

## 2.6 Distance and Midpoint Formulas

### Learning Objectives

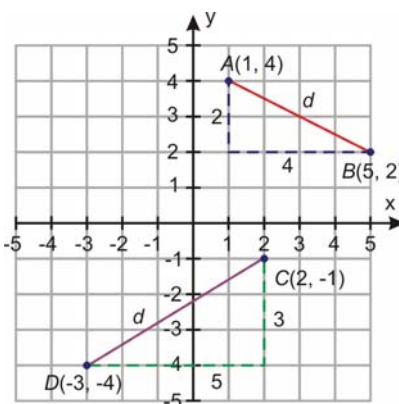
- Find the distance between two points in the coordinate plane.
- Find the missing coordinate of a point given the distance from another known point.
- Find the midpoint of a line segment.
- Solve real-world problems using distance and midpoint formulas.

### Introduction

In the last section, we saw how to use the Pythagorean Theorem in order to find lengths of sides of right triangles. In this section, you will learn how to use the Pythagorean Theorem to find the distance between two coordinate points.

#### Example 1

Find distance between points  $A = (1, 4)$  and  $B = (5, 2)$ .



#### Solution

Plot the two points on the coordinate plane. In order to get from point  $A = (1, 4)$  to point  $B = (5, 2)$ , we need to move 4 units to the right and 2 units down.

To find the distance between  $A$  and  $B$  we find the value of  $d$  using the Pythagorean Theorem.

$$d^2 = 2^2 + 4^2 = 20$$

$$d = \sqrt{20} = 2\sqrt{5} = 4.47$$

**Example 2:** Find the distance between points  $C = (2, -1)$  and  $D = (-3, -4)$ .

**Solution:** We plot the two points on the graph above.

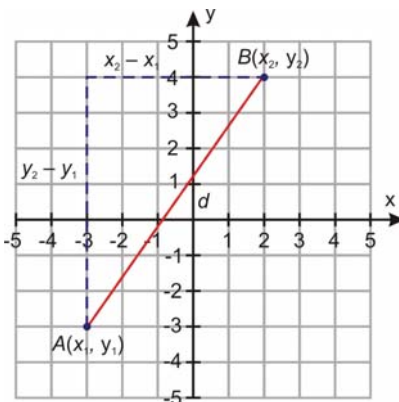
In order to get from point  $C$  to point  $D$ , we need to move 3 units down and 5 units to the left.

We find the distance from  $C$  to  $D$  by finding the length of  $d$  with the Pythagorean Theorem.

$$d^2 = 3^2 + 5^2 = 34$$

$$d = \sqrt{34} = 5.83$$

### The Distance Formula



This procedure can be generalized by using the Pythagorean Theorem to derive a formula for the distance between two points on the coordinate plane.

Let's find the distance between two general points  $A = (x_1, y_1)$  and  $B = (x_2, y_2)$ .

Start by plotting the points on the coordinate plane.

In order to move from point  $A$  to point  $B$  in the coordinate plane, we move  $x_2 - x_1$  units to the right and  $y_2 - y_1$  units up. We can find the length  $d$  by using the Pythagorean Theorem.

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

This equation leads us to the Distance Formula.

Given two points  $(x_1, y_1)$  and  $(x_2, y_2)$  the distance between them is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We can use this formula to find the distance between two points on the coordinate plane. Notice that the distance is the same if you are going from point  $A$  to point  $B$  or if you are going from point  $B$  to point  $A$ . Thus, it does not matter which order you plug the points into the distance formula.

### Find the Distance Between Two Points in the Coordinate Plane

Let's now apply the distance formula to the following examples.

#### Example 2

Find the distance between the following points.

- $(-3, 5)$  and  $(4, -2)$
- $(12, 16)$  and  $(19, 21)$

c) (11.5, 2.3) and (-4.2, -3.9)

### Solution

Plug the values of the two points into the distance formula. Be sure to simplify if possible.

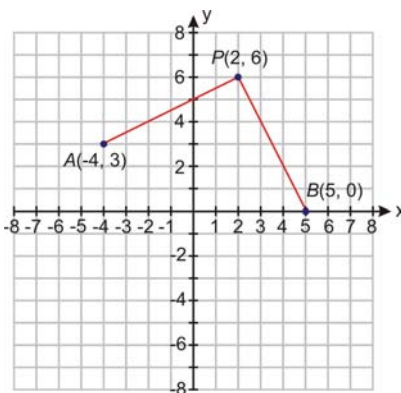
$$a) d = \sqrt{(-3 - 4)^2 + (5 - (-2))^2} = \sqrt{(-7)^2 + (7)^2} = \sqrt{49 + 49} = \sqrt{98} = 7\sqrt{2}$$

$$b) d = \sqrt{(12 - 19)^2 + (16 - 21)^2} = \sqrt{(-7)^2 + (-5)^2} = \sqrt{49 + 25} = \sqrt{74}$$

$$c) d = \sqrt{(11.5 + 4.2)^2 + (2.3 + 3.9)^2} = \sqrt{(15.7)^2 + (6.2)^2} = \sqrt{284.93} = 16.88$$

### Example 3

Show that point  $P = (2, 6)$  is equidistant for  $A = (-4, 3)$  and  $B = (5, 0)$ .



### Solution

To show that the point  $P$  is equidistant from points  $A$  and  $B$ , we need to show that the distance from  $P$  to  $A$  is equal to the distance from  $P$  to  $B$ .

Before we apply the distance formula, let's graph the three points on the coordinate plane to get a visual representation of the problem.

From the graph we see that to get from point  $P$  to point  $A$ , we move 6 units to the left and 3 units down. To move from point  $P$  to point  $B$ , we move 6 units down and 3 units to the left. From this information, we should expect  $P$  to be equidistant from  $A$  and  $B$ .

Now, let's apply the distance formula for find the lengths  $PA$  and  $PB$ .

$$PA = \sqrt{(2 + 4)^2 + (6 - 3)^2} = \sqrt{(6)^2 + (3)^2} = \sqrt{39 + 9} = \sqrt{45}$$

$$PB = \sqrt{(2 - 5)^2 + (6 - 0)^2} = \sqrt{(-3)^2 + (6)^2} = \sqrt{9 + 36} = \sqrt{45}$$

$PA = PB$ , so  $P$  is equidistant from points  $A$  and  $B$ .

## Find the Missing Coordinate of a Point Given Distance From Another Known Point

### Example 4

Point  $A = (6, -4)$  and point  $B = (2, k)$ . What is the value of  $k$  such that the distance between the two points is 5?

### Solution

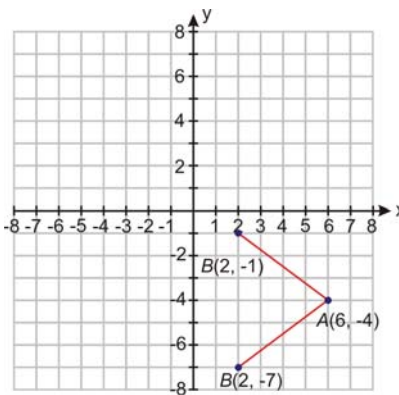
Let's use the distance formula.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \Rightarrow 5 = \sqrt{(6 - 2)^2 + (-4 - k)^2}$$

Square both sides of the equation.	$5^2 = \left[ \sqrt{(6 - 2)^2 + (-4 - k)^2} \right]^2$
Simplify.	$25 = 16 + (-4 - k)^2$
Eliminate the parentheses.	$0 = -9 + k^2 + 8k + 16$
Simplify.	$0 = k^2 + 8k + 7$
Factor to solve.	$0 = (k + 1)(k + 7)$
Use the Zero Product Property.	$k = -1$ or $k = -7$

**Answer**  $k = -7$  or  $k = -1$ .

Therefore, there are two possibilities for the value of  $k$ . Let's graph the points to get a visual representation of our results.



From the figure, we can see that both answers make sense because they are both positions of B are equidistant from point A.

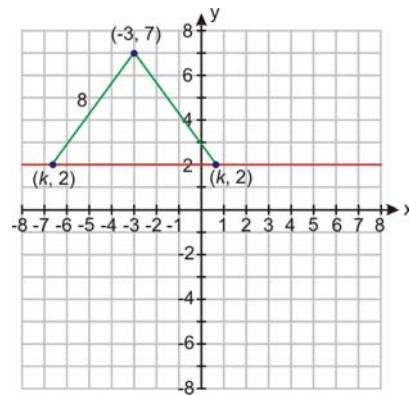
### Example 5

Find all points on line  $y = 2$  that are a distance of 8 units away from point  $(-3, 7)$ .

### Solution

Let's make a sketch of the given situation. Draw line segments from point  $(-3, 7)$  to the line  $y = 2$ . Let  $k$  be the missing value of  $x$  we are seeking. Let's use the distance formula.

$$8 = \sqrt{(-3 - k)^2 + (7 - 2)^2}$$



Now let's solve using the distance formula.

Square both sides of the equation

Therefore.

Or

Use the quadratic formula.

Therefore.

$$64 = (-3 - k)^2 + 25$$

$$0 = 9 + 6k + k^2 - 39$$

$$0 = k^2 + 6k - 30$$

$$k = \frac{-6 \pm \sqrt{36 + 120}}{2} = \frac{-6 \pm \sqrt{156}}{2}$$

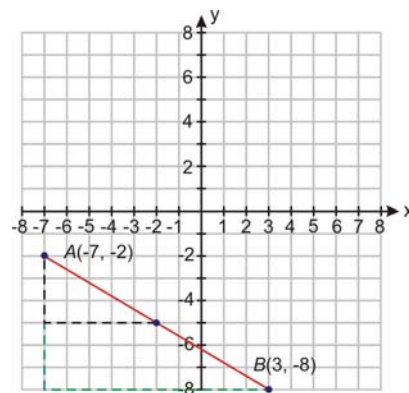
$$k \approx 3.24 \text{ or } k \approx -9.24$$

**Answer** The points are  $(-9.24, 2)$  and  $(3.24, 2)$

### Find the Midpoint of a Line Segment

#### Example 6

Find the coordinates of the point that is in the middle of the line segment connecting points  $A = (-7, -2)$  and  $B = (3, -8)$ .



#### Solution

Let's start by graphing the two points.

We see that to get from point  $A$  to point  $B$  we move 6 units down and 10 units to the right.

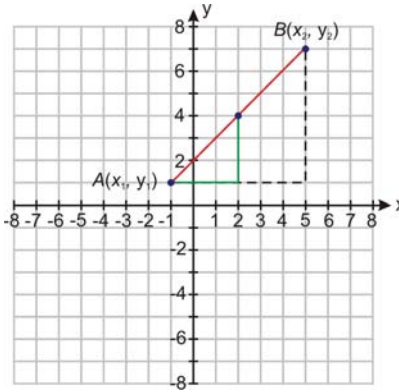
In order to get to the point that is half-way between the two points, it makes sense that we should move half the vertical and half the horizontal distance, that is 3 units down and 5 units to the right from point  $A$ .

The midpoint is  $M = \left(\frac{-7+3}{2}, \frac{-2+(-8)}{2}\right) = (-2, -5)$

For the x-coordinate, half way between -7 and 3 is -2 or  $\frac{-7+3}{2} = -2$ . Notice this is the average of the x-coordinates. For the y-coordinate, halfway between -2 and -8 is -5 or  $\frac{-2+(-8)}{2} = -5$ . Notice this is the average of the y-coordinates.

### The Midpoint Formula:

We now want to generalize this method in order to find a formula for the midpoint of a line segment.



It should hopefully make sense that the midpoint of a line is found by taking the average values of the x and y-values of the endpoints.

### Midpoint Formula

The midpoint of the segment connecting points  $(x_1, y_1)$  and  $(x_2, y_2)$  has coordinates

$$M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right).$$

### Example 7

Find the midpoint between the following points.

- $(-10, 2)$  and  $(3, 5)$
- $(3, 6)$  and  $(7, 6)$
- $(4, -5)$  and  $(-4, 5)$

### Solution

Let's apply the Midpoint Formula.

$$M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

- The midpoint of  $(-10, 2)$  and  $(3, 5)$  is  $\left(\frac{-10+3}{2}, \frac{2+5}{2}\right) = \left(\frac{-7}{2}, \frac{7}{2}\right) = (-3.5, 3.5)$ .
- The midpoint of  $(3, 6)$  and  $(7, 6)$  is  $\left(\frac{3+7}{2}, \frac{6+6}{2}\right) = \left(\frac{10}{2}, \frac{12}{2}\right) = (5, 6)$ .
- The midpoint of  $(4, -5)$  and  $(-4, 5)$  is  $\left(\frac{4+(-4)}{2}, \frac{-5+5}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0, 0)$ .

### Example 8

A line segment whose midpoint is  $(2, -6)$  has an endpoint of  $(9, -2)$ . What is the other endpoint?

**Solution**

In this problem we know the midpoint and we are looking for the missing endpoint.

The midpoint is  $(2, -6)$ .

One endpoint is  $(x_1, y_1) = (9, -2)$

Let's call the missing point  $(x, y)$ .

We know that the  $x$ -coordinate of the midpoint is 2, so

$$2 = \frac{9 + x_2}{2} \Rightarrow 4 = 9 + x_2 \Rightarrow x_2 = -5$$

We know that the  $y$ -coordinate of the midpoint is -6, so

$$-6 = \frac{-2 + y_2}{2} \Rightarrow -12 = -2 + y_2 \Rightarrow y_2 = -10$$

**Answer** The missing endpoint is  $(-5, -10)$ .

**Solve Real-World Problems Using Distance and Midpoint Formulas**

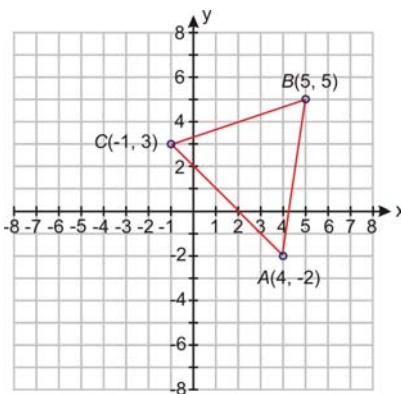
The distance and midpoint formula are applicable in geometry situations where we desire to find the distance between two points or the point halfway between two points.

**Example 9**

Plot the points  $A = (4, -2)$ ,  $B = (5, 5)$ , and  $C = (-1, 3)$  and connect them to make a triangle. Show  $\triangle ABC$  is isosceles. An isosceles triangle has at least two sides with the same length.

**Solution**

Let's start by plotting the three points on the coordinate plane and making a triangle.



We use the distance formula three times to find the lengths of the three sides of the triangle.

$$AB = \sqrt{(4 - 5)^2 + (-2 - 5)^2} = \sqrt{(-1)^2 + (-7)^2} = \sqrt{1 + 49} = \sqrt{50} = 5\sqrt{2}$$

$$BC = \sqrt{(5 + 1)^2 + (5 - 3)^2} = \sqrt{(6)^2 + (2)^2} = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10}$$

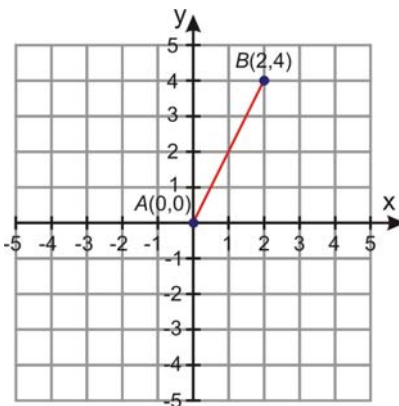
$$AC = \sqrt{(4 + 1)^2 + (-2 - 3)^2} = \sqrt{(5)^2 + (-5)^2} = \sqrt{25 + 25} = \sqrt{50} = 5\sqrt{2}$$

Notice that  $AB = AC$ , therefore  $\triangle ABC$  is isosceles.

### Example 10

At 8 AM one day, Amir decides to walk in a straight line on the beach. After two hours of making no turns and traveling at a steady rate, Amir was two mile east and four miles north of his starting point. How far did Amir walk and what was his walking speed?

### Solution



Let's start by plotting Amir's route on a coordinate graph. We can place his starting point at the origin  $A = (0,0)$ . Then, his ending point will be at point  $B = (2,4)$ . The distance can be found with the distance formula.

$$d = \sqrt{(2-0)^2 + (4-0)^2} = \sqrt{(2)^2 + (4)^2} = \sqrt{4+16} = \sqrt{20}$$

$$d = 4.47 \text{ miles.}$$

Since Amir walked 4.47 miles in 2 hours, his speed is

$$\text{Speed} = \frac{4.47 \text{ miles}}{2 \text{ hours}} = 2.24 \text{ mi/h}$$

### Review Questions

Find the distance between the two points.

- (3, -4) and (6, 0)
- (-1, 0) and (4, 2)
- (-3, 2) and (6, 2)
- (0.5, -2.5) and (4, -4)
- (12, -10) and (0, -6)
- (2.3, 4.5) and (-3.4, -5.2)
- Find all points having an  $x$  coordinate of -4 and whose distance from point (4, 2) is 10.
- Find all points having a  $y$  coordinate of 3 and whose distance from point (-2, 5) is 8.

Find the midpoint of the line segment joining the two points.

- (3, -4) and (6, 1)



10. (2, -3) and (2, 4)
11. (4, -5) and (8, 2)
12. (1.8, -3.4) and (-0.4, 1.4)
13. (5, -1) and (-4, 0)
14. (10, 2) and (2, -4)
15. An endpoint of a line segment is (4, 5) and the midpoint of the line segment is (3, -2). Find the other endpoint.
16. An endpoint of a line segment is (-10, -2) and the midpoint of the line segment is (0, 4). Find the other endpoint.
17. Plot the points  $A = (1, 0)$ ,  $B = (6, 4)$ ,  $C = (9, -2)$  and  $D = (-6, -4)$ ,  $E = (-1, 0)$ ,  $F = (2, -6)$ . Prove that triangles  $ABC$  and  $DEF$  are congruent.
18. Plot the points  $A = (4, -3)$ ,  $B = (3, 4)$ ,  $C = (-2, -1)$ ,  $D = (-1, -8)$ . Show that  $ABCD$  is a rhombus (all sides are equal).
19. Plot points  $A = (-5, 3)$ ,  $B = (6, 0)$ ,  $C = (5, 5)$ . Find the length of each side. Show that it is a right triangle. Find the area.
20. Find the area of the circle with center (-5, 4) and the point on the circle (3, 2).
21. Michelle decides to ride her bike one day. First she rides her bike due south for 12 miles, then the direction of the bike trail changes and she rides in the new direction for a while longer. When she stops, Michelle is 2 miles south and 10 miles west from her starting point. Find the total distance that Michelle covered from her starting point.

### Review Answers

1. 5
2.  $\sqrt{29}$
3. 9
4. 3.81
5.  $4\sqrt{10}$
6. 11.25
7. (-4, -4) and (-4, 8)
8. (-9.75, 3) and (5.75, 3)
9. (4.5, -1.5)
10. (2, 0.5)
11. (6, -1.5)
12. (.7, -1)
13. (0.5, -0.5)
14. (6, -1)
15. (2, -9)
16. (10, 10)
17.  $AB = DE = 6.4$ ,  $AC = DF = 8.25$ ,  $BC = EF = 6.71$
18.  $AB = BC = CD = DA = 7.07$
19.  $AB = \sqrt{130}$ ,  $AC = \sqrt{104}$ ,  $BC = \sqrt{26}$  and  $(\sqrt{26})^2 + (\sqrt{104})^2 = (\sqrt{130})^2$ . Right triangle.
20. Radius =  $2\sqrt{17}$ , Area =  $68\pi$
21. 26.14 miles