2.4 Multiplication and Division of Integers

Before we begin, let it be known that the integers satisfy the same properties of multiplication as do the whole numbers.

**Integer Properties of Multiplication.**

**Commutative Property.** If $a$ and $b$ are integers, then their product commutes. That is,

$$a \cdot b = b \cdot a.$$

**Associative Property.** If $a$, $b$, and $c$ are integers, then their product is associative. That is,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

**Multiplicative Identity Property.** If $a$ is any integer, then

$$a \cdot 1 = a \quad \text{and} \quad 1 \cdot a = a.$$

Because multiplying any integer by 1 returns the identical integer, the integer 1 is called the **multiplicative identity**.

In Section 1.3, we learned that multiplication is equivalent to repeated addition. For example,

$$3 \cdot 4 = 4 + 4 + 4.$$  

On the number line, three sets of four is equivalent to walking three sets of four units to the right, starting from zero, as shown in Figure 2.14.

![Figure 2.14: Note that $3 \cdot 4 = 4 + 4 + 4$. That is, $3 \cdot 4 = 12$.](image)

This example and a little thought should convince readers that the product of two positive integers will always be a positive integer.

**The Product of Two Positive Integers.** If $a$ and $b$ are two positive integers, then their product $ab$ is also a positive integer.

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For example, $2 \cdot 3 = 6$ and $13 \cdot 117 = 1521$. In each case, the product of two positive integers is a positive integer.

**The Product of a Positive Integer and a Negative Integer**

If we continue with the idea that multiplication is equivalent to repeated addition, then it must be that

$$3 \cdot (-4) = -4 + (-4) + (-4).$$

Pictured on the number line, $3 \cdot (-4)$ would then be equivalent to walking three sets of negative four units (to the left), starting from zero, as shown in Figure 2.15.

![Number Line](image)

Figure 2.15: Note that $3 \cdot (-4) = -4 + (-4) + (-4)$. That is, $3 \cdot (-4) = -12$.

Note, at least in this particular case, that the product of a positive integer and a negative integer is a negative integer.

We’ve shown that $3 \cdot (-4) = -12$. However, integer multiplication is commutative, so it must also be true that $-4 \cdot 3 = -12$. That is, the product of a negative integer and a positive integer is also a negative integer. Although not a proof, this argument motivates the following fact about integer multiplication.

The Product of a Positive Integer and a Negative Integer. Two facts are true:

1. If $a$ is a positive integer and $b$ is a negative integer, then the product $ab$ is a negative integer.
2. If $a$ is a negative integer and $b$ is a positive integer, then the product $ab$ is a negative integer.

Thus, for example, $5 \cdot (-12) = -60$ and $-13 \cdot 2 = -26$. In each case the answer is negative because we are taking a product where one of the factors is positive and the other is negative.

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2.4. MULTIPLICATION AND DIVISION OF INTEGERS

The Distributive Property

The integers satisfy the *distributive property*.

**The Distributive Property.** Let \( a, b, \) and \( c \) be integers. Then,

\[
a \cdot (b + c) = a \cdot b + a \cdot c.
\]

We say that “multiplication is distributive with respect to addition.”

Note how the \( a \) is “distributed.” The \( a \) is multiplied times each term in the parentheses.

For example, consider the expression \( 3 \cdot (4 + 5) \). We can evaluate this expression according to the order of operations, simplifying the expression inside the parentheses first.

\[
3 \cdot (4 + 5) = 3 \cdot 9 \tag{2.1}
\]

\[
= 27 \tag{2.2}
\]

But we can also use the distributive property, multiplying each term inside the parentheses by three, then simplifying the result.

\[
3 \cdot (4 + 5) = 3 \cdot 4 + 3 \cdot 5 \quad \text{Distribute the 3.}
\]

\[
= 12 + 15 \quad \text{Perform multiplications first:}
\]

\[
3 \cdot 4 = 12 \quad \text{and} \quad 3 \cdot 5 = 15.
\]

\[
= 27 \quad \text{Add:} \quad 12 + 15 = 27.
\]

Note that evaluating \( 3 \cdot (4 + 5) \) using the distributive property provides the same result as the evaluation (2.1) using the order of operations.

**The Multiplicative Property of Zero**

The distributive property can be used to provide proofs of a number of important properties of integers. One important property is the fact that if you multiply an integer by zero, the product is zero. Here is a proof of that fact that uses the distributive property.

Let \( a \) be any integer. Then,

\[
a \cdot 0 = a \cdot (0 + 0) \quad \text{Additive Identity Property:} \quad 0 + 0 = 0.
\]

\[
a \cdot 0 = a \cdot 0 + a \cdot 0 \quad \text{Distribute} \ a \text{ times each zero in the parentheses.}
\]

Next, to “undo” the effect of adding \( a \cdot 0 \), subtract \( a \cdot 0 \) from both sides of the equation.

\[
a \cdot 0 - a \cdot 0 = a \cdot 0 + a \cdot 0 - a \cdot 0 \quad \text{Subtract} \ a \cdot 0 \text{ from both sides.}
\]

\[
0 = a \cdot 0 \quad \text{Add:} \quad a \cdot 0 - a \cdot 0 = 0 \text{ on each side.}
\]

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**Multiplicative Property of Zero.** Let \( a \) represent any integer. Then
\[
a \cdot 0 = 0 \quad \text{and} \quad 0 \cdot a = 0.
\]

Thus, for example, \(-18 \cdot 0 = 0\) and \(0 \cdot 122 = 0\).

**Multiplying by Minus One**

Here is another useful application of the distributive property.

\[
(-1)a + a = (-1)a + 1a \quad \text{Replace } a \text{ with } 1a.
\]
\[
= (-1 + 1)a \quad \text{Use the distributive property to factor out } a.
\]
\[
= 0a \quad \text{Replace } -1 + 1 \text{ with } 0.
\]
\[
= 0 \quad \text{Replace } 0a \text{ with } 0.
\]

Thus, \((-1)a + a = 0\). That is, if you add \((-1)a\) to \(a\) you get zero. However, the Additive Inverse Property says that \(-a\) is the unique number that you add to \(a\) to get zero. The conclusion must be that \((-1)a = -a\).

**Multiplying by Minus One.** If \(a\) is any integer, then

\[
(-1)a = -a.
\]

Thus, for example, \(-1(4) = -4\) and \(-1(-4) = -(-4) = 4\).

This property is rather important, as we will see in future work. Not only does it tell us that \((-1)a = -a\), but it also tells us that if we see \(-a\), then it can be interpreted to mean \((-1)a\).

**The Product of Two Negative Integers**

We can employ the multiplicative property of \(-1\), that is, \((-1)a = -a\) to find the product of two negative numbers.

\[
(-4)(-3) = [(-1)(4)](-3) \quad \text{Replace } -4 \text{ with } (-1)(4).
\]
\[
\quad = (-1)[(4)(-3)] \quad \text{Use the associative property to regroup.}
\]
\[
\quad = (-1)(-12) \quad \text{We know: } (4)(-3) = -12.
\]
\[
\quad = -(-12) \quad (-1)a = -a. \text{ Here } (-1)(-12) = -(-12).
\]
\[
\quad = 12 \quad -(a) = a. \text{ Here } -(12) = 12.
\]

Thus, at least in the case of \((-4)(-3)\), the product of two negative integers is a positive integer. This is true in general.

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The Product of Two Negative Integers. If both \( a \) and \( b \) are negative integers, then their product \( ab \) is a positive integer.

Thus, for example, \((-5)(-7) = 35\) and \((-112)(-6) = 672\). In each case the answer is positive, because the product of two negative integers is a positive integer.

Memory Device

Here’s a simple memory device to help remember the rules for finding the product of two integers.

Like and Unlike Signs. There are two cases:

Unlike Signs. The product of two integers with \textit{unlike} signs is negative. That is:

\[
(+)(-) = - \\
(-)(+) = -
\]

Like Signs. The product of two integers with \textit{like} signs is positive. That is:

\[
(+) (+) = + \\
(-) (-) = +
\]

EXAMPLE 1. Simplify: (a) \((-3)(-2)\), (b) \((4)(-10)\), and (c) \((12)(-3)\).

Solution. In each example, we use the “like” and “unlike” signs approach.

a) Like signs gives a positive result. Hence, \((-3)(-2) = 6\).

b) Unlike signs gives a negative result. Hence, \((4)(-10) = -40\).

c) Unlike signs gives a negative result. Hence, \((12)(-3) = -36\).

You Try It!

Simplify: (a) \((-12)(4)\) and (b) \((-3)(-11)\).

Answer: (a) \(-48\), (b) \(33\)
You Try It!  EXAMPLE 2. Simplify \((-3)(2)(-4)(-2)\).

Solution. Order of operations demands that we work from left to right.

\[
(-3)(2)(-4)(-2) = (-6)(-2) = 12
\]

Work left to right: \((-3)(2) = -6\).

\[
= (24)(-2) = -48
\]

Work left to right: \((-6)(-4) = 24\).

Multiply: \((24)(-2) = -48\).

Hence, \((-3)(2)(-4)(-2) = -48\).

Answer: \(-24\).

Example 3 motivates the following fact.

**Even and Odd.** Two facts are apparent.
1. If a product contains an odd number of negative factors, then the product is negative.

2. If a product contains an even number of negative factors, then the product is positive.

Thus, for example,

\[ (-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32 \]

quickly evaluates as \(-32\) as it has an odd number of negative factors. On the other hand,

\[ (-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = 64 \]

quickly evaluates as \(64\) as it has an even number of negative factors.

**Division of Integers**

Consider that

\[ \frac{12}{3} = 4 \text{ because } 3(4) = 12 \quad \text{and} \quad \frac{-12}{-3} = 4 \text{ because } -3(4) = -12. \]

In like manner,

\[ \frac{12}{-3} = -4 \text{ because } -3(-4) = 12 \quad \text{and} \quad \frac{-12}{3} = -4 \text{ because } 3(-4) = -12. \]

Thus, the rules for dividing integers are the same as the rules for multiplying integers.

**Like and Unlike Signs.** There are two cases:

**Unlike Signs.** The quotient of two integers with **unlike** signs is negative. That is,

\[
\begin{align*}
\frac{(+)}{(-)} &= - \\
\frac{(-)}{(+)} &= -
\end{align*}
\]

**Like Signs.** The quotient of two integers with **like** signs is positive. That is,

\[
\begin{align*}
\frac{(+)}{(+)} &= + \\
\frac{(-)}{(-)} &= +
\end{align*}
\]
Thus, for example, $12/(-6) = -2$ and $-44/(-4) = 11$. In the first case, unlike signs gives a negative quotient. In the second case, like signs gives a positive quotient.

One final reminder.

**Division by Zero is Undefined.** If $a$ is any integer, the quotient

$$\frac{a}{0}$$

is undefined. Division by zero is meaningless.

See the discussion in Section 1.3 for a discussion on division by zero.

**You Try It!**

EXAMPLE 4. Simplify: (a) $-12/(-4)$, (b) $6/(-3)$, and (c) $-15/0$.

**Solution.** In each example, we use the “like” and “unlike” signs approach.

a) Like signs gives a positive result. Hence,

$$\frac{-12}{-4} = 3.$$

b) Unlike signs gives a negative result. Hence,

$$\frac{6}{-3} = -2.$$

c) Division by zero is undefined. Hence,

$$\frac{-15}{0}$$

is undefined.

Answer: (a) $-6$, (b) $4$
### Exercises

In Exercises 1-16, state the property of multiplication depicted by the given identity.

1. \((-2)[(-16)(13)] = [(-2)(-16)](13)\) 
2. \((10)[(-15)(-6)] = [(10)(-15)](-6)\) 
3. \((-17)(-10) = (-10)(-17)\) 
4. \((-5)(3) = (3)(-5)\) 
5. \((4)(11) = (11)(4)\) 
6. \((-5)(-11) = (-11)(-5)\) 
7. \(16 \cdot (8 + (-15)) = 16 \cdot 8 + 16 \cdot (-15)\) 
8. \(1 \cdot (-16 + (-6)) = 1 \cdot (-16) + 1 \cdot (-6)\) 
9. \((17)(20)(11) = [(17)(20)](11)\) 
10. \((14)(-20)(-18) = [(14)(-20)](-18)\) 
11. \(-19 \cdot 1 = -19\) 
12. \(-17 \cdot 1 = -17\) 
13. \(8 \cdot 1 = 8\) 
14. \(-20 \cdot 1 = -20\) 
15. \(14 \cdot (-12 + 7) = 14 \cdot (-12) + 14 \cdot 7\) 
16. \(-14 \cdot (-3 + 6) = -14 \cdot (-3) + (-14) \cdot 6\)

In Exercises 17-36, simplify each given expression.

17. \(4 \cdot 7\) 
18. \(4 \cdot 2\) 
19. \(3 \cdot (-3)\) 
20. \(7 \cdot (-9)\) 
21. \(-1 \cdot 10\) 
22. \(-1 \cdot 11\) 
23. \(-1 \cdot 0\) 
24. \(-8 \cdot 0\) 
25. \(-1 \cdot (-14)\) 
26. \(-1 \cdot (-13)\) 
27. \(-1 \cdot (-19)\) 
28. \(-1 \cdot (-17)\) 
29. \(2 \cdot 0\) 
30. \(-6 \cdot 0\) 
31. \(-3 \cdot 8\) 
32. \(7 \cdot (-3)\) 
33. \(7 \cdot 9\) 
34. \(6 \cdot 3\) 
35. \(-1 \cdot 5\) 
36. \(-1 \cdot 2\)

In Exercises 37-48, simplify each given expression.

37. \((-7)(-1)(3)\) 
38. \((10)(6)(3)\) 
39. \((-7)(9)(10)(-10)\) 
40. \((-8)(-5)(7)(-9)\) 
41. \((6)(5)(8)\) 
42. \((7)(-1)(-9)\)

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43. \((-10)(4)(-3)(8)\) \hspace{1cm} 46. \((-5)(-4)(1)\)  
44. \((8)(-2)(-5)(2)\) \hspace{1cm} 47. \((2)(1)(3)(4)\)  
45. \((6)(-3)(-8)\) \hspace{1cm} 48. \((7)(5)(1)(4)\)

In Exercises 49-60, compute the exact value.

49. \((-4)^4\) \hspace{1cm} 55. \((-6)^2\)  
50. \((-3)^4\) \hspace{1cm} 56. \((-6)^4\)  
51. \((-5)^4\) \hspace{1cm} 57. \((-4)^5\)  
52. \((-2)^2\) \hspace{1cm} 58. \((-4)^2\)  
53. \((-5)^2\) \hspace{1cm} 59. \((-5)^3\)  
54. \((-3)^3\) \hspace{1cm} 60. \((-3)^2\)

In Exercises 61-84, simplify each given expression.

61. \(-16 \div (-8)\) \hspace{1cm} 73. \(\frac{63}{21}\)  
62. \(-33 \div (-3)\) \hspace{1cm} 74. \(\frac{-6}{-1}\)  
63. \(-8 \div 1\) \hspace{1cm} 75. \(\frac{78}{13}\)  
64. \(\frac{40}{-20}\) \hspace{1cm} 76. \(\frac{-84}{-14}\)  
65. \(-1 \div 0\) \hspace{1cm} 77. \(0 \div 5\)  
66. \(\frac{2}{0}\) \hspace{1cm} 78. \(0 \div (-16)\)  
67. \(-3 \div 3\) \hspace{1cm} 79. \(\frac{17}{0}\)  
68. \(-58 \div 29\) \hspace{1cm} 80. \(\frac{-20}{0}\)  
69. \(\frac{56}{-28}\) \hspace{1cm} 81. \(-45 \div 15\)  
70. \(\frac{60}{-12}\) \hspace{1cm} 82. \(-28 \div 28\)  
71. \(0 \div 15\) \hspace{1cm} 83. \(12 \div 3\)  
72. \(0 \div (-4)\) \hspace{1cm} 84. \(-22 \div (-22)\)
85. **Scuba.** A diver goes down 25 feet, and then dives down 5 times further. Write the final depth of the diver as an integer.

86. **Investing Loss.** An investing club of five friends has lost $4400 on a trade. If they share the loss equally, write each members’ loss as an integer.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Associative property of multiplication</td>
<td>45. 144</td>
</tr>
<tr>
<td>3.</td>
<td>Commutative property of multiplication</td>
<td>47. 24</td>
</tr>
<tr>
<td>5.</td>
<td>Commutative property of multiplication</td>
<td>49. 256</td>
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<tr>
<td>7.</td>
<td>Distributive property</td>
<td>51. 625</td>
</tr>
<tr>
<td>9.</td>
<td>Associative property of multiplication</td>
<td>53. 25</td>
</tr>
<tr>
<td>11.</td>
<td>Multiplicative identity property</td>
<td>55. 36</td>
</tr>
<tr>
<td>13.</td>
<td>Multiplicative identity property</td>
<td>57. −1024</td>
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<tr>
<td>15.</td>
<td>Distributive property</td>
<td>59. −125</td>
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<td>17.</td>
<td>28</td>
<td>61. 2</td>
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<tr>
<td>19.</td>
<td>−9</td>
<td>63. −8</td>
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<td>21.</td>
<td>−10</td>
<td>65. Division by zero is undefined.</td>
</tr>
<tr>
<td>23.</td>
<td>0</td>
<td>67. −1</td>
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<td>25.</td>
<td>14</td>
<td>69. −2</td>
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<td>27.</td>
<td>19</td>
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<tr>
<td>29.</td>
<td>0</td>
<td>73. 3</td>
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<td>31.</td>
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<td>75. 6</td>
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<tr>
<td>33.</td>
<td>63</td>
<td>77. 0</td>
</tr>
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<td>−5</td>
<td>79. Division by zero is undefined.</td>
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<td>21</td>
<td>81. −3</td>
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<td>83. 4</td>
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<td>41.</td>
<td>240</td>
<td>85. −125 feet</td>
</tr>
<tr>
<td>43.</td>
<td>960</td>
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