

## 2.2 Radical Expressions I

### Learning objectives

- Use the product and quotient properties of radicals to simplify radicals.
- Add and subtract radical expressions.
- Solve real-world problems using square root functions.

### Introduction

A radical reverses the operation of raising a number to a power. For example, to find the square of 4 we write  $4^2 = 4 \cdot 4 = 16$ . The reverse process is called finding the square root. The symbol for a square root is  $\sqrt{\phantom{x}}$ . This symbol is also called the **radical**. When we take the square root of a number, the result is a number which when squared gives the number under the square root. For example,

$$\sqrt{9} = 3 \quad \text{since} \quad 3^2 = 3 \cdot 3 = 9$$

The index of a radical indicates which root of the number we are seeking. Square roots have an index of 2 but most times this index is not written.

$$\sqrt{36} = \sqrt[2]{36} = 6 \quad \text{since} \quad 6^2 = 36$$

The cube root of a number gives a number which when raised to the third power gives the number under the radical.

$$\sqrt[3]{64} = 4 \quad \text{since} \quad 4^3 = 4 \cdot 4 \cdot 4 = 64$$

The fourth root of number gives a number which when raised to the power four gives the number under the radical sign.

$$\sqrt[4]{81} = 3 \quad \text{since} \quad 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

The index is 4 and the 81 is called the **radicand**.

### Even and odd roots

Radical expressions that have even indices are called **even roots** and radical expressions that have odd indices are called **odd roots**. There is a very important difference between even and odd roots in that they give drastically different results when the number inside the radical sign is negative.

Any real number raised to an even power results in a positive number. Therefore, when the index of a radical is even, the number inside the radical sign must be non-negative in order to get a real number answer.

On the other hand, a positive number raised to an odd power is positive and a negative number raised to an odd power is negative. Thus, a negative number inside the radical with an odd index is not a problem. It results in a negative number.

**Example 1**

*Evaluate each radical expression.*

a)  $\sqrt{121}$

b)  $\sqrt[3]{125}$

c)  $\sqrt[4]{-625}$

d)  $\sqrt[5]{-32}$

**Solution**

a)  $\sqrt{121} = 11$

b)  $\sqrt[3]{125} = 5$

c)  $\sqrt[4]{-625}$  is not a real number

d)  $\sqrt[5]{-32} = -2$

**Use the Product and Quotient Properties of Radicals**

Radicals can be rewritten as exponents with rational powers. The radical  $y = \sqrt[m]{a^n}$  is defined as  $a^{\frac{n}{m}}$ .

**Example 2**

*Write each expression as an exponent with a rational value for the exponent.*

a)  $\sqrt{5}$

b)  $\sqrt[4]{a}$

c)  $\sqrt[3]{4xy}$

d)  $\sqrt[6]{x^5}$

**Solution**

a)  $\sqrt{5} = 5^{\frac{1}{2}}$

b)  $\sqrt[4]{a} = a^{\frac{1}{4}}$

c)  $\sqrt[3]{4xy} = (4xy)^{\frac{1}{3}}$

d)  $\sqrt[6]{x^5} = x^{\frac{5}{6}}$

As a result of this property, for any non-negative number  $\sqrt[n]{a^n} = a^{\frac{n}{n}} = a^1 = a$ .

Since roots of numbers can be treated as powers, we can use exponent rules to simplify and evaluate radical expressions. Let's review the product and quotient rule of exponents.

Raising a product to a power

$$(x \cdot y)^n = x^n \cdot y^n$$

Raising a quotient to a power

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

In radical notation, these properties are written as

Raising a product to a power

$$\sqrt[m]{x \cdot y} = \sqrt[m]{x} \cdot \sqrt[m]{y}$$

Raising a quotient to a power

$$\sqrt[m]{\frac{x}{y}} = \frac{\sqrt[m]{x}}{\sqrt[m]{y}}$$

A very important application of these rules is reducing a radical expression to its simplest form. This means that we apply the root on all the factors of the number that are perfect roots and leave all factors that are not perfect roots inside the radical sign.

For example, in the expression  $\sqrt{16}$ , the number is a perfect square because  $16 = 4^2$ . This means that we can simplify.

$$\sqrt{16} = \sqrt{4^2} = 4$$

Thus, the square root disappears completely. This happens when the index and exponent are the same.

On the other hand, in the expression, the number  $\sqrt{32}$  is not a perfect square so we cannot remove the square root. However, we notice that  $32 = 16 \cdot 2$ , so we can write 32 as the product of a perfect square and another number.

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2}$$

If we apply the “raising a product to a power” rule we obtain

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4 \cdot \sqrt{2} = 4\sqrt{2}$$

### Example 3

Write the following expression in the simplest radical form.

a)  $\sqrt{8}$

b)  $\sqrt{50}$

c)  $\sqrt{\frac{125}{36}}$

### Solution

The strategy is to write the number under the square root as the product of a perfect square and another number. The goal is to find the highest perfect square possible, however, if we don't we can repeat the procedure until we cannot simplify any longer.

a) We can write  $8 = 4 \cdot 2$  so  $\sqrt{8} = \sqrt{4 \cdot 2}$

Use the rule for raising a product to a power  $\sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$

b) We can write  $50 = 25 \cdot 2$  so  $\sqrt{50} = \sqrt{25 \cdot 2}$

Use the rule for raising a product to a power  $= \sqrt{25} \cdot \sqrt{2} = \underline{\underline{5\sqrt{2}}}$

c) Use the rule for raising a quotient to a power to separate the fraction.

$$\sqrt{\frac{125}{36}} = \frac{\sqrt{125}}{\sqrt{36}}$$

Rewrite each radical as a product of a perfect square and another number.

$$= \frac{\sqrt{25 \cdot 5}}{\sqrt{6 \cdot 6}} = \frac{5\sqrt{5}}{6}$$

In algebra, when simplifying quotients with radical, we often don't want a radical in the denominator. The process eliminating a radical from the denominator is often called rationalizing the denominator. We will see this process in the next set of examples.

The same method can be applied to reduce radicals of different indices to their simplest form.

#### Example 4

Write the following expression in the simplest radical form.

a)  $\sqrt[3]{40}$

b)  $\sqrt{\frac{125}{27}}$

c)  $\sqrt[3]{135}$

#### Solution

In these cases we look for the highest possible perfect cube, fourth power, etc. as indicated by the index of the radical.

a) Here we are looking for the product of the highest perfect cube and another number. We write

$$\sqrt[3]{40} = \sqrt[3]{8 \cdot 5} = \sqrt[3]{2^3 \cdot 5} = 2\sqrt[3]{5}$$

b) Here we are looking for the product of the highest perfect fourth power and another number.

Rewrite as the quotient of two radicals

$$\sqrt{\frac{125}{27}} = \frac{\sqrt{125}}{\sqrt{27}}$$

Simplify each radical separately

$$= \frac{\sqrt{25 \cdot 5}}{\sqrt{9 \cdot 3}} = \frac{\sqrt{25} \cdot \sqrt{5}}{\sqrt{9} \cdot \sqrt{3}} = \frac{5\sqrt{5}}{3\sqrt{3}}$$

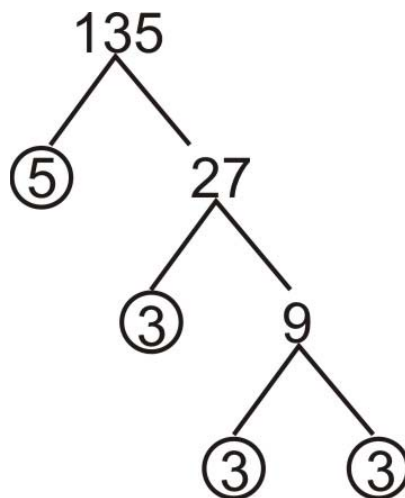
Eliminate the radical from the denominator

$$= \frac{5\sqrt{5}}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{5 \cdot 3}}{3\sqrt{3 \cdot 3}} = \frac{5\sqrt{15}}{9}$$

c) Here we are looking for the product of the highest perfect cube root and another number.

Often it is not very easy to identify the perfect root in the expression under the radical sign.

In this case, we can factor the number under the radical sign completely by using a factor tree.



We see that  $135 = 3 \cdot 3 \cdot 3 \cdot 5 = 3^3 \cdot 5$

Therefore  $\sqrt[3]{135} = \sqrt[3]{3^3 \cdot 5} = \sqrt[3]{3^3} \cdot \sqrt[3]{5} = 3\sqrt[3]{5}$

Here are some examples involving variables.

### Example 5

Write the following expression in the simplest radical form.

a)  $\sqrt{12x^3y^5}$

b)  $\sqrt[4]{\frac{1250x^7}{405y^9}}$

### Solution

Treat constants and each variable separately and write each expression as the products of a perfect power as indicated by the index of the radical and another number.

a)

Rewrite as a product of radicals.

$$\sqrt{12x^3y^5} = \sqrt{12} \cdot \sqrt{x^3} \cdot \sqrt{y^5}$$

Simplify each radical separately.

$$(\sqrt{4 \cdot 3}) \cdot (\sqrt{x^2 \cdot x}) \cdot (\sqrt{y^4 \cdot y}) = (2\sqrt{3}) \cdot (x\sqrt{x}) \cdot (y^2\sqrt{y})$$

Combine all factors outside and inside the radical sign

$$= 2xy^2\sqrt{3xy}$$

b)

Rewrite as a quotient of radicals

$$\sqrt[4]{\frac{1250x^7}{405y^9}} = \frac{\sqrt[4]{1250x^7}}{\sqrt[4]{405y^9}}$$

Simplify each radical separately

$$= \frac{\sqrt[4]{625 \cdot 2} \cdot \sqrt[4]{x^4 \cdot x^3}}{\sqrt[4]{81 \cdot 5} \cdot \sqrt[4]{y^4 \cdot y^4 \cdot y}} = \frac{5\sqrt[4]{2} \cdot x \cdot \sqrt[4]{x^3}}{3\sqrt[4]{5} \cdot y \cdot y \cdot \sqrt[4]{y}} = \frac{5x\sqrt[4]{2x^3}}{3y^2\sqrt[4]{5y}}$$

Eliminate the radical from the denominator

$$= \frac{5x\sqrt[4]{2x^3}}{3y^2\sqrt[4]{5y}} \cdot \frac{\sqrt[4]{5 \cdot 5 \cdot 5 \cdot y \cdot y \cdot y}}{\sqrt[4]{5 \cdot 5 \cdot 5 \cdot y \cdot y \cdot y}} = \frac{5x\sqrt[4]{250x^3y^3}}{15y^3} = \frac{x\sqrt[4]{250x^3y^3}}{3y^3}$$

## Add and Subtract Radical Expressions

When we add and subtract radical expressions, we can combine radical terms only when they have the same index and the same expression under the radical sign. This is a similar procedure to combining like terms in variable expressions. For example,

$$4\sqrt{2} + 5\sqrt{2} = 9\sqrt{2}$$

or

$$2\sqrt{3} - \sqrt{2} + 5\sqrt{3} + 10\sqrt{2} = 7\sqrt{3} + 9\sqrt{2}$$

It is important to simplify all radicals to their simplest form in order to make sure that we are combining all possible like terms in the expression. For example, the expression  $\sqrt{8} - 2\sqrt{50}$  looks like it cannot be simplified any more because it has no like terms. However, when we write each radical in its simplest form we have

$$\sqrt{8} - 2\sqrt{50} = 2\sqrt{2} - 10\sqrt{2} = -8\sqrt{2}$$

Notice how after simplify, we have like radicals because the index and radicand is the same.

If it is helpful we could factor our the  $\sqrt{2}$

$$2\sqrt{2} - 10\sqrt{2} = (2 - 10)\sqrt{2} = -8\sqrt{2}$$

### Example 6

*Simplify the following expressions as much as possible.*

a)  $4\sqrt{3} + 2\sqrt{12}$

b)  $10\sqrt{24} - \sqrt{28}$

### Solution

a)

$$\text{Simplify } \sqrt{12} \text{ to its simplest form.} = 4\sqrt{3} + 2\sqrt{4 \cdot 3} = 4\sqrt{3} + 2 \cdot 2\sqrt{3} = 4\sqrt{3} + 4\sqrt{3}$$

$$\text{Combine like terms.} = 8\sqrt{3}$$

b)

$$\text{Simplify } \sqrt{24} \text{ and } \sqrt{28} \text{ to their simplest form.} = 10\sqrt{6 \cdot 4} - \sqrt{7 \cdot 4} = 20\sqrt{6} - 2\sqrt{7}$$

There are no like terms.

### Example 7

*Simplify the following expressions as much as possible.*

a)  $4\sqrt[3]{128} - 3\sqrt[3]{250}$

b)  $3\sqrt{x^3} - 4x\sqrt{9x}$

### Solution

a)

$$\text{Rewrite radicals in simplest terms.} = 4\sqrt[3]{64 \cdot 2} - \sqrt[3]{125 \cdot 2} = 16\sqrt[3]{2} - 5\sqrt[3]{2}$$

$$\text{Combine like terms.} = 11\sqrt[3]{2}$$

b)

$$\text{Rewrite radicals in simplest terms.} = 3\sqrt{x^2 \cdot x} - 4x\sqrt{9x} = 3x\sqrt{x} - 12x\sqrt{x}$$

$$\text{Combine like terms.} = -9x\sqrt{x}$$

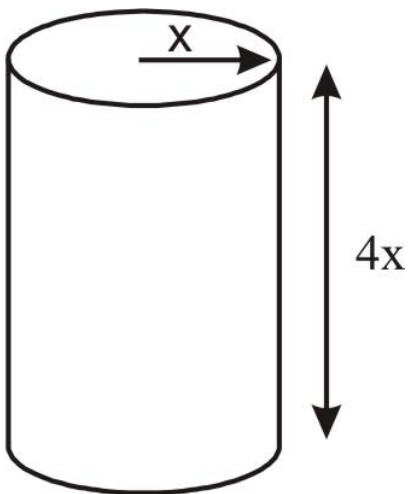
### Solve Real-World Problems Using Radical Expressions

Radicals often arise in problems involving areas and volumes of geometrical figures.

#### Example 10

The volume of a soda can is  $355 \text{ cm}^3$ . The height of the can is four times the radius of the base. Find the radius of the base of the cylinder.

#### Solution



1. Make a sketch.
2. Let  $x$  = the radius of the cylinder base
3. Write an equation.

The volume of a cylinder is given by

$$V = \pi r^2 \cdot h$$

4. Solve the equation.

$$355 = \pi x^2(4x)$$

$$355 = 4\pi x^3$$

$$x^3 = \frac{355}{4\pi}$$

$$x = \sqrt[3]{\frac{355}{4\pi}} = 3.046 \text{ cm}$$

5. Check by substituting the result back into the formula.

$$V = \pi r^2 \cdot h = \pi(3.046)^2 \cdot (4 \cdot 3 \cdot 046) = 355 \text{ cm}^3$$

So the volume is  $355 \text{ cm}^3$ .

**The answer checks.**

## Review Questions

Evaluate each radical expression.

1.  $\sqrt{169}$
2.  $\sqrt[4]{81}$
3.  $\sqrt[3]{-125}$
4.  $\sqrt[5]{1024}$

Write each expression as a rational exponent.

5.  $\sqrt[3]{14}$
6.  $\sqrt[4]{zw}$
7.  $\sqrt{a}$
8.  $\sqrt[9]{y^3}$

Write the following expressions in simplest radical form.

9.  $\sqrt{24}$
10.  $\sqrt{300}$
11.  $\sqrt[5]{96}$
12.  $\sqrt{\frac{240}{567}}$
13.  $\sqrt[3]{500}$
14.  $\sqrt[6]{64x^8}$
15.  $\sqrt[3]{48a^3b^7}$
16.  $\sqrt[3]{\frac{16x^5}{135y^4}}$

Simplify the following expressions as much as possible.

17.  $3\sqrt{8} - 6\sqrt{32}$
18.  $\sqrt{180} + 6\sqrt{405}$
19.  $\sqrt{6} - \sqrt{27} + 2\sqrt{54} + 3\sqrt{48}$
20.  $\sqrt{8x^3} - 4x\sqrt{98x}$
21.  $\sqrt{48a} + \sqrt{27a}$
22.  $\sqrt[3]{4x^3} + x\sqrt[3]{256}$
23. The volume of a spherical balloon is  $950 \text{ cm-cubed}$ . Find the radius of the balloon. (Volume of a sphere  $= \frac{4}{3}\pi R^3$ ).



**Review Answers**

1. 13
2. not a real solution
3. -5
4. 4
5.  $14^{\frac{1}{3}}$
6.  $z^{\frac{1}{4}}w^{\frac{1}{4}}$
7.  $a^{\frac{1}{2}}$
8.  $y^{\frac{1}{3}}$
9.  $2\sqrt{6}$
10.  $10\sqrt{3}$
11.  $2\sqrt[5]{3}$
12.  $\frac{4\sqrt{135}}{63}$
13.  $5\sqrt[3]{4}$
14.  $2x\sqrt[6]{x^2}$
15.  $2ab^2\sqrt[3]{3b}$
16.  $\frac{2x\sqrt{5x^2y}}{15y^2}$
17.  $-18\sqrt{2}$
18.  $60\sqrt{5}$
19.  $7\sqrt{6} + 9\sqrt{3}$
20.  $-26x\sqrt{2x}$
21.  $7\sqrt{3a}$
22.  $5x\sqrt[3]{4}$
23.  $R = 6.1 \text{ cm}$