2.4 Multiplication and Division of Integers

Before we begin, let it be known that the integers satisfy the same properties of multiplication as do the whole numbers.

Integer Properties of Multiplication.

Commutative Property. If a and b are integers, then their product commutes. That is,

$$a \cdot b = b \cdot a$$
.

Associative Property. If a, b, and c are integers, then their product is associative. That is,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Multiplicative Identity Property. If a is any integer, then

$$a \cdot 1 = a$$
 and $1 \cdot a = a$.

Because multiplying any integer by 1 returns the identical integer, the integer 1 is called the **multiplicative identity**.

In Section 1.3, we learned that multiplication is equivalent to *repeated addition*. For example,

$$3 \cdot 4 = \underbrace{4 + 4 + 4}_{\text{three fours}}.$$

On the number line, three sets of four is equivalent to walking three sets of four units to the right, starting from zero, as shown in Figure 2.14.

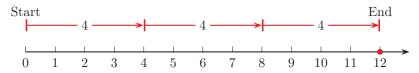


Figure 2.14: Note that $3 \cdot 4 = 4 + 4 + 4$. That is, $3 \cdot 4 = 12$.

This example and a little thought should convince readers that the product of two positive integers will always be a positive integer.

The Product of Two Positive Integers. If a and b are two positive integers, then their product ab is also a positive integer.

For example, $2 \cdot 3 = 6$ and $13 \cdot 117 = 1521$. In each case, the product of two positive integers is a positive integer.

The Product of a Positive Integer and a Negative Integer

If we continue with the idea that multiplication is equivalent to repeated addition, then it must be that

$$3 \cdot (-4) = \underbrace{-4 + (-4) + (-4)}_{\text{three negative fours}}.$$

Pictured on the number line, $3 \cdot (-4)$ would then be equivalent to walking three sets of negative four units (to the left), starting from zero, as shown in Figure 2.15.



Figure 2.15: Note that $3 \cdot (-4) = -4 + (-4) + (-4)$. That is, $3 \cdot (-4) = -12$.

Note, at least in this particular case, that the product of a positive integer and a negative integer is a negative integer.

We've shown that $3 \cdot (-4) = -12$. However, integer multiplication is commutative, so it must also be true that $-4 \cdot 3 = -12$. That is, the product of a negative integer and a positive integer is also a negative integer. Although not a proof, this argument motivates the following fact about integer multiplication.

The Product of a Positive Integer and a Negative Integer. Two facts are true:

- 1. If a is a positive integer and b is a negative integer, then the product ab is a negative integer.
- 2. If a is a negative integer and b is a positive integer, then the product ab is a negative integer.

Thus, for example, $5 \cdot (-12) = -60$ and $-13 \cdot 2 = -26$. In each case the answer is negative because we are taking a product where one of the factors is positive and the other is negative.

The Distributive Property

The integers satisfy the distributive property.

The Distributive Property. Let a, b, and c be integers. Then,

$$a \cdot (b+c) = a \cdot b + a \cdot c.$$

We say that "multiplication is distributive with respect to addition."

Note how the a is "distributed." The a is multiplied times each term in the parentheses.

For example, consider the expression $3 \cdot (4+5)$. We can evaluate this expression according to the order of operations, simplifying the expression inside the parentheses first.

$$3 \cdot (4+5) = 3 \cdot 9 \tag{2.1}$$

$$=27 \tag{2.2}$$

But we can also use the distributive property, multiplying each term inside the parentheses by three, then simplifying the result.

$$3 \cdot (4+5) = 3 \cdot 4 + 3 \cdot 5$$
 Distribute the 3.
 $= 12+15$ Perform multiplications first:
 $3 \cdot 4 = 12$ and $3 \cdot 5 = 15$.
 $= 27$ Add: $12+15=27$.

Note that evaluating $3 \cdot (4+5)$ using the distributive property provides the same result as the evaluation (2.1) using the order of operations.

The Multiplicative Property of Zero

The distributive property can be used to provide proofs of a number of important properties of integers. One important property is the fact that if you multiply an integer by zero, the product is zero. Here is a proof of that fact that uses the distributive property.

Let a be any integer. Then,

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a \cdot 0 = a \cdot (0+0) Additive Identity Property: 0+0=0.

a \cdot 0 = a \cdot 0 + a \cdot 0 Distribute a times each zero in the parentheses.
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Next, to "undo" the effect of adding $a \cdot 0$, subtract $a \cdot 0$ from both sides of the equation.

$$a \cdot 0 - a \cdot 0 = a \cdot 0 + a \cdot 0 - a \cdot 0$$
 Subtract $a \cdot 0$ from both sides.
 $0 = a \cdot 0$ $a \cdot 0 - a \cdot 0 = 0$ on each side.

Multiplicative Property of Zero. Let a represent any integer. Then

$$a \cdot 0 = 0$$
 and $0 \cdot a = 0$.

Thus, for example, $-18 \cdot 0 = 0$ and $0 \cdot 122 = 0$.

Multiplying by Minus One

Here is another useful application of the distributive property.

$$(-1)a + a = (-1)a + 1a$$
 Replace a with $1a$.
 $= (-1+1)a$ Use the distributive property to factor out a .
 $= 0a$ Replace $-1+1$ with 0 .
 $= 0$ Replace $0a$ with 0 .

Thus, (-1)a + a = 0. That is, if you add (-1)a to a you get zero. However, the Additive Inverse Property says that -a is the *unique* number that you add to a to get zero. The conclusion must be that (-1)a = -a.

Multiplying by Minus One. If a is any integer, then

$$(-1)a = -a$$
.

Thus, for example, -1(4) = -4 and -1(-4) = -(-4) = 4.

This property is rather important, as we will see in future work. Not only does it tell us that (-1)a = -a, but it also tells us that if we see -a, then it can be interpreted to mean (-1)a.

The Product of Two Negative Integers

We can employ the multiplicative property of -1, that is, (-1)a = -a to find the product of two negative numbers.

$$(-4)(-3) = [(-1)(4)](-3) \qquad \text{Replace } -4 \text{ with } (-1)(4).$$

$$= (-1)[(4)(-3)] \qquad \text{Use the associative property to regroup.}$$

$$= (-1)(-12) \qquad \text{We know: } (4)(-3) = -12.$$

$$= -(-12) \qquad (-1)a = -a. \text{ Here } (-1)(-12) = -(-12).$$

$$= 12 \qquad -(-a) = a. \text{ Here } -(-12) = 12.$$

Thus, at least in the case of (-4)(-3), the product of two negative integers is a positive integer. This is true in general.

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The Product of Two Negative Integers. If both a and b are negative integers, then their product ab is a positive integer.

Thus, for example, (-5)(-7) = 35 and (-112)(-6) = 672. In each case the answer is positive, because the product of two negative integers is a positive integer.

Memory Device

Here's a simple memory device to help remember the rules for finding the product of two integers.

Like and Unlike Signs. There are two cases:

Unlike Signs. The product of two integers with *unlike* signs is negative. That is:

$$(+)(-) = -$$

 $(-)(+) = -$

Like Signs. The product of two integers with *like* signs is positive. That is:

$$(+)(+) = +$$

 $(-)(-) = +$

You Try It!

EXAMPLE 1. Simplify: (a) (-3)(-2), (b) (4)(-10), and (c) (12)(-3).

Solution. In each example, we use the "like" and "unlike" signs approach.

Simplify: (a) (-12)(4) and (b) (-3)(-11).

- a) Like signs gives a positive result. Hence, (-3)(-2) = 6.
- b) Unlike signs gives a negative result. Hence, (4)(-10) = -40.
- c) Unlike signs gives a negative result. Hence, (12)(-3) = -36.

Answer: (a) -48, (b) 33

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You Try It!

Simplify: (-2)(-3)(4)(-1). **EXAMPLE 2.** Simplify (-3)(2)(-4)(-2).

Solution. Order of operations demands that we work from left to right.

$$(-3)(2)(-4)(-2) = (-6)(-4)(-2)$$
 Work left to right: $(-3)(2) = -6$.
 $= (24)(-2)$ Work left to right: $(-6)(-4) = 24$.
 $= -48$ Multiply: $(24)(-2) = -48$.

Answer: -24. Hence, (-3)(2)(-4)(-2) = -48.

You Try It!

Simplify: (a) $(-2)^2$ and (b) **EXAMPLE 3.** Simplify: (a) $(-2)^3$ and (c) $(-3)^4$.

Solution. In each example, use

$$a^m = \underbrace{a \cdot a \cdot a \cdot \dots \cdot a}_{m \text{ times}},$$

then work left to right with the multiplication.

a) Use the definition of an exponent, then order of operations.

$$(-2)^3 = (-2)(-2)(-2)$$
 Write -2 as a factor three times.
 $= 4(-2)$ Work left to right: $(-2)(-2) = 4$.
 $= -8$

b) Use the definition of an exponent, then order of operations.

$$(-3)^4 = (-3)(-3)(-3)(-3)$$
 Write -3 as a factor four times.
 $= 9(-3)(-3)$ Work left to right: $(-3)(-3) = 9$.
 $= -27(-3)$ Work left to right: $9(-3) = -27$.
 $= 81$

Answer: (a) 4 and (b) -4.

Example 3 motivates the following fact.

Even and Odd. Two facts are apparent.

- 1. If a product contains an odd number of negative factors, then the product is negative.
- 2. If a product contains an even number of negative factors, then the product is positive.

Thus, for example,

$$(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$$

quickly evaluates as -32 as it has an odd number of negative factors. On the other hand,

$$(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = 64$$

quickly evaluates as 64 as it has an even number of negative factors.

Division of Integers

Consider that

$$\frac{12}{3} = 4$$
 because $3(4) = 12$ and $\frac{-12}{-3} = 4$ because $-3(4) = -12$.

In like manner,

$$\frac{12}{-3} = -4$$
 because $-3(-4) = 12$ and $\frac{-12}{3} = -4$ because $3(-4) = -12$.

Thus, the rules for dividing integers are the same as the rules for multiplying integers.

Like and Unlike Signs. There are two cases:

Unlike Signs. The quotient of two integers with *unlike* signs is negative. That is,

$$\frac{(+)}{(-)} = -$$

$$\frac{(-)}{(+)} = -$$

Like Signs. The quotient of two integers with *like* signs is positive. That is,

$$\frac{(+)}{(+)} = +$$

$$\frac{(-)}{(-)} = +$$

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Thus, for example, 12/(-6) = -2 and -44/(-4) = 11. In the first case, unlike signs gives a negative quotient. In the second case, like signs gives a positive quotient.

One final reminder.

Division by Zero is Undefined. If a is any integer, the quotient

 $\frac{a}{0}$

is undefined. Division by zero is meaningless.

See the discussion in Section 1.3 for a discussion on division by zero.

You Try It!

Simplify: (a) -24/4 and (b) -28/(-7).

EXAMPLE 4. Simplify: (a) -12/(-4), (b) 6/(-3), and (c) -15/0.

Solution. In each example, we use the "like" and "unlike" signs approach.

a) Like signs gives a positive result. Hence,

$$\frac{-12}{-4} = 3.$$

b) Unlike signs gives a negative result. Hence,

$$\frac{6}{-3} = -2.$$

c) Division by zero is undefined. Hence,

$$\frac{-15}{0}$$

Answer: (a) -6, (b) 4

is undefined.

Exercises & &

In Exercises 1-16, state the property of multiplication depicted by the given identity.

1.
$$(-2)[(-16)(13)] = [(-2)(-16)](13)$$

2.
$$(10)[(-15)(-6)] = [(10)(-15)](-6)$$

3.
$$(-17)(-10) = (-10)(-17)$$

4.
$$(-5)(3) = (3)(-5)$$

5.
$$(4)(11) = (11)(4)$$

6.
$$(-5)(-11) = (-11)(-5)$$

7.
$$16 \cdot (8 + (-15)) = 16 \cdot 8 + 16 \cdot (-15)$$

8.
$$1 \cdot (-16 + (-6)) = 1 \cdot (-16) + 1 \cdot (-6)$$

9.
$$(17)[(20)(11)] = [(17)(20)](11)$$

10.
$$(14)[(-20)(-18)] = [(14)(-20)](-18)$$

11.
$$-19 \cdot 1 = -19$$

12.
$$-17 \cdot 1 = -17$$

13.
$$8 \cdot 1 = 8$$

14.
$$-20 \cdot 1 = -20$$

15.
$$14 \cdot (-12 + 7) = 14 \cdot (-12) + 14 \cdot 7$$

16.
$$-14 \cdot (-3+6) = -14 \cdot (-3) + (-14) \cdot 6$$

In Exercises 17-36, simplify each given expression.

19.
$$3 \cdot (-3)$$

20.
$$7 \cdot (-9)$$

21.
$$-1 \cdot 10$$

22.
$$-1 \cdot 11$$

23.
$$-1 \cdot 0$$

24.
$$-8 \cdot 0$$

25.
$$-1 \cdot (-14)$$

26.
$$-1 \cdot (-13)$$

27.
$$-1 \cdot (-19)$$

28.
$$-1 \cdot (-17)$$

30.
$$-6 \cdot 0$$

31.
$$-3 \cdot 8$$

32.
$$7 \cdot (-3)$$

35.
$$-1 \cdot 5$$

36.
$$-1 \cdot 2$$

In Exercises 37-48, simplify each given expression.

37.
$$(-7)(-1)(3)$$

39.
$$(-7)(9)(10)(-10)$$

40.
$$(-8)(-5)(7)(-9)$$

42.
$$(7)(-1)(-9)$$

43.
$$(-10)(4)(-3)(8)$$

44.
$$(8)(-2)(-5)(2)$$

45.
$$(6)(-3)(-8)$$

46.
$$(-5)(-4)(1)$$

In Exercises 49-60, compute the exact value.

49.
$$(-4)^4$$

50.
$$(-3)^4$$

51.
$$(-5)^4$$

52.
$$(-2)^2$$

53.
$$(-5)^2$$

54.
$$(-3)^3$$

55.
$$(-6)^2$$

56.
$$(-6)^4$$

57.
$$(-4)^5$$

58.
$$(-4)^2$$

59.
$$(-5)^3$$

60.
$$(-3)^2$$

In Exercises 61-84, simplify each given expression.

61.
$$-16 \div (-8)$$

62.
$$-33 \div (-3)$$

63.
$$\frac{-8}{1}$$

64.
$$\frac{40}{-20}$$

65.
$$\frac{-1}{0}$$

66.
$$\frac{2}{0}$$

67.
$$-3 \div 3$$

68.
$$-58 \div 29$$

69.
$$\frac{56}{-28}$$

70.
$$\frac{60}{-12}$$

71.
$$0 \div 15$$

72.
$$0 \div (-4)$$

73.
$$\frac{63}{21}$$

74.
$$\frac{-6}{-1}$$

75.
$$\frac{78}{13}$$

76.
$$\frac{-84}{-14}$$

77.
$$0 \div 5$$

78.
$$0 \div (-16)$$

79.
$$\frac{17}{0}$$

80.
$$\frac{-20}{0}$$

81.
$$-45 \div 15$$

82.
$$-28 \div 28$$

83.
$$12 \div 3$$

84.
$$-22 \div (-22)$$

- **85**. **Scuba.** A diver goes down 25 feet, and then dives down 5 times further. Write the final depth of the diver as an integer.
- **86. Investing Loss.** An investing club of five friends has lost \$4400 on a trade. If they share the loss equally, write each members' loss as an integer.

a a a An	iswers 🐠 🐠
1. Associative property of multiplication	45. 144
3. Commutative property of multiplication	47. 24
5. Commutative property of multiplication	49. 256
7. Distributive property	51. 625
9. Associative property of multiplication	53. 25
11. Multiplicative identity property	55. 36
13. Multiplicative identity property	57. -1024
15. Distributive property	59. -125
17. 28	61. 2
19. –9	63. -8
21. –10	65. Division by zero is undefined.
23. 0	67. –1
25. 14	69. –2
27. 19	71. 0
29. 0	73. 3
31. -24	75. 6
33. 63	77. 0
35. –5	79. Division by zero is undefined.
37. 21	813
39. 6300	83. 4
41. 240	85125 feet
43. 960	⊙ 0. −120 teet