

1.1 Polynomial Equations in Factored Form

Learning Objectives

- Use the zero-product property
- Find greatest common monomial factor
- Solve simple polynomial equations by factoring

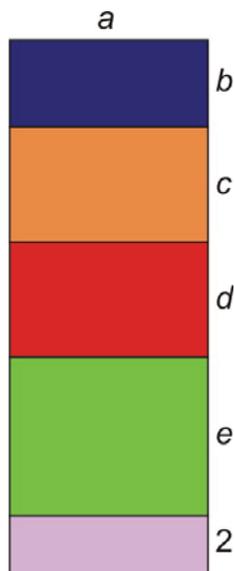
Introduction

In the last few sections, we learned how to multiply polynomials. We did that by using the Distributive Property. All the terms in one polynomial must be multiplied by all terms in the other polynomial. In this section, you will start learning how to do this process in reverse. Similar to what you have done in the past with positive integers, the factors of 15 are 1, 3, 5, and 15 since:

$$1 \times 15 = 15$$

$$3 \times 5 = 15$$

The factors are the parts of multiplication problem. In this section you will be breaking down polynomials into the multiplication parts or factors. This process is called **factoring polynomials**.



Lets look at the areas of the rectangles again: Area = length \cdot width. The total area of the figure on the right can be found in two ways.

Method 1 Find the areas of all the small rectangles and add them

$$\text{Blue rectangle} = ab$$

$$\text{Orange rectangle} = ac$$

$$\text{Red rectangle} = ad$$

$$\text{Green rectangle} = ae$$

$$\text{Pink rectangle} = 2a$$

$$\text{Total area} = ab + ac + ad + ae + 2a$$

Method 2 Find the area of the big rectangle all at once

$$\text{Length} = a$$

$$\text{Width} = b + c + d + e + 2$$

$$\text{Area} = a(b + c + d + e + 2)$$

Since the area of the rectangle is the same no matter what method you use then the answers are the same:

$$ab + ac + ad + ae + 2a = a(b + c + d + e + 2)$$

Factoring out the greatest common factor (GCF) means that you take the factors that are common to all the terms in a polynomial. Then, multiply them by a parenthesis containing all the terms that are left over when you divide out the common factors.

Use the Zero-Product Property

Polynomials can be written in **expanded form** or in **factored form**. Expanded form means that you have sums and differences of different terms:

$$6x^4 + 7x^3 - 26x^2 + 17x + 30$$

Notice that the degree of the polynomials is four. It is written in standard form because the terms are written in order of decreasing power.

Factored form means that the polynomial is written as a product of factors. The factors are also polynomials, usually of lower degree.

$$6x^4 + 7x^3 - 26x^2 + 17x + 30 = (x - 1)(x + 2)(2x - 3)(3x + 5)$$

Factored form of a polynomial expression:

$$\underbrace{(x - 1)}_{1^{\text{st}} \text{ factor}} \underbrace{(x + 2)}_{2^{\text{nd}} \text{ factor}} \underbrace{(2x - 3)}_{3^{\text{rd}} \text{ factor}} \underbrace{(3x + 5)}_{4^{\text{th}} \text{ factor}}$$

Notice that each factor in this polynomial is a binomial. Writing polynomials in factored form is very useful because it helps us solve polynomial equations. Before we talk about how we can solve polynomial equations of degree 2 or higher, let's review how to solve a linear equation (degree 1).

Example 1

Solve the following equations

a) $x - 4 = 0$

b) $3x - 5 = 0$

Solution

Remember that to solve an equation you are trying to find the value of x :

a)

$$\begin{array}{r} x - 4 = 0 \\ +4 = +4 \\ \hline x = 4 \end{array}$$

b)

$$\begin{array}{r} 3x - 5 = 0 \\ +5 = +5 \\ \hline 3x = 5 \\ \frac{3x}{3} = \frac{5}{3} \\ x = \frac{5}{3} \end{array}$$

Now we are ready to think about solving equations like

$$2x^2 + 5x = 42$$

Notice we can't isolate x in any way that you have already learned. But, we can subtract 42 on both sides to get

$$2x^2 + 5x - 42 = 0$$

Now, the left hand side of this equation can be factored!

Factoring a polynomial allows us to break up the problem into easier chunks. For example,

$$2x^2 + 5x - 42 = (x + 6)(2x - 7)$$

So now we want to solve:

$$(x + 6)(2x - 7) = 0$$

How would we solve this? If we multiply two numbers together and their product is zero, what can we say about these numbers? The only way a product is zero is if one or both of the terms are zero. This property is called the **Zero-product Property** and is the property used to solve factorable polynomials. In general, if

If

$$a \times b = 0$$

then

$$a = 0 \text{ OR } b = 0$$

How does that help us solve the polynomial equation? Since the product equals 0, then either of the terms or factors in the product must equal zero. We set each factor equal to zero and we solve. You will learn the factoring process later in this chapter.

$$(x + 6) = 0 \quad \text{OR} \quad (2x - 7) = 0$$

We can now solve each part individually and we obtain:

$$\begin{array}{ccc} x + 6 = 0 & \text{or} & 2x - 7 = 0 \\ & & 2x = 7 \\ x = -6 & \text{or} & x = -\frac{7}{2} \end{array}$$

Notice that the solution is $x = -6$ **OR** $x = \frac{7}{2}$. The **OR** says that either of these values of x would make the product of the two factors equal to zero. Lets plug the solutions back into the equation and check that this is correct.

Check $x = -6$

$$\begin{aligned}(x+6)(2x-7) &= \\ (-6+6)(2(6)-7) &= \\ (0)(5) &= 0\end{aligned}$$

Check $x = \frac{7}{2}$

$$\begin{aligned}(x+6)(2x-7) &= \\ \left(\frac{7}{2}+6\right)\left(2\cdot\frac{7}{2}-7\right) &= \\ \left(\frac{19}{2}\right)(7-7) &= \\ \left(\frac{19}{2}\right)(0) &= 0\end{aligned}$$

We can also substitute the solutions back into the original equation $2x^2 + 5x = 42$.

Check $x = -6$

$$\begin{aligned}2(-6)^2 + 5(-6) &= 42 = \\ 2(36) + (-30) &= 42 = \\ 72 + (-30) &= 42 \\ 42 &= 42\end{aligned}$$

Check $x = \frac{7}{2}$

$$\begin{aligned}2\left(\frac{7}{2}\right)^2 + 5\left(\frac{7}{2}\right) &= 42 \\ 2\left(\frac{49}{4}\right) + 5\left(\frac{7}{2}\right) &= 42 \\ \frac{49}{2} + \frac{35}{2} &= 42 \\ \frac{84}{2} &= 42 \\ 42 &= 42\end{aligned}$$

Both solutions check out. You should notice that the product equals to zero because each solution makes one of the factors simplify to zero.

If we are not able to factor a polynomial the problem becomes harder and we must use other methods that you will learn later.

As a last note in this section, keep in mind that the Zero-product Property only works when a product equals zero. For example, if you multiplied two numbers and the answer was nine you could not say that each of the numbers was nine.

$$(x+2)(x-3) = 9$$

$$x \neq -2 \text{ or } x \neq 3$$

In order to use the property, you must have the factored polynomial equal to zero.

$$(x+2)(x-3) = 0$$

$$x = -2 \text{ or } x = 3$$

Example 2

Solve each of the polynomials

a) $(x-9)(3x+4) = 0$

b) $x(5x-4) = 0$

c) $4x(x+6)(4x-9) = 0$

Solution

Since all polynomials are in factored form, we set each factor equal to zero and solve the simpler equations separately

a) $(x-9)(3x+4) = 0$ can be split up into two linear equations

$$x - 9 = 0 \qquad \text{or} \qquad 3x + 4 = 0$$

$$x = 9 \qquad \text{or} \qquad 3x = -4$$

$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad x = -\frac{4}{3}$$

b) $x(5x-4) = 0$ can be split up into two linear equations

$$x = 0 \qquad \text{or} \qquad 5x - 4 = 0$$

$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad 5x = 4$$

$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad x = \frac{4}{5}$$

c) $4x(x+6)(4x-9) = 0$ can be split up into three linear equations.

$$4x = 0 \qquad \qquad \qquad 4x - 9 = 0$$

$$x = \frac{0}{4} \qquad \text{or} \qquad x + 6 = 0 \qquad \text{or} \qquad 4x = 9$$

$$x = 0 \qquad \qquad \qquad x = -6 \qquad \qquad \qquad x = \frac{9}{4}$$

Find Greatest Common Monomial Factor

Once we get a polynomial in factored form, it is easier to solve the polynomial equation. But first, we need to learn how to factor. There are several factoring methods you will be learning in the next few sections. In most cases,

factoring can take several steps to complete because we want to **factor completely**. That means that we factor until we cannot factor anymore.

Lets start with the simplest case, finding the greatest common factor or GCF. When we want to factor, we always look for common monomials first. Consider the following polynomial, written in expanded form.

$$ax + bx + cx + dx$$

A common factor can be a number, a variable or a combination of numbers and variables that are contained in all terms of the polynomial. We are looking for expressions that divide out evenly from each term in the polynomial. Notice that in our example, the factor x appears in all terms. Therefore x is a **common factor**.

$$x \cdot (a) + x \cdot (b) + x \cdot (c) + x \cdot (d)$$

Since x is a common factor, we factor it by writing to the left of a set of parenthesis:

$$x (\quad)$$

Inside the parenthesis, we write what is left over when we divide or factor out x from each term.

$$x(a + b + c + d)$$

Lets look at more examples.

Example 3

Factor

a) $2x + 8$

b) $15x - 25$

c) $3a + 9b + 6$

Solution

a) We see that the factor 2 divides evenly from both terms.

$$2x + 8 = 2(x) + 2(4)$$

We factor the 2 by writing it to the left of a parenthesis.

$$2(\quad)$$

Inside the parenthesis, we write what is left from each term when we divide by 2 or factor out 2.

$$2(x + 4)$$
 This is the factored form.

b) We see that the factor of 5 divides evenly from all terms.

$$\text{Rewrite } 15x - 25 = 5(3x) - 5(5).$$

Factor 5 to get $5(3x - 5)$.

c) We see that the factor of 3 divides evenly from all terms.

$$\text{Rewrite } 3a + 9b + 6 = 3(a) + 3(3b) + 3(2)$$

Factor 3 to get $3(a + 3b + 2)$.

Here are examples where different powers of the common factor appear in the polynomial.

Example 4

Find the greatest common factor

a) $a^3 - 3a^2 + 4a$

b) $12a^4 - 5a^3 + 7a^2$

Solution

a) $a^3 - 3a^2 + 4a$

Notice that a appears in all terms of $a^3 - 3a^2 + 4a$ but each term has a different power of a . The common variable factor is the lowest power of the variable that appears in each term of the expression. In this case the common factor is a .

Lets rewrite $a^3 - 3a^2 + 4a = a(a^2) + a(-3a) + a(4)$

Factor a to get $a(a^2 - 3a + 4)$.

b) $12a^4 - 5a^3 + 7a^2$

The factor a appears in all the term and the lowest power is a^2 .

We rewrite the expression as $12a^4 - 5a^3 + 7a^2 = 12a^2 \cdot a^2 - 5a \cdot a^2 + 7 \cdot a^2$

Factor a^2 to get $a^2(12a^2 - 5a + 7)$

Lets look at some examples where there is more than one common factor.

Example 5:

Factor completely

a) $3ax + 9a$

b) $x^3y + xy$

c) $5x^3y - 15x^2y^2 + 25xy^3$

Solution

a) Notice that 3 is common to both terms.

When we factor 3 we get $3(ax + 3a)$

This is not completely factored though because if you look inside the parenthesis, we notice that a is also a common factor.

When we factor a we get $3 \cdot a(x + 3)$, which is equivalent to $3a(x + 3)$.

This is the final answer because there are no more common factors.

A different option is to factor **all** common factors at once.

Since both 3 and a are common we factor the term $3a$ and get $3a(x + 3)$.

b) Notice that both x and y are common factors.

Lets rewrite the expression $x^3y + xy = xy(x^2) + xy(1)$

When we factor xy we obtain $xy(x^2 + 1)$.

c) The common factors are $5xy$.

When we factor $5xy$ we obtain $5xy(x^2 - 3xy + 5y^2)$.

Note: Always look for both the common number and variable factors for each term in the expression.

Solve Simple Polynomial Equations by Factoring

Now that we know the basics of factoring, we can solve some simple polynomial equations. We already saw how we can use the Zero-product Property to solve polynomials in factored form. Here you will learn how to solve polynomials in expanded form. These are the steps for this process.

Step 1

If necessary, **re-write** the equation in standard form such that:

Polynomial expression = 0.

Step 2

Factor the polynomial completely.

Step 3

Use the zero-product rule to set **each factor equal to zero**.

Step 4

Solve each equation from step 3.

Step 5

Check your answers by substituting your solutions into the original equation.

Example 6

Solve the following polynomial equations

a) $x^2 - 2x = 0$

b) $2x^2 = 5x$

c) $9x^2 - 6x = 0$

Solution:

a) $x^2 - 2x = 0$

Rewrite. This is not necessary since the equation is in the standard form.

Factor. The common factor is x , so this factors as: $x(x - 2) = 0$.

Set each factor equal to zero.

$$x = 0 \quad \text{or} \quad x - 2 = 0$$

Solve.

$$x = 0 \quad \text{or} \quad x = 2$$

Check Substitute each solution back into the original equation.

$$x = 0 \quad \Rightarrow \quad (0)^2 - 2(0) = 0 \quad \text{check}$$

$$x = 2 \quad \Rightarrow \quad (2)^2 - 2(2) = 4 - 4 = 0 \quad \text{check}$$

Answer

$$x = 0 \qquad x = 2$$

b) $2x^2 = 5x$

Rewrite. $2x^2 = 5x \Rightarrow 2x^2 - 5x = 0$.**Factor.** The common factor is x , so this factors as: $x(2x - 5) = 0$.**Set each factor equal to zero..**

$$x = 0 \qquad \text{or} \qquad 2x - 5 = 0$$

Solve.

$$x = 0 \qquad \text{or} \qquad 2x = 5$$

$$\qquad \qquad \qquad \qquad \qquad \qquad x = \frac{5}{2}$$

Check. Substitute each solution back into the original equation.

$$x = 0 \Rightarrow 2(0)^2 = 5(0) \Rightarrow 0 = 0 \qquad \text{works out}$$

$$x = \frac{5}{2} \Rightarrow 2\left(\frac{5}{2}\right)^2 = 5 \cdot \frac{5}{2} \Rightarrow 2 \cdot \frac{25}{4} = \frac{25}{2} \Rightarrow \frac{25}{2} = \frac{25}{2} \qquad \text{works out}$$

Answer

$$x = 0 \qquad \text{or} \qquad x = \frac{5}{2}$$

c) $9x^2 - 6x = 0$

Rewrite. This step is not necessary.**Factor.** The common factor is $3x$, so this factors as $3x(3x - 2) = 0$.**Set each factor equal to zero.**

$$3x = 0 \qquad \text{or} \qquad 3x - 2 = 0$$

Solve.

$$x = 0 \qquad \text{or} \qquad 3x = 2$$

$$\qquad \qquad \qquad \qquad \qquad \qquad x = \frac{2}{3}$$

Check. Substitute each solution back into the original equation.

$$x = 0 \Rightarrow 9(0) - 6(0) = 0 - 0 = 0 \quad \text{checks}$$

$$x = \frac{2}{3} \Rightarrow 9 \cdot \left(\frac{2}{3}\right)^2 - 6 \cdot \frac{2}{3} = 9 \cdot \frac{4}{9} - 4 = 4 - 4 = 0 \quad \text{checks}$$

Answer

$$x = 0 \quad \text{or} \quad x = \frac{2}{3}$$

Review Questions

Factor the common factor in the following polynomials.

1. $3x^3 - 21x$
2. $5x^6 + 15x^4$
3. $4x^3 + 10x^2 - 2x$
4. $-10x^6 + 12x^5 - 4x^4$
5. $12xy + 24xy^2 + 36xy^3$
6. $5a^3 - 7a$
7. $45y^{12} + 30y^{10}$
8. $16xy^2z + 4x^3y$

Solve the following polynomial equations.

9. $x(x + 12) = 0$
10. $(2x + 1)(2x - 1) = 0$
11. $(x - 5)(2x + 7)(3x - 4) = 0$
12. $2x(x + 9)(7x - 20) = 0$
13. $18y - 3y^2 = 0$
14. $9x^2 = 27x$
15. $4a^2 + a = 0$
16. $3b^2 - 5b = 0$

Review Answers

1. $3x(x^2 - 7)$
2. $5x^4(x^2 + 3)$
3. $2x(2x^2 + 5x - 1)$
4. $2x^4(-5x^2 + 6x - 2)$
5. $12xy(1 + 2y + 3y^2)$
6. $a(5a^2 - 7)$
7. $15y^{10}(3y^2 + 2)$
8. $4xy(4yz + x^2)$
9. $x = 0, x = -12$

10. $x = -\frac{1}{2}, x = \frac{1}{2}$
11. $x = 5, x = -\frac{7}{2}, x = \frac{4}{3}$
12. $x = 0, x = -9, x = \frac{20}{7}$
13. $y = 0, y = 6$
14. $x = 0, x = 3$
15. $a = 0, a = -\frac{1}{4}$
16. $b = 0, b = \frac{5}{3}$