1.5 Order of Operations

The order in which we evaluate expressions can be ambiguous. Take for example, the expression $4 + 3 \cdot 2$. If we do the addition first, then

$$4 + 3 \cdot 2 = 7 \cdot 2$$
$$= 14.$$

On the other hand, if we do the multiplication first, then

$$4 + 3 \cdot 2 = 4 + 6$$

= 10.

So, what are we to do?

Of course, grouping symbols can remove the ambiguity.

Grouping Symbols. Parentheses, brackets, or curly braces can be used to group parts of an expression. Each of the following are equivalent:

$$(4+3) \cdot 2$$
 or $[4+3] \cdot 2$ or $\{4+3\} \cdot 2$

In each case, the rule is "evaluate the expression inside the grouping symbols first." If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

Thus, for example,

$$(4+3) \cdot 2 = 7 \cdot 2$$
$$= 14.$$

Note how the expression contained in the parentheses was evaluated first.

Another way to avoid ambiguities in evaluating expressions is to establish an order in which operations should be performed. The following guidelines should always be strictly enforced when evaluating expressions.

Rules Guiding Order of Operations. When evaluating expressions, proceed in the following order.

- 1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
- 2. Evaluate all exponents that appear in the expression.
- 3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.

4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

You Try It!

Simplify: $8 + 2 \cdot 5$.



EXAMPLE 1. Evaluate $4 + 3 \cdot 2$.

Solution. Because of the established *Rules Guiding Order of Operations*, this expression is no longer ambiguous. There are no grouping symbols or exponents, so we immediately go to rule three, evaluate all multiplications and divisions in the order that they appear, moving left to right. After that we invoke rule four, performing all additions and subtractions in the order that they appear, moving left to right.

$$4 + 3 \cdot 2 = 4 + 6$$

= 10

Thus, $4 + 3 \cdot 2 = 10$.

You Try It!

Simplify: 17 - 8 + 2.

Answer: 18



EXAMPLE 2. Evaluate 18 - 2 + 3.

Solution. Follow the *Rules Guiding Order of Operations*. Addition has no precedence over subtraction, nor does subtraction have precedence over addition. We are to perform additions and subtractions as they occur, moving left to right.

$$18-2+3=16+3$$
 Subtract: $18-2=16$.
= 19 Add: $16+3=19$.

Thus, 18 - 2 + 3 = 19.

Answer: 11



EXAMPLE 3. Evaluate $54 \div 9 \cdot 2$.

Solution. Follow the *Rules Guiding Order of Operations*. Division has no precedence over multiplication, nor does multiplication have precedence over division. We are to perform divisions and multiplications as they occur, moving

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You Try It!

Simplify: $72 \div 9 \cdot 2$.

left to right.

$$54 \div 9 \cdot 2 = 6 \cdot 2$$
 Divide: $54 \div 9 = 6$.
= 12 Multiply: $6 \cdot 2 = 12$.

Answer: 16

Thus, $54 \div 9 \cdot 2 = 12$.

You Try It!

Simplify: $14 + 3 \cdot 4^2$. **EXAMPLE 4.** Evaluate $2 \cdot 3^2 - 12$.

 $\textbf{Solution.} \ \ \textbf{Follow the} \ \ \textit{Rules} \ \ \textit{Guiding} \ \ \textit{Order of Operations}, \ \text{exponents first, then} \\ \text{multiplication, then subtraction.}$



$$2 \cdot 3^2 - 12 = 2 \cdot 9 - 12$$
 Evaluate the exponent: $3^2 = 9$.
 $= 18 - 12$ Perform the multiplication: $2 \cdot 9 = 18$.
 $= 6$ Perform the subtraction: $18 - 12 = 6$.

Answer: 62

Thus, $2 \cdot 3^2 - 12 = 6$.

You Try It!

Simplify: $3(2+3\cdot4)^2 - 11$. **EXAMPLE 5.** Evaluate $12 + 2(3+2\cdot5)^2$.

Solution. Follow the Rules Guiding Order of Operations, evaluate the expression inside the parentheses first, then exponents, then multiplication, then addition.



$$12 + 2(3 + 2 \cdot 5)^2 = 12 + 2(3 + 10)^2$$
 Multiply inside parentheses: $2 \cdot 5 = 10$.
 $= 12 + 2(13)^2$ Add inside parentheses: $3 + 10 = 13$.
 $= 12 + 2(169)$ Exponents are next: $(13)^2 = 169$.
 $= 12 + 338$ Multiplication is next: $2(169) = 338$.
 $= 350$ Time to add: $12 + 338 = 350$.

Answer: 577 Thus, $12 + 2(3 + 2 \cdot 5)^2 = 350$.

You Try It!

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EXAMPLE 6. Evaluate $2\{2 + 2[2 + 2]\}$.

Simplify: $2{3+2[3+2]}$.

Solution. When grouping symbols are nested, evaluate the expression between the pair of innermost grouping symbols first.

$$2\{2+2[2+2]\} = 2\{2+2[4]\}$$
 Innermost grouping first: $2+2=4$.
 $= 2\{2+8\}$ Multiply next: $2[4] = 8$.
 $= 2\{10\}$ Add inside braces: $2+8=10$.
 $= 20$ Multiply: $2\{10\} = 20$

Thus, $2\{2+2[2+2]\}=20$. Answer: 26

Fraction Bars

Consider the expression

$$\frac{6^2 + 8^2}{(2+3)^2}.$$

Because a fraction bar means division, the above expression is equivalent to

$$(6^2 + 8^2) \div (2+3)^2$$
.

The position of the grouping symbols signals how we should proceed. We should simplify the numerator, then the denominator, then divide.

Fractional Expressions. If a fractional expression is present, evaluate the numerator and denominator first, then divide.

You Try It!

Simplify: $\frac{12+3\cdot 2}{6}$.

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EXAMPLE 7. Evaluate the expression

$$\frac{6^2 + 8^2}{(2+3)^2}.$$

Solution. Simplify the numerator and denominator first, then divide.

$$\frac{6^2 + 8^2}{(2+3)^2} = \frac{6^2 + 8^2}{(5)^2}$$
 Parentheses in denominator first: $2 + 3 = 5$.
$$= \frac{36 + 64}{25}$$
 Exponents are next: $6^2 = 36$, $8^2 = 64$, $5^2 = 25$.
$$= \frac{100}{25}$$
 Add in numerator: $36 + 64 = 100$.
$$= 4$$
 Divide: $100 \div 25 = 4$.

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Thus,
$$\frac{6^2 + 8^2}{(2+3)^2} = 4$$
.

The Distributive Property

Consider the expression $2 \cdot (3+4)$. If we follow the "Rules Guiding Order of Operations," we would evaluate the expression inside the parentheses first.

$$2 \cdot (3+4) = 2 \cdot 7$$
 Parentheses first: $3+4=7$.
= 14 Multiply: $2 \cdot 7 = 14$.

However, we could also choose to "distribute" the 2, first multiplying 2 times each addend in the parentheses.

$$2 \cdot (3+4) = 2 \cdot 3 + 2 \cdot 4$$
 Multiply 2 times both 3 and 4.
 $= 6+8$ Multiply: $2 \cdot 3 = 6$ and $2 \cdot 4 = 8$.
 $= 14$ Add: $6+8=14$.

The fact that we get the same answer in the second approach is an illustration of an important property of whole numbers.¹

The Distributive Property. Let a, b, and c be any whole numbers. Then,

$$a \cdot (b+c) = a \cdot b + a \cdot c.$$

We say that "multiplication is distributive with respect to addition."

Multiplication is distributive with respect to addition. If you are not computing the product of a number and a sum of numbers, the distributive property does not apply.

Caution! Wrong Answer Ahead! If you are calculating the product of a number and the product of two numbers, the distributive property must not be used. For example, here is a *common misapplication* of the distributive property.

$$2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot (2 \cdot 4)$$

= $6 \cdot 8$
= 48

¹Later, we'll see that this property applies to all numbers, not just whole numbers.

This result is quite distant from the correct answer, which is found by computing the product within the parentheses first.

$$2 \cdot (3 \cdot 4) = 2 \cdot 12$$
$$= 24.$$

In order to apply the distributive property, you must be multiplying times a sum.

You Try It!

Adada Movie **EXAMPLE 8.** Use the distributive property to calculate $4 \cdot (5 + 11)$.

Distribute: $5 \cdot (11 + 8)$.

Solution. This is the product of a number and a sum, so the distributive property may be applied.

$$4 \cdot (5+11) = 4 \cdot 5 + 4 \cdot 11$$
 Distribute the 4 times each addend in the sum.
= $20 + 44$ Multiply: $4 \cdot 5 = 20$ and $4 \cdot 11 = 44$.
= 64 Add: $20 + 44 = 64$.

Readers should check that the same answer is found by computing the sum within the parentheses first.

Answer: 95

The distributive property is the underpinning of the multiplication algorithm learned in our childhood years.

You Try It!

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EXAMPLE 9. Multiply: $6 \cdot 43$.

Use the distributive property to evaluate $8 \cdot 92$.

$$6 \cdot 43 = 6 \cdot (40 + 3)$$
 Express 43 as a sum: $43 = 40 + 3$.
 $= 6 \cdot 40 + 6 \cdot 3$ Distribute the 6.
 $= 240 + 18$ Multiply: $6 \cdot 40 = 240$ and $6 \cdot 3 = 18$.
 $= 258$ Add: $240 + 18 = 258$.

Solution. We'll express 43 as sum, then use the distributive property.

Readers should be able to see this application of the distributive property in the more familiar algorithmic form:

$$\begin{array}{r}
 43 \\
 \times 6 \\
 \hline
 18 \\
 240 \\
 \hline
 258
 \end{array}$$

Or in the even more condensed form with "carrying:"

 $^{1}43$ $\times 6$

 $\frac{\times 6}{258}$

Answer: 736

Multiplication is also distributive with respect to subtraction.

The Distributive Property (Subtraction). Let a, b, and c be any whole numbers. Then,

$$a \cdot (b - c) = a \cdot b - a \cdot c.$$

We say the multiplication is "distributive with respect to subtraction."

You Try It!

Distribute: $8 \cdot (9-2)$. **EXAMPLE 10.** Use the distributive property to simplify: $3 \cdot (12-8)$.

property may be applied.

Solution. This is the product of a number and a difference, so the distributive

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 $3 \cdot (12 - 8) = 3 \cdot 12 - 3 \cdot 8$ Distribute the 3 times each term in the difference. = 36 - 24 Multiply: $3 \cdot 12 = 36$ and $3 \cdot 8 = 24$. = 12 Subtract: 36 - 24 = 12.

Alternate solution. Note what happens if we use the usual "order of operations" to evaluate the expression.

 $3 \cdot (12 - 8) = 3 \cdot 4$ Parentheses first: 12 - 8 = 4. = 12 Multiply: $3 \cdot 4 = 12$.

Answer: 56 Same answer.

Exercises & &

In Exercises 1-12, simplify the given expression.

1.
$$5 + 2 \cdot 2$$

2.
$$5 + 2 \cdot 8$$

3.
$$23 - 7 \cdot 2$$

4.
$$37 - 3 \cdot 7$$

5.
$$4 \cdot 3 + 2 \cdot 5$$

6.
$$2 \cdot 5 + 9 \cdot 7$$

7.
$$6 \cdot 5 + 4 \cdot 3$$

8.
$$5 \cdot 2 + 9 \cdot 8$$

9.
$$9 + 2 \cdot 3$$

10.
$$3 + 6 \cdot 6$$

11.
$$32 - 8 \cdot 2$$

12.
$$24 - 2 \cdot 5$$

In Exercises 13-28, simplify the given expression.

13.
$$45 \div 3 \cdot 5$$

14.
$$20 \div 1 \cdot 4$$

15.
$$2 \cdot 9 \div 3 \cdot 18$$

16.
$$19 \cdot 20 \div 4 \cdot 16$$

17.
$$30 \div 2 \cdot 3$$

18.
$$27 \div 3 \cdot 3$$

19.
$$8 - 6 + 1$$

20.
$$15 - 5 + 10$$

21.
$$14 \cdot 16 \div 16 \cdot 19$$

22.
$$20 \cdot 17 \div 17 \cdot 14$$

23.
$$15 \cdot 17 + 10 \div 10 - 12 \cdot 4$$

24.
$$14 \cdot 18 + 9 \div 3 - 7 \cdot 13$$

25.
$$22 - 10 + 7$$

26.
$$29 - 11 + 1$$

27.
$$20 \cdot 10 + 15 \div 5 - 7 \cdot 6$$

28.
$$18 \cdot 19 + 18 \div 18 - 6 \cdot 7$$

In Exercises 29-40, simplify the given expression.

29.
$$9 + 8 \div \{4 + 4\}$$

30.
$$10 + 20 \div \{2 + 2\}$$

31.
$$7 \cdot [8 - 5] - 10$$

32.
$$11 \cdot [12 - 4] - 10$$

33.
$$(18+10) \div (2+2)$$

34.
$$(14+7) \div (2+5)$$

35.
$$9 \cdot (10+7) - 3 \cdot (4+10)$$

36.
$$9 \cdot (7+7) - 8 \cdot (3+8)$$

37.
$$2 \cdot \{8 + 12\} \div 4$$

38.
$$4 \cdot \{8+7\} \div 3$$

39.
$$9 + 6 \cdot (12 + 3)$$

40.
$$3 + 5 \cdot (10 + 12)$$

In Exercises 41-56, simplify the given expression.

41.
$$2 + 9 \cdot [7 + 3 \cdot (9 + 5)]$$

42.
$$6 + 3 \cdot [4 + 4 \cdot (5 + 8)]$$

43.
$$7 + 3 \cdot [8 + 8 \cdot (5 + 9)]$$

44.
$$4 + 9 \cdot [7 + 6 \cdot (3 + 3)]$$

45.
$$6 - 5[11 - (2 + 8)]$$

46.
$$15 - 1[19 - (7 + 3)]$$

47.
$$11 - 1[19 - (2 + 15)]$$

48.
$$9 - 8[6 - (2 + 3)]$$

49.
$$4\{7[9+3]-2[3+2]\}$$

50.
$$4\{8[3+9]-4[6+2]\}$$

51.
$$9 \cdot [3 + 4 \cdot (5 + 2)]$$

52.
$$3 \cdot [4 + 9 \cdot (8 + 5)]$$

53.
$$3\{8[6+5] - 8[7+3]\}$$

54.
$$2\{4[6+9]-2[3+4]\}$$

55.
$$3 \cdot [2 + 4 \cdot (9 + 6)]$$

56.
$$8 \cdot [3 + 9 \cdot (5 + 2)]$$

In Exercises 57-68, simplify the given expression.

57.
$$(5-2)^2$$

58.
$$(5-3)^4$$

59.
$$(4+2)^2$$

60.
$$(3+5)^2$$

61.
$$2^3 + 3^3$$

62.
$$5^4 + 2^4$$

63.
$$2^3 - 1^3$$

64.
$$3^2 - 1^2$$

65.
$$12 \cdot 5^2 + 8 \cdot 9 + 4$$

66.
$$6 \cdot 3^2 + 7 \cdot 5 + 12$$

67.
$$9 - 3 \cdot 2 + 12 \cdot 10^2$$

68.
$$11 - 2 \cdot 3 + 12 \cdot 4^2$$

In Exercises 69-80, simplify the given expression.

69.
$$4^2 - (13 + 2)$$

70.
$$3^3 - (7+6)$$

71.
$$3^3 - (7 + 12)$$

72.
$$4^3 - (6+5)$$

73.
$$19 + 3[12 - (2^3 + 1)]$$

74.
$$13 + 12[14 - (2^2 + 1)]$$

75.
$$17 + 7[13 - (2^2 + 6)]$$

76.
$$10 + 1[16 - (2^2 + 9)]$$

77.
$$4^3 - (12 + 1)$$

78.
$$5^3 - (17 + 15)$$

79.
$$5 + 7[11 - (2^2 + 1)]$$

80.
$$10 + 11[20 - (2^2 + 1)]$$

In Exercises 81-92, simplify the given expression.

81.
$$\frac{13+35}{3(4)}$$

82.
$$\frac{35+28}{7(3)}$$

83.
$$\frac{64 - (8 \cdot 6 - 3)}{4 \cdot 7 - 9}$$

84.
$$\frac{19 - (4 \cdot 3 - 2)}{6 \cdot 3 - 9}$$

85.
$$\frac{2+13}{4-1}$$

86.
$$\frac{7+1}{8-4}$$

87.
$$\frac{17+14}{9-8}$$

88.
$$\frac{16+2}{13-11}$$

89.
$$\frac{37+27}{8(2)}$$

90.
$$\frac{16+38}{6(3)}$$

91.
$$\frac{40 - (3 \cdot 7 - 9)}{8 \cdot 2 - 2}$$

92.
$$\frac{60 - (8 \cdot 6 - 3)}{5 \cdot 4 - 5}$$

In Exercises 93-100, use the distributive property to evaluate the given expression.

93.
$$5 \cdot (8+4)$$

97.
$$6 \cdot (7-2)$$

94.
$$8 \cdot (4+2)$$

98.
$$4 \cdot (8-6)$$

95.
$$7 \cdot (8-3)$$

99.
$$4 \cdot (3+2)$$

96.
$$8 \cdot (9-7)$$

100.
$$4 \cdot (9+6)$$

In Exercises 101-104, use the distributive property to evaluate the given expression using the technique shown in Example 9.

104.
$$7 \cdot 57$$

	Answers & &	
1. 9	7. 42	
3. 9	9. 15	
5. 22	11. 16	

13. 75	59. 36
15. 108	61. 35
17. 45	63. 7
19. 3	65. 376
21. 266	67. 1203
23. 208	69. 1
25. 19	71. 8
27. 161	73. 28
29. 10	75. 38
31. 11	77. 51
33. 7	79. 47
35. 111	81. 4
37. 10	83. 1
39. 99	85. 5
41. 443	87. 31
43. 367	89. 4
45. 1	91. 2
47. 9	93. 60
49. 296	95. 35
51. 279	97. 30
53. 24	99. 20
55. 186	101. 558
57. 9	103. 174