425

5.7 Introduction to Square Roots

Recall that

$$x^2 = x \cdot x$$
.

The Square of a Number. The number x^2 is called the *square* of the number x

Thus, for example:

- $9^2 = 9 \cdot 9 = 81$. Therefore, the number 81 is the square of the number 9.
- $(-4)^2 = (-4)(-4) = 16$. Therefore, the number 16 is the square of the number -4.

In the margin, we've placed a "List of Squares" of the whole numbers ranging from 0 through 25, inclusive.

Square Roots

Once you've mastered the process of squaring a whole number, then you are ready for the inverse of the squaring process, taking the *square root* of a whole number.

- Above, we saw that $9^2 = 81$. We called the number 81 the *square* of the number 9. Conversely, we call the number 9 a *square root* of the number 81.
- Above, we saw that $(-4)^2 = 16$. We called the number 16 the square of the number -4. Conversely, we call the number -4 a square root of the number 16.

Square Root. If $a^2 = b$, then a is called a square root of the number b.

List of Squares

DISC OI	5quai cs
\boldsymbol{x}	x^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

You Try It!

EXAMPLE 1. Find the square roots of the number 49.

Solution. To find a square root of 49, we must think of a number a such that $a^2 = 49$. Two numbers come to mind.

- $(-7)^2 = 49$. Therefore, -7 is a square root of 49.
- $7^2 = 49$. Therefore, 7 is a square root of 49.

Note that 49 has two square roots, one of which is positive and the other one is negative.

Find the square roots of 256.

Answer: -16, 16

You Try It!

Find the square roots of 625.

EXAMPLE 2. Find the square roots of the number 196.

Solution. To find a square root of 196, we must think of a number a such that $a^2 = 196$. With help from the "List of Squares," two numbers come to mind.

- $(-14)^2 = 196$. Therefore, -14 is a square root of 196.
- $14^2 = 196$. Therefore, 14 is a square root of 196.

Note that 196 has two square roots, one of which is positive and the other one is negative.

Answer: -25, 25

You Try It!

Find the square roots of 9.

EXAMPLE 3. Find the square roots of the number 0.

Solution. To find a square root of 0, we must think of a number a such that $a^2 = 0$. There is only one such number, namely zero. Hence, 0 is the square root of 0.

Answer: -3, 3

-3, 3

You Try It!

Find the square roots of -81.

EXAMPLE 4. Find the square roots of the number -25.

Solution. To find a square root of -25, we must think of a number a such that $a^2 = -25$. This is impossible because no square of a real number (whole number, integer, fraction, or decimal) can be negative. Positive times positive is positive and negative times negative is also positive. You cannot square and get a negative answer. Therefore, -25 has no square roots².

Answer: There are none.

Radical Notation

Because $(-3)^2 = 9$ and $3^2 = 9$, both -3 and 3 are square roots of 9. Special notation, called *radical notation*, is used to request these square roots.

Version: Fall 2010

Square Roots

Square	Roots
x	\sqrt{x}
0	0
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10
121	11
144	12
169	13
196	14
225	15
256	16
289	17
324	18
361	19
400	20
441	21
484	22
529	23
576	24
625	25

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 $^{^2}$ At least not in Prealgebra. In later courses, you will be introduced to the set of complex numbers, where -25 will have two square roots

• The radical notation $\sqrt{9}$, pronounced "the nonnegative square root of 9," calls for the nonnegative³ square root of 9. Hence,

$$\sqrt{9} = 3$$

• The radical notation $-\sqrt{9}$, pronounced "the negative square root of 9," calls for the negative square root of 9. Hence,

$$-\sqrt{9} = -3.$$

Radical Notation. In the expression $\sqrt{9}$, the symbol $\sqrt{}$ is called a radical and the number within the radical, in this case the number 9, is called the radicand.

List of Squares	
x	x^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

For example,

- In the expression $\sqrt{529}$, the number 529 is the radicand.
- In the expression $\sqrt{a^2+b^2}$, the expression a^2+b^2 is the radicand.

Radical Notation and Square Root. If b is a positive number, then

- 1. \sqrt{b} calls for the nonnegative square root of b.
- 2. $-\sqrt{b}$ calls for the *negative* square root of b.

Note: Nonnegative is equivalent to saying "not negative;" i.e., positive or zero.

You Try It!

Simplify:

b) $-\sqrt{324}$

EXAMPLE 5. Simplify: (a) $\sqrt{121}$, (b) $-\sqrt{625}$, and (c) $\sqrt{0}$.

Solution.

(a) Referring to the list of squares, we note that $11^2 = 121$ and $(-11)^2 = 121$. Therefore, both 11 and -11 are square roots of 121. However, $\sqrt{121}$ calls for the nonnegative square root of 121. Thus,

$$\sqrt{121} = 11.$$

(b) Referring to the list of squares, we note that $25^2 = 625$ and $(-25)^2 = 625$. Therefore, both 25 and -25 are square roots of 625. However, $-\sqrt{625}$ calls for the negative square root of 625. Thus,

$$-\sqrt{625} = -25.$$

³Nonnegative is equivalent to saying "not negative;" i.e., positive or zero.

(c) There is only one square root of zero. Therefore,

$$\sqrt{0} = 0.$$

Answer: (a) 12 (b) -18

You Try It!

Simplify:

Answer: (a) -6 (b) undefined

EXAMPLE 6. Simplify: (a) $-\sqrt{25}$, and (b) $\sqrt{-25}$.

Solution.

(a) Because $5^2 = 25$ and $(-5)^2 = 25$, both 5 and -5 are square roots of 25. However, the notation $-\sqrt{25}$ calls for the negative square root of 25. Thus,

$$-\sqrt{25} = -5$$
.

(b) It is not possible to square a real number (whole number, integer, fraction, or decimal) and get -25. Therefore, there is no real square root of -25. That is,

$$\sqrt{-25}$$

is not a real number. It is undefined⁴.

Order of Operations

With the addition of radical notation, the Rules Guiding Order of Operations change slightly.

Rules Guiding Order of Operations. When evaluating expressions, proceed in the following order.

- 1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
- 2. Evaluate all exponents and radicals that appear in the expression.
- 3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
- 4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

The only change in the rules is in item #2, which says: "Evaluate all exponents and radicals, that, appear in the expression," putting radicals on the same level as exponents.

Square	Roots
x	\sqrt{x}
0	0
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10
121	11
144	12
169	13
196	14
225	15
256	16
289	17
324	18
361	19
400	20
441	21
484	22
529	23
576	24
625	25

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⁴At least in Prealgebra. In later courses you will be introduced to the set of complex numbers, where $\sqrt{-25}$ will take on a new meaning.

You Try It!

EXAMPLE 7. Simplify: $-3\sqrt{9} + 12\sqrt{4}$.

Solution. According to the *Rules Guiding Order of Operations*, we must evaluate the radicals in this expression first.

$$2\sqrt{4} - 3\sqrt{9}$$

Simplify:

$$-3\sqrt{9} + 12\sqrt{4} = -3(3) + 12(2)$$
 Evaluate radicals first: $\sqrt{9} = 3$ and $\sqrt{4} = 2$.
$$= -9 + 24$$
 Multiply: $-3(3) = -9$ and $12(2) = 24$.
$$= 15$$
 Add: $-9 + 24 = 15$.

Answer: -5

List of Squares

LIST OI	5quares
\boldsymbol{x}	x^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

EXAMPLE 8. Simplify: $-2 - 3\sqrt{36}$.

Solution. According to the *Rules Guiding Order of Operations*, we must evaluate the radicals in this expression first, moving left to right.

You Try It!

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$$5 - 8\sqrt{169}$$

$$-2 - 3\sqrt{36} = -2 - 3(6)$$
 Evaluate radicals first: $\sqrt{36} = 6$
= $-2 - 18$ Multiply: $3(6) = 18$.
= -20 Subtract: $-2 - 18 = -2 + (-18) = -20$.

Answer: -99

Simplify:

a) $\sqrt{25+144}$

b) $\sqrt{25} + \sqrt{144}$

You Try It!

EXAMPLE 9. Simplify: (a) $\sqrt{9+16}$ and (b) $\sqrt{9}+\sqrt{16}$.

Solution. Apply the Rules Guiding Order of Operations.

a) In this case, the radical acts like grouping symbols, so we must evaluate what is inside the radical first.

$$\sqrt{9+16} = \sqrt{25}$$
 Add: $9+16=25$.
= 5 Take nonnegative square root: $\sqrt{25} = 5$.

b) In this example, we must evaluate the square roots first.

$$\sqrt{9} + \sqrt{16} = 3 + 4$$
 Square root: $\sqrt{9} = 3$ and $\sqrt{16} = 4$.
= 7 Add: $3 + 4 = 7$.

Answer: (a) 13 (b) 17

Fractions and Decimals

We can also find square roots of fractions and decimals.

You Try It!

Simplify:

b) $\sqrt{0.36}$

EXAMPLE 10. Simplify: (a)
$$\sqrt{\frac{4}{9}}$$
, and (b) $-\sqrt{0.49}$.

Solution.

(a) Because
$$\left(\frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9}$$
, then

$$\sqrt{\frac{4}{9}} = \frac{2}{3}.$$

(b) Because $(0.7)^2 = (0.7)(0.7) = 0.49$ and $(-0.7)^2 = (-0.7)(-0.7) = 0.49$, both 0.7 and -0.7 are square roots of 0.49. However, $-\sqrt{0.49}$ calls for the negative square root of 0.49. Hence,

$$-\sqrt{0.49} = -0.7.$$

Answer: (a) 5/7 (b) 0.6

Estimating Square Roots

The squares in the "List of Squares" are called perfect squares. Each is the square of a whole number. Not all numbers are perfect squares. For example, in the case of $\sqrt{24}$, there is no whole number whose square is equal to 24. However, this does not prevent $\sqrt{24}$ from being a perfectly good number.

We can use the "List of Squares" to find decimal approximations when the radicand is not a perfect square.

You Try It!

Estimate: $\sqrt{83}$

EXAMPLE 11. Estimate $\sqrt{24}$ by guessing. Use a calculator to find a more accurate result and compare this result with your guess.

Solution. From the "List of Squares," note that 24 lies betwen 16 and 25, so $\sqrt{24}$ will lie between 4 and 5, with $\sqrt{24}$ much closer to 5 than it is to 4.



Version: Fall 2010

Square Roots

Square	Roots
x	\sqrt{x}
0	0
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10
121	11
144	12
169	13
196	14
225	15
256	16
289	17
324	18
361	19
400	20
441	21
484	22
529	23
576	24
625	25

Let's guess

$$\sqrt{24} \approx 4.8$$
.

As a check, let's square 4.8.

$$(4.8)^2 = (4.8)(4.8) = 23.04$$

Not quite 24! Clearly, $\sqrt{24}$ must be a little bit bigger than 4.8.

Let's use a scientific calculator to get a better approximation. From our calculator, using the square root button, we find

$$\sqrt{24} \approx 4.89897948557.$$

Even though this is better than our estimate of 4.8, it is still only an approximation. Our calculator was only capable of providing 11 decimal places. However, the exact decimal representation of $\sqrt{24}$ is an infinite decimal that never terminates and never establishes a pattern of repetition.

Just for fun, here is a decimal approximation of $\sqrt{24}$ that is accurate to 1000 places, courtesy of http://www.wolframalpha.com/.

4.898979485566356196394568149411782783931894961313340256865385134501920754914630053079718866209280469637189202453228378249717734487226082885405404323484036766001631796156760261794014573879872616743161888016008874773750983290293078782900240894528962666325870180186871723358639673313325338182638130717275322105163123587324722463325946466056580995478209485272014102527539509377735401281985911851434656929005776183028851492605205905926474151050068455119830908525625960061293441598848506045756852410681358957200931938799598711950812334271730930691249641651255377273856188261274486701772960314496926744648947590909762887695867274018394820295570465751182126319692156620734019070649453

If you were to multiply this number by itself (square the number), you would get a number that is extremely close to 24, but it would not be exactly 24. There would still be a little discrepancy.

Answer: 9.1

Important Observation. A calculator can only produce a finite number of decimal places. If the decimal representation of your number does not terminate within this limited number of places, then the number in your calculator window is only an approximation.

- \bullet The decimal representation of 1/8 will terminate within three places, so most calculators will report the exact answer, 0.125.
- For contrast, 2/3 does not terminate. A calculator capable of reporting 11 places of accuracy produces the number 0.66666666667. However, the exact decimal representation of 2/3 is 0.6. Note that the calculator has rounded in the last place and only provides an approximation of 2/3. If your instructor asks for an exact answer on an exam or quiz then 0.666666666667, being an approximation, is not acceptable. You must give the exact answer 2/3.

Exercises & &

In Exercises 1-16, list all square roots of the given number. If the number has no square roots, write "none".

- 1. 256
- **2**. 361
- **3**. −289
- **4**. -400
- **5**. 441
- **6**. 36
- **7**. 324
- **8**. 0

- **9**. 144
- **10**. 100
- **11**. -144
- **12**. -100
- **13**. 121
- **14**. -196
- **15**. 529
- **16**. 400

In Exercises 17-32, compute the exact square root. If the square root is undefined, write "undefined".

- 17. $\sqrt{-9}$
- 18. $-\sqrt{-196}$
- **19**. $\sqrt{576}$
- **20**. $\sqrt{289}$
- **21**. $\sqrt{-529}$
- **22**. $\sqrt{-256}$
- **23**. $-\sqrt{25}$
- **24**. $\sqrt{225}$

- **25**. $-\sqrt{484}$
- **26**. $-\sqrt{36}$
- **27**. $-\sqrt{196}$
- **28**. $-\sqrt{289}$
- **29**. $\sqrt{441}$
- **30**. $\sqrt{324}$
- **31**. $-\sqrt{4}$
- **32**. $\sqrt{100}$

In Exercises 33-52, compute the exact square root.

- **33**. $\sqrt{0.81}$
- **34**. $\sqrt{5.29}$
- **35**. $\sqrt{3.61}$
- **36**. $\sqrt{0.09}$

- **37**. $\sqrt{\frac{225}{16}}$
- **38.** $\sqrt{\frac{100}{81}}$
- **39**. $\sqrt{3.24}$

40.
$$\sqrt{5.76}$$

41.
$$\sqrt{\frac{121}{49}}$$

42.
$$\sqrt{\frac{625}{324}}$$

43.
$$\sqrt{\frac{529}{121}}$$

44.
$$\sqrt{\frac{4}{121}}$$

45.
$$\sqrt{2.89}$$

46.
$$\sqrt{4.41}$$

47.
$$\sqrt{\frac{144}{25}}$$

48.
$$\sqrt{\frac{49}{36}}$$

49.
$$\sqrt{\frac{256}{361}}$$

50.
$$\sqrt{\frac{529}{16}}$$

51.
$$\sqrt{0.49}$$

52.
$$\sqrt{4.84}$$

In Exercises 53-70, compute the exact value of the given expression.

53.
$$6 - \sqrt{576}$$

54.
$$-2 - 7\sqrt{576}$$

55.
$$\sqrt{8^2+15^2}$$

56.
$$\sqrt{7^2 + 24^2}$$

57.
$$6\sqrt{16} - 9\sqrt{49}$$

58.
$$3\sqrt{441} + 6\sqrt{484}$$

59.
$$\sqrt{5^2+12^2}$$

60.
$$\sqrt{15^2 + 20^2}$$

61.
$$\sqrt{3^2+4^2}$$

62.
$$\sqrt{6^2+8^2}$$

63.
$$-2\sqrt{324} - 6\sqrt{361}$$

64.
$$-6\sqrt{576} - 8\sqrt{121}$$

65.
$$-4 - 3\sqrt{529}$$

66.
$$-1 + \sqrt{625}$$

67.
$$-9\sqrt{484} + 7\sqrt{81}$$

68.
$$-\sqrt{625} - 5\sqrt{576}$$

69.
$$2 - \sqrt{16}$$

70.
$$8 - 6\sqrt{400}$$

In Exercises 71-76, complete the following tasks to estimate the given square root.

- a) Determine the two integers that the square root lies between.
- b) Draw a number line, and locate the approximate location of the square root between the two integers found in part (a).
- c) Without using a calculator, estimate the square root to the nearest tenth.

71.
$$\sqrt{58}$$

72.
$$\sqrt{27}$$

73.
$$\sqrt{79}$$

74.
$$\sqrt{12}$$

75.
$$\sqrt{44}$$

76.
$$\sqrt{88}$$

In Exercises 77-82, use a calculator to approximate the square root to the nearest tenth.

	/100
7.7	$\frac{1}{4}$

78. $\sqrt{73}$

79. $\sqrt{615}$

80.
$$\sqrt{162}$$

81. $\sqrt{444}$

82. $\sqrt{223}$

	a a Answers & S
1. 16, -16	35. 1.9
3. none	37. $\frac{15}{4}$
5. 21, -21	39. 1.8
7. 18, -18	41. $\frac{11}{7}$
9. 12, -12	43. $\frac{23}{11}$
11. none	11 45. 1.7
13. 11, -11	47. $\frac{12}{5}$
15. 23, -23	
17. undefined	49. $\frac{16}{19}$
19. 24	51. 0.7
21. undefined	53. −18
23. -5	55. 17
25. –22	57. −39
27. –14	59. 13
	61. 5
29. 21	63. -150
31. –2	65. –73
33. 0.9	67. −135

CHAPTER 5. DECIMALS

71. 7.6 79. 24.8 73. 8.9	69. –2	77. 21.7
	71. 7.6	79. 24.8
75. 6.6		81. 21.1

436