

## 1.4 Factoring Polynomials Completely and Solving Polynomial Equations by Factoring

### Learning Objectives

- Factor out a common binomial.
- Factor by grouping.
- Factor a quadratic trinomial where  $a \neq 1$ .
- Solve polynomial equations by factoring.
- Solve real world problems using polynomial equations.

### Introduction

We say that a polynomial is **factored completely** when we factor as much as we can and we can't factor any more. Here are some suggestions that you should follow to make sure that you factor completely.

- Factor all common monomials first.
- Identify special products such as difference of squares or the square of a binomial. Factor according to their formulas.
- If there are no special products, factor using the methods we learned in the previous sections.
- Look at each factor and see if any of these can be factored further.

Here are some examples.

#### Example 1

*Factor the following polynomials completely.*

a)  $6x^2 - 30x + 36$

b)  $2x^2 - 8$

c)  $x^3 + 6x^2 + 9x$

#### Solution

a)  $6x^2 - 30x + 36$

Factor the common monomial. In this case 6 can be factored from each term.

$$6(x^2 - 5x + 6)$$

There are no special products. We factor  $x^2 - 5x + 6$  as a product of two binomials  $(x \pm \underline{\quad})(x \pm \underline{\quad})$ .

The two numbers that multiply to 6 and add to -5 are -2 and -3. Let's substitute them into the two parenthesis. The 6 is outside because it is factored out.

$$6(x^2 - 5x + 6) = 6(x - 2)(x - 3)$$

If we look at each factor we see that we can't factor anything else.

The factored form is  $6(x-2)(x-3)$ .

b)  $2x^2 - 8$

Factor common monomials  $2x^2 - 8 = 2(x^2 - 4)$ .

We recognize  $x^2 - 4$  as a difference of squares. We factor as  $2(x^2 - 4) = 2(x+2)(x-2)$ .

If we look at each factor we see that we can't factor anything else.

The factored form is  $2(x+2)(x-2)$ .

c)  $x^3 + 6x^2 + 9x$

Factor common monomials  $x^3 + 6x^2 + 9x = x(x^2 + 6x + 9)$ .

We recognize as a perfect square and factor as  $x(x+3)^2$ .

If we look at each factor we see that we can't factor anything else.

The factored form is  $x(x+3)^2$ .

### Example 2

Factor the following polynomials completely.

a)  $-2x^4 + 162$

b)  $x^5 - 8x^3 + 16x$

#### Solution

a)  $-2x^4 + 162$

Factor the common monomial. In this case, factor -2 rather than 2. It is always easier to factor when leading coefficient is positive.

$$-2x^4 + 162 = -2(x^4 - 81)$$

We recognize the expression in parenthesis as a difference of squares. We factor and get this result.

$$-2(x^2 - 9)(x^2 + 9)$$

If we look at each factor, we see that the first parenthesis is a difference of squares. After factoring we get:

$$-2(x+3)(x-3)(x^2+9)$$

If we look at each factor, we see that we can factor no more.

The complete factored form is  $-2(x+3)(x-3)(x^2+9)$ .

b)  $x^5 - 8x^3 + 16x$

Factor out the common monomial  $x^5 - 8x^3 + 16x = x(x^4 - 8x^2 + 16)$ .

We recognize  $x^4 - 8x^2 + 16$  as a perfect square and we factor it as  $x(x^2 - 4)^2$ .

We look at each term and recognize that the term in parenthesis is a difference of squares.

We factor and get:  $x[(x+2)(x-2)]^2 = x[(x+2)^2(x-2)^2] = x(x+2)^2(x-2)^2$ .

We use square brackets [ and ] in this expression because x is multiplied by the expression  $(x+2)^2(x-2)$ . When we have nested grouping symbols we use brackets [ and ] to show the levels of nesting.

If we look at each factor now we see that we can't factor anything else.

The complete factored form is:  $x(x+2)^2(x-2)^2$ .

### Factor out a Common Binomial

The first step in the factoring process is often factoring the common monomials from a polynomial. Sometimes polynomials have common terms that are binomials. For example, consider the following expression.

$$x(3x+2) - 5(3x+2)$$

You can see that the binomial  $(3x+2)$  appears in both products of the polynomial. This common term can be factored by writing it in front of a parenthesis. Inside the parenthesis, we write all the terms that are left over when we divide them by the common factor.

$$(3x+2)(x-5)$$

This expression is now completely factored.

Lets look at some more examples.

#### Example 3

*Factor the common binomials.*

a)  $3x(x-1) + 4(x-1)$

b)  $x(4x+5) + (4x+5)$

#### Solution

a)  $3x(x-1) + 4(x-1)$  has a common binomial of  $(x-1)$ .

When we factor the common binomial out, we get  $(x-1)(3x+4)$ .

b)  $x(4x+5) + (4x+5)$  has a common binomial of  $(4x+5)$ .

When we factor the common binomial out, we get  $(4x+5)(x+1)$ .

### Factor by Grouping

It may be possible to factor a polynomial containing four or more terms by factoring common monomials from groups of terms. This method is called **factor by grouping**.

The next example illustrates how this process works.

#### Example 4

*Factor  $2x + 2y + ax + ay$ .*

#### Solution

There isn't a common factor for all four terms in this example. However, there is a factor of 2 that is common to the first two terms and there is a factor of a that is common to the last two terms. Factor 2 from the first two terms and factor  $a$  from the last two terms.

$$2x + 2y + ax + ay = 2(x+y) + a(x+y)$$

Now we notice that the binomial  $(x + y)$  is common to both terms. We factor the common binomial and get:

$$(x + y)(2 + a)$$

Our polynomial is now factored completely.

### Example 5

Factor  $3x^2 + 6x + 4x + 8$ .

### Solution

We factor  $3x$  from the first two terms and factor 4 from the last two terms.

$$3x(x + 2) + 4(x + 2)$$

Now factor  $(x + 2)$  from both terms.

$$(x + 2)(3x + 4).$$

Now the polynomial is factored completely.

## Factor Quadratic Trinomials Where $a \neq 1$

Factoring by grouping is a very useful method for factoring quadratic trinomials where  $a \neq 1$ . A quadratic polynomial of this type is written in the form:

$$ax^2 + bx + c$$

This does not factor as  $(x \pm m)(x \pm n)$ , so it is not as simple as looking for two numbers that multiply to give  $c$  and add to give  $b$ . In this case, we must take into account the coefficient that appears in the first term.

To factor a quadratic polynomial where  $a \neq 1$ , we follow the following steps.

1. We find the product  $ac$ .
2. We look for two numbers that multiply to give  $ac$  and add to give  $b$ .
3. We rewrite the middle term using the two numbers we just found.
4. We factor the expression by grouping.

## Factor Quadratic Trinomials Where $a = 1$ By Trial and Error

Trial and error can also be a very useful method for factoring quadratic trinomials where  $a \neq 1$ . A quadratic polynomial such as this one.

$$ax^2 + bx + c$$

Using the trial and error process is like undoing the FOIL process used to multiply two binomials. For example the factored form of  $2x^2 - 7x - 15$  is  $(2x + 3)(x - 5)$ . The first terms in the binomial factors,  $2x$  and  $x$ , are factors of

the first terms of the original trinomial. Also the second terms of the binomial factors, 3 and  $-5$ , are factors of the third term or the constant term of the original trinomial. Finally notice the sum of the product of the inner terms and outer terms give the middle term of the original trinomial:  $3x + (-10x) = -7x$ . When performing trial and error to factor a quadratic trinomial, the goal is to find the correct factors of  $ax^2$  and  $c$  for the binomial factors.

**Steps to solving a trinomial by trial and error.**

**Step 1** Set up a product of two binomials.

$$(\quad)(\quad)$$

**Step 2** Place the possible factors of  $ax^2$  in the first positions of the binomials.

**Step 3** Place the possible factors of  $c$  in the second positions of the binomial factors.

**Step 4** Keep trying different factors of  $ax^2$  and  $c$  until the sum of the inner product and outer product of the binomials is equal to the middle term,  $bx$ , or the original trinomial.

Lets apply this method to the following examples.

**Example 6**

*Factor the following quadratic trinomials by grouping.*

a)  $3x^2 + 8x + 4$

b)  $6x^2 - 11x + 4$

c)  $5x^2 - 6x + 1$

**Solution**

Lets follow the steps outlined above.

a)  $3x^2 + 8x + 4$

**Step 1**  $ac = 3 \cdot 4 = 12$

**Step 2** The number 12 can be written as a product of two numbers in any of these ways:

$12 = 1 \cdot 12$	and	$1 + 12 = 13$	
$12 = 2 \cdot 6$	and	$2 + 6 = 8$	This is the correct choice.
$12 = 3 \cdot 4$	and	$3 + 4 = 7$	

**Step 3** Re-write the middle term as:  $8x = 2x + 6x$ , so the problem becomes the following.

$$3x^2 + 8x + 4 = 3x^2 + 2x + 6x + 4$$

**Step 4:** Factor an  $x$  from the first two terms and 2 from the last two terms.

$$x(3x + 2) + 2(3x + 2)$$

Now factor the common binomial  $(3x + 2)$ .

$$(3x + 2)(x + 2)$$

The factored form is  $(3x + 2)(x + 2)$ .

To check if this is correct we multiply  $(3x + 2)(x + 2)$ .

$$(3x + 2)(x + 2) = 3x^2 + 6x + 2x + 4 = 3x^2 + 8x + 4$$

**The answer checks out.**

b)  $6x^2 - 11x + 4$

**Step 1**  $ac = 6 \cdot 4 = 24$

**Step 2** The number 24 can be written as a product of two numbers in any of these ways.

$24 = 1 \cdot 24$	and	$1 + 24 = 25$	
$24 = -1 \cdot (-24)$	and	$-1 + (-24) = -25$	
$24 = 2 \cdot 12$	and	$2 + 12 = 14$	
$24 = -2 \cdot (-12)$	and	$-2 + (-12) = -14$	
$24 = 3 \cdot 8$	and	$3 + 8 = 11$	
$24 = -3 \cdot (-8)$	and	$-3 + (-8) = -11$	← This is the correct choice.
$24 = 4 \cdot 6$	and	$4 + 6 = 10$	
$24 = -4 \cdot (-6)$	and	$-4 + (-6) = -10$	

**Step 3** Re-write the middle term as  $-11x = -3x - 8x$ , so the problem becomes

$$6x^2 - 11x + 4 = 6x^2 - 3x - 8x + 4$$

**Step 4** Factor by grouping. Factor a  $3x$  from the first two terms and factor  $-4$  from the last two terms.

$$3x(2x - 1) - 4(2x - 1)$$

Now factor the common binomial  $(2x - 1)$ .

$$(2x - 1)(3x - 4)$$

Our factored form is  $(2x - 1)(3x - 4)$ .

c)  $5x^2 - 6x + 1$

**Step 1**  $ac = 5 \cdot 1 = 5$

**Step 2** The number 5 can be written as a product of two numbers in any of these ways:

$$\begin{array}{llll}
 5 = 1 \cdot 5 & \text{and} & 1 + 5 = 6 & \\
 5 = -1 \cdot (-5) & \text{and} & -1 + (-5) = -6 & \leftarrow \text{ This is the correct choice}
 \end{array}$$

**Step 3** Rewrite the middle term as  $-6x = -x - 5x$ . The problem becomes

$$5x^2 - 6x + 1 = 5x^2 - x - 5x + 1$$

**Step 4** Factor by grouping. Factor an  $x$  from the first two terms and a factor of  $-1$  from the last two terms:

$$x(5x - 1) - 1(5x - 1)$$

Now factor the common binomial  $(5x - 1)$ .

$$(5x - 1)(x - 1).$$

Our factored form is  $(5x - 1)(x - 1)$ .

### Solve Quadratic Equations by Factoring

Now that we know the basics of factoring, we can solve some simple polynomial equations. We already saw how we can use the Zero-product Property to solve polynomials in factored form. Here you will learn how to solve polynomials in expanded form. These are the steps for this process.

#### Step 1

If necessary, **re-write** the equation in standard form such that:

$$\text{Polynomial expression} = 0.$$

#### Step 2

**Factor** the polynomial completely.

#### Step 3

Use the zero-product property to set **each factor equal to zero**.

#### Step 4

**Solve** each equation from step 3

#### Step 5

**Check** your answers by substituting your solutions into the original equation.

#### Example 7

*Solve the following polynomial equations.*

a)  $3x^2 - 24x + 36 = 0$

b)  $2x^2 = 50$

c)  $12x^2 - 7x - 10 = 0$

**Solution:**

a)  $3x^2 - 24x + 36 = 0$

**Rewrite.** This is not necessary since the equation is in the correct form.**Factor out the GCF of the trinomial.** Notice 3 is the greatest common factor.

**Factor**  $3(x^2 - 8x + 12) = 0 \Rightarrow 3(x - 6)(x - 2) = 0$

**Set each factor equal to zero.**

$$x - 6 = 0 \quad \text{or} \quad x - 2 = 0$$

**Solve.**

$$x = 6 \quad \text{or} \quad x = 2$$

**Check.** Substitute each solution back into the original equation.

$x = 6$	$\Rightarrow$	$3(6)^2 - 24(6) + 36 = 0$	checks
$x = 2$	$\Rightarrow$	$3(2)^2 - 24(2) + 36 = 0$	checks

**Answer**  $x = 6, x = 2$ 

b)  $2x^2 = 50$

**Rewrite.**  $2x^2 - 50 = 0$ .

**Factor.**  $2(x^2 - 25) = 0 \Rightarrow 2(x + 5)(x - 5) = 0$ .

**Set each factor equal to zero.**

$$2 \neq 0 \quad \text{or} \quad x - 5 = 0 \quad \text{or} \quad x + 5 = 0$$

**Solve.**

$$x = 5 \quad \text{or} \quad x = -5$$

**Notice the factor of 2 does not contain a variable and does not give us a solution. We normally will not set the constant factors equal to zero.****Check.** Substitute each solution back into the original equation.

$x = 5 \Rightarrow 2(5)^2 - 50 = 0$		checks
$x = -5 \Rightarrow 2(-5)^2 - 50 = 0$		checks

**Answer**  $x = 5, x = -5$



c)  $12x^2 - 7x - 10 = 0$

**Rewrite.** This step is not needed.**Factor.**

$$(3x + 2)(4x - 5)$$

**Set each factor equal to zero.**

$$3x + 2 = 0 \quad \text{or} \quad 4x - 5 = 0$$

**Solve.**

$$x = -\frac{2}{3} \quad \text{or} \quad x = \frac{5}{4}$$

**Check** Substitute each solution back into the original equation.

$$x = -2/3 \Rightarrow 12(-2/3)^2 - 7(-2/3) - 10 = 0$$

$$x = -2/3 \Rightarrow 12(4/9) - 7(-2/3) - 10 = 0$$

$$x = -2/3 \Rightarrow (48/9) + (14/3) - 10 = 0$$

$$x = -2/3 \Rightarrow (48/9) + (42/9) - 90/9 = 0$$

checks

$$x = 5/4 \Rightarrow 12(5/4)^2 - 7(5/4) - 10 = 0$$

$$x = 5/4 \Rightarrow 12(25/16) - 7(5/4) - 10 = 0$$

$$x = 5/4 \Rightarrow (300/16) - (35/4) - 10 = 0$$

$$x = 5/4 \Rightarrow (300/16) - (140/16) - (160/16) = 0$$

checks

**Answer**  $x = -\frac{2}{3}, x = \frac{5}{4}$

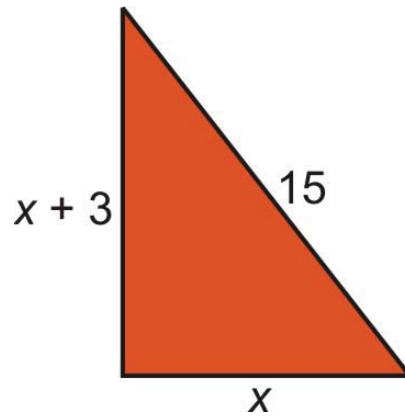
### Solve Real-World Problems Using Polynomial Equations

Now that we know most of the factoring strategies for quadratic polynomials we can see how these methods apply to solving real world problems.

#### Example 8 Pythagorean Theorem

*One leg of a right triangle is 3 feet longer than the other leg. The hypotenuse is 15 feet. Find the dimensions of the right triangle.*

**Solution**



Let  $x$  = the length of one leg of the triangle, then the other leg will measure  $x + 3$ .

Let's draw a diagram.

Use the Pythagorean Theorem  $(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2$  or  $a^2 + b^2 = c^2$ .

Here  $a$  and  $b$  are the lengths of the legs and  $c$  is the length of the hypotenuse.

Let's substitute the values from the diagram.

$$a^2 + b^2 = c^2$$

$$x^2 + (x + 3)^2 = 15^2$$

In order to solve, we need to get the polynomial in standard form. We must first distribute, collect like terms and **re-write** in the form polynomial = 0.

$$x^2 + (x + 3)^2 = 15^2$$

$$x^2 + x^2 + 6x + 9 = 225$$

$$2x^2 + 6x + 9 = 225$$

$$2x^2 + 6x - 216 = 0$$

**Factor** the common monomial  $2(x^2 + 3x - 108) = 0$ .

To factor the trinomial inside the parenthesis we need to two numbers that multiply to -108 and add to 3. It would take a long time to go through all the options so let's try some of the bigger factors.

$$\begin{array}{lll} -108 = -12 \cdot & \text{and} & -12 + 9 = -3 \\ -108 = 12 \cdot (-9) & \text{and} & 12 + (-9) = 3 \quad \leftarrow \quad \text{This is the correct choice.} \end{array}$$

We factor as:  $2(x - 9)(x + 12) = 0$ .

**Set each term equal to zero and solve.**

$$\begin{array}{ll} x - 9 = 0 & x + 12 = 0 \\ & \text{or} \\ x = 9 & x = -12 \end{array}$$

It makes no sense to have a negative answer for the length of a side of the triangle, so the answer must be the following.

**Answer**  $x = 9$  and  $x + 3 = 12$ , so one leg is 9 feet and the other leg is 12 feet.

**Check**  $9^2 + 12^2 = 81 + 144 = 225 = 15^2$  so the answer checks.

### Example 9 Number Problems

*The product of two positive numbers is 60. Find the two numbers if one of the numbers is 4 more than the other.*

#### Solution

Let  $x =$  one of the numbers and  $x + 4$  equals the other number.

The product of these two numbers equals 60. We can write the equation.

$$x(x + 4) = 60$$

In order to solve we must write the polynomial in standard form. Distribute, collect like terms and re-write in the form polynomial = 0.

$$x^2 + 4x = 60$$

$$x^2 + 4x - 60 = 0$$

**Factor** by finding two numbers that multiply to -60 and add to 4. List some numbers that multiply to -60:

$-60 = -4 \cdot 15$	and	$-4 + 15 = 11$	
$-60 = 4 \cdot (-15)$	and	$4 + (-15) = -11$	
$-60 = -5 \cdot 12$	and	$-5 + 12 = 7$	
$-60 = 5 \cdot (-12)$	and	$5 + (-12) = -7$	
$-60 = -6 \cdot 10$	and	$-6 + 10 = 4$	← This is the correct choice.
$-60 = 6 \cdot (-10)$	and	$6 + (-10) = -4$	

The expression factors as  $(x + 10)(x - 6) = 0$ .

**Set each term equal to zero and solve.**

$$x + 10 = 0$$

$$x = -10$$

$$x - 6 = 0$$

$$x = 6$$

or

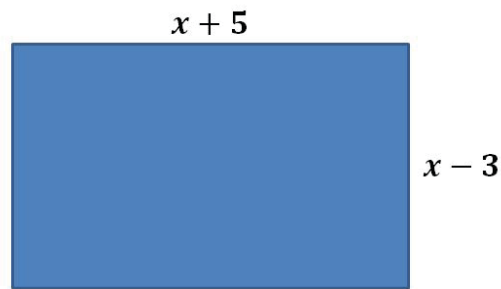
Since we are looking for positive numbers, the answer must be the following.

**Answer**  $x = 6$  and  $x + 4 = 10$ , so the two numbers are 6 and 10.

**Check**  $6 \cdot 10 = 60$  so the answer checks.

### Example 10 Area of a rectangle

*A rectangle has sides of  $x + 5$  and  $x - 3$ . What value of  $x$  gives an area of 48?*



Area of the rectangle = length  $\times$  width  
Make a sketch of this situation.

$$(x + 5)(x - 3) = 48$$

In order to solve, we must write the polynomial in standard form. Distribute, collect like terms and **rewrite** in the form polynomial = 0.

$$x^2 + 2x - 15 = 48$$

$$x^2 + 2x - 63 = 0$$

**Factor** by finding two numbers that multiply to -63 and add to 2. List some numbers that multiply to -63.

$$-63 = 7 \cdot (-9)$$

and

$$7 + (-9) = -2$$

$$-63 = -7 \cdot 9$$

and

$$-7 + 9 = 2$$

←

This is the correct choice.

The expression factors as  $(x + 9)(x - 7) = 0$ .

**Set each term equal to zero and solve.**

$$x + 9 = 0$$

$$x - 7 = 0$$

or

$$x = -9$$

$$x = 7$$

Since we are looking for positive numbers, the answer must be  $x = 7$ .

**Answer** The width is  $x - 3 = 4$  and  $x + 5 = 12$ , so the width is 4 and the length is 7.

**Check**  $4 \cdot 12 = 48$  so the answer checks out.

### Review Questions

Factor completely.

1.  $2x^2 + 16x + 30$

2.  $-x^3 + 17x^2 - 70x$

3.  $2x^4 - 512$
4.  $12x^3 + 12x^2 + 3x$

Factor by grouping.

5.  $6x^2 - 9x + 10x - 15$
6.  $5x^2 - 35x + x - 7$
7.  $9x^2 - 9x - x + 1$
8.  $4x^2 + 32x - 5x - 40$

Factor the following quadratic binomials by grouping or trial and error.

9.  $4x^2 + 25x - 21$
10.  $6x^2 + 7x + 1$
11.  $4x^2 + 8x - 5$
12.  $3x^2 + 16x + 21$

Solve.

13.  $3x^2 + 24x + 36 = 0$
14.  $5x^2 - 45 = 0$
15.  $20x^2 - 39x + 18 = 0$
16.  $4x^2 + 12x + 9 = 0$

Solve the following application problems:

17. One leg of a right triangle is 7 feet longer than the other leg. The hypotenuse is 13 feet. Find the dimensions of the right triangle.
18. A rectangle has sides of  $x + 2$  and  $x - 1$ . What value of  $x$  gives an area of 108?
19. The product of two positive numbers is 120. Find the two numbers if one number is 7 more than the other.
20. Framing Warehouse offers a picture framing service. The cost for framing a picture is made up of two parts. The cost of glass is \$1 per square foot. The cost of the frame is \$2 per linear foot. If the frame is a square, what size picture can you get framed for \$20?

### Review Answers

1.  $2(x + 3)(x + 5)$
2.  $-x(x - 7)(x - 10)$
3.  $2(x - 4)(x + 4)(x^2 + 16)$
4.  $3x(2x + 1)^2$
5.  $(2x - 3)(3x + 5)$
6.  $(x - 7)(5x + 1)$
7.  $(9x - 1)(x - 1)$
8.  $(x + 8)(4x - 5)$
9.  $(4x - 3)(x + 7)$
10.  $(6x + 1)(x + 1)$
11.  $(2x - 1)(2x + 5)$
12.  $(x + 3)(3x + 7)$
13.  $x = -2, x = -6$

14.  $x = 3, x = -3$
15.  $x = \frac{3}{4}, x = \frac{6}{5}$
16.  $x = -\frac{3}{2}$  (double solution or double root)
17. Leg 1 = 5 ft, Leg 2 = 12 ft
18.  $x = 10$
19. Numbers are 8 and 15.
20. You can frame a 2 foot  $\times$  2 foot picture.

### Texas Instruments Resources

*In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9619>.*