x = 2$  and  $x = 1$

Substitute  $y$  to find the first solution at  $(2,1)$

Find the second solution at  $(1,-2)$

# Systems of Equations in Three Variables

A solution to a system of three equations in three variables  $(x, y, z)$  is called an **ordered triple**.

**Given a linear system of three equations, solve for three unknowns.**

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

# Solving a System of Three Equations in Three Variables by Elimination

**Example:** Find a solution to the following system

$$x - 2y + 3z = 9 \quad (\text{a})$$

$$-x + 3y - z = -6 \quad (\text{b})$$

$$2x - 5y + 5z = 17 \quad (\text{c})$$

**Solution:**

$$y + 2z = 3 \quad (\text{d}) \quad \text{Add equations (a) and (b) to eliminate } x$$

$$-y - z = -1 \quad (\text{e}) \quad \text{Multiply (a) by } -2 \text{ and add to (c) to eliminate } x$$

$$z = 2 \quad (\text{f}) \quad \text{Add equations (d) and (e) to eliminate } y \text{ and solve for } z$$

Back-substitute  $z$  into equation (e) to solve for  $y = -1$  and then  $x = 1$

# Linear Factors

The **partial fraction decomposition** of  $\frac{P(x)}{Q(x)}$  when  $Q(x)$  has nonrepeated linear factors and the degree of  $P(x)$  is less than the degree of  $Q(x)$  is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_2x + b_2)} + \frac{A_3}{(a_3x + b_3)} + \cdots + \frac{A_n}{(a_nx + b_n)}$$

If  $Q(x)$  has a repeated linear factor occurring  $n$  times, the decomposition is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(ax + b)} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n}$$

# Decomposing a Rational Expression

**Given a rational expression with distinct linear factors in the denominator, decompose it.**

1. Use a variable for the original numerators, placing each over a single factor.
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

# Decomposing a Rational Expression

**Example:** decompose the rational expression  $\frac{3x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

$3x = A(x - 1) + B(x + 2)$  Multiply by the common denominator

$3x = (A + B)x - A + 2B$  Expand and collect like terms

$\begin{cases} 3 = A + B \\ 0 = -A + 2B \end{cases}$  System of equations from the coefficients

$B = 1, A = 2$  Solve the system

$\frac{3x}{(x+2)(x-1)} = \frac{2}{x+2} + \frac{1}{x-1}$  Substitute  $A$  and  $B$  for the partial fraction decomposition

# Quadratic Factors

The partial fraction decomposition of  $\frac{P(x)}{Q(x)}$  such that  $Q(x)$  has a nonrepeated irreducible quadratic factor and the degree of  $P(x)$  is less than the degree of  $Q(x)$  is

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{(a_1x^2 + b_1x + c_1)} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)} + \cdots + \frac{A_nx + B_n}{(a_nx^2 + b_nx + c_n)}$$

Using the same method as with linear factors, a system of equations will be used to solve the coefficients for the numerators.

# Quick Review

- What are the two methods for solving a system of two equations in two variables?
- If the solution to a system of equations results in an identity equation, what type of system is it, and how many solutions are there?
- What is the break-even point for a revenue-cost model?
- What is the feasible region in the solution to a system of nonlinear inequalities?
- What are the steps in solving a system of three equations in three variables?
- What is a solution set?
- What is a partial fraction decomposition?