

3.1 Graphs of Quadratic Functions

Learning Objectives

- Graph quadratic functions.
- Compare graphs of quadratic functions.
- Graph quadratic functions in intercept form.
- Analyze graphs of real-world quadratic functions.

Introduction

The graphs of quadratic functions are curved lines called **parabolas**. You don't have to look hard to find parabolic shapes around you. Here are a few examples.

- The path that a ball or a rocket takes through the air.
- Water flowing out of a drinking fountain.
- The shape of a satellite dish.
- The shape of the mirror in car headlights or a flashlight.

Graph Quadratic Functions

Let's see what a parabola looks like by graphing the simplest quadratic function, $y = x^2$.

We will graph this function by making a table of values. Since the graph will be curved we need to make sure that we pick enough points to get an accurate graph.

$$y = x^2$$

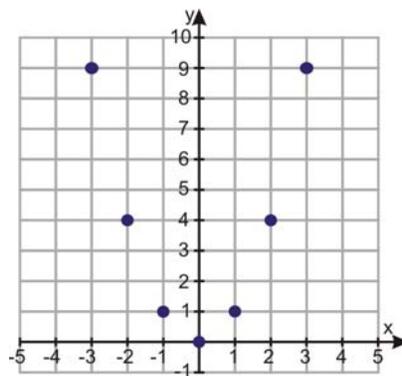
TABLE 3.1:

x	y
-3	$(-3)^2 = 9$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$

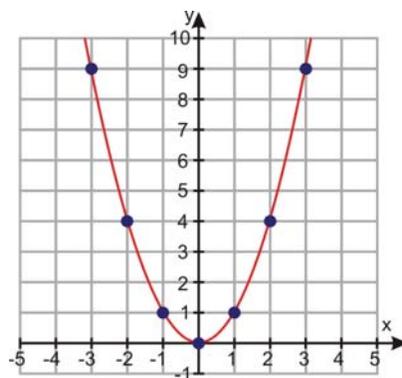
TABLE 3.1: (continued)

x	y
1	$(1)^2 = 1$
2	$(2)^2 = 4$
3	$(3)^2 = 9$

Each x -value and corresponding y -value represent an ordered pair (x,y) , which would be a point on the graph of the function. We plot these points on a coordinate graph.



To draw the parabola, draw a smooth curve through all the points. (Do not connect the points with straight lines).



Let's graph a few more examples.

Example 1

Graph the following parabolas.

a) $y = 2x^2 + 4x + 1$

b) $y = -x^2$

c) $y = x^2 - 2x + 3$

Solution

a) $y = 2x^2 + 4x + 1$

Make a table of values.

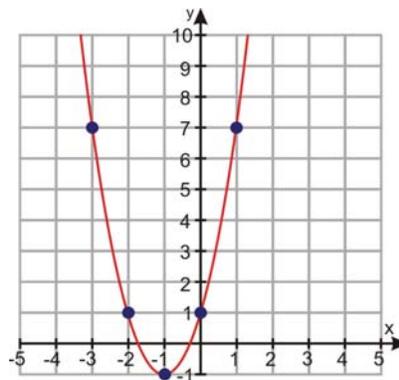
$$y = 2x^2 + 4x + 1$$

TABLE 3.2:

x	y
-3	$2(-3)^2 + 4(-3) + 1 = 7$
-2	$2(-2)^2 + 4(-2) + 1 = 1$
-1	$2(-1)^2 + 4(-1) + 1 = -1$
0	$2(0)^2 + 4(0) + 1 = 1$
1	$2(1)^2 + 4(1) + 1 = 7$
2	$2(2)^2 + 4(2) + 1 = 17$
3	$2(3)^2 + 4(3) + 1 = 31$

Each x -value and corresponding y -value represent an ordered pair (x,y) , which would be a point on the graph of the function. We plot these points on a coordinate graph.

Notice that the last two points have large y -values. We will not graph them since that will make our y -scale too big. Now plot the remaining points and join them with a smooth curve.



b) $y = -x^2 + 3$

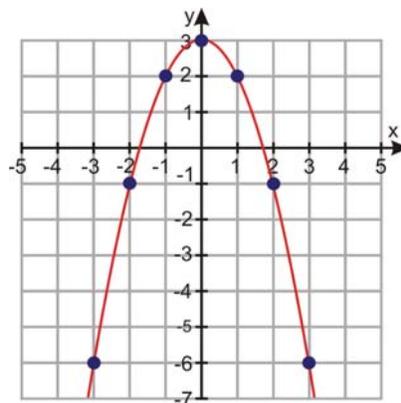
Make a table of values.

$$y = -x^2 + 3$$

TABLE 3.3:

x	y
-3	$-(-3)^2 + 3 = -6$
-2	$-(-2)^2 + 3 = -1$
-1	$-(-1)^2 + 3 = 2$
0	$-(0)^2 + 3 = 3$
1	$-(1)^2 + 3 = 2$
2	$-(2)^2 + 3 = -1$
3	$-(3)^2 + 3 = -6$

Each x -value and corresponding y -value represent an ordered pair (x,y) , which would be a point on the graph of the function. We plot these points on a coordinate graph.



Notice that this graph makes an “upside down” parabola. Our equation has a negative sign in front of the x^2 term. The sign of the coefficient of the x^2 term determines whether the parabola open up or down.

If the coefficient of x^2 ; is positive, then the parabola open up.

If the coefficient of x^2 ; is negative, then the parabola opens down.

c) $y = x^2 - 8x + 3$

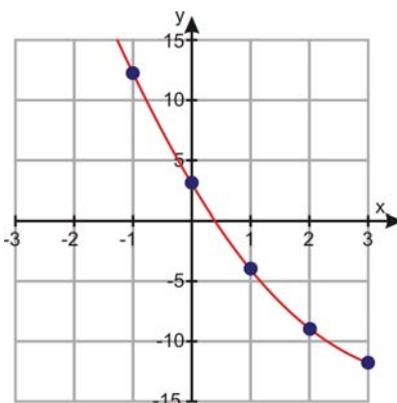
Make a table of values.

$$y = x^2 - 8x + 3$$

TABLE 3.4:

x	y
-3	$(-3)^2 - 8(-3) + 3 = 36$
-2	$(-2)^2 - 8(-2) + 3 = 23$
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$

Let's not graph the first two points in the table. Plot the other points and join them with a smooth curve.



This does not look like the graph of a parabola. What is happening here? If it is not clear what the graph looks like

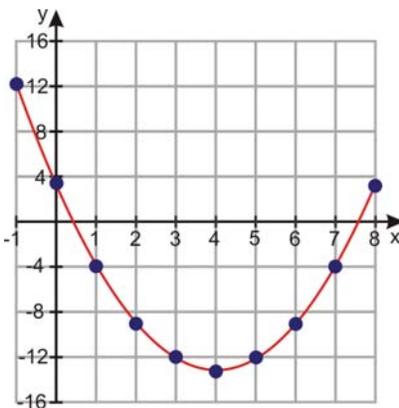
choose more points to graph until you can see a familiar curve. For negative values of x it looks like the values of y are getting bigger and bigger. Let's pick more positive values of x beyond $x = 3$.

$$y = x^2 - 8x + 3$$

TABLE 3.5:

x	y
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$
4	$(4)^2 - 8(4) + 3 = -13$
5	$(5)^2 - 8(5) + 3 = -12$
6	$(6)^2 - 8(6) + 3 = -9$
7	$(7)^2 - 8(7) + 3 = -4$
8	$(8)^2 - 8(8) + 3 = 3$

Plot the points again and join them with a smooth curve.



We now see the familiar parabolic shape. Graphing by making a table of values can be very tedious, especially in problems like this example. In the next few sections, we will learn some techniques that will simplify this process greatly, but first we need to learn more about the properties of parabolas.

Compare Graphs of Quadratic Functions

The **general form** (or **standard form**) of a quadratic function is:

$$y = ax^2 + bx + c$$

Here a , b and c are the **coefficients**. Remember a coefficient is just a number or constant factor of a term, which is to the left of a variable or it can be alone. You should know that if you have a quadratic function, its graph is always a parabola. While the graph of a quadratic is always the same basic shape, we have different situations where the graph could be upside down. It could be shifted to different locations or it could be “wider” or “narrower”. These situations are determined by the values of the coefficients. Let’s see how changing the coefficient changes the orientation, location or shape of the parabola.

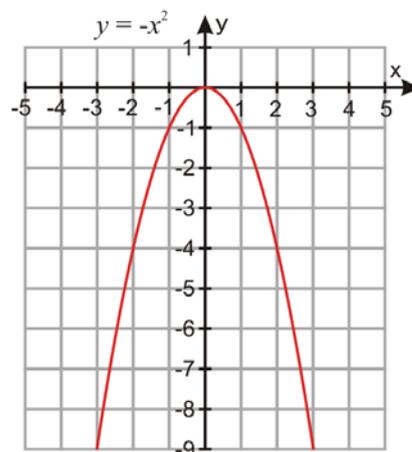
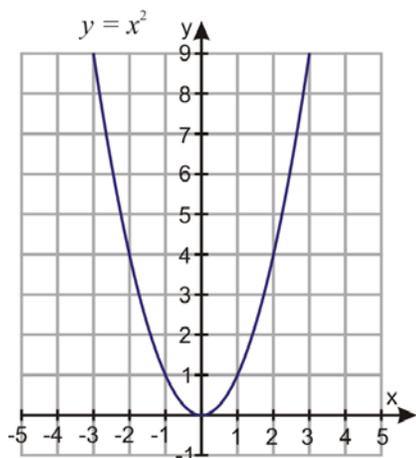
Orientation

Does the parabola open up or down?

The answer to that question is pretty simple:

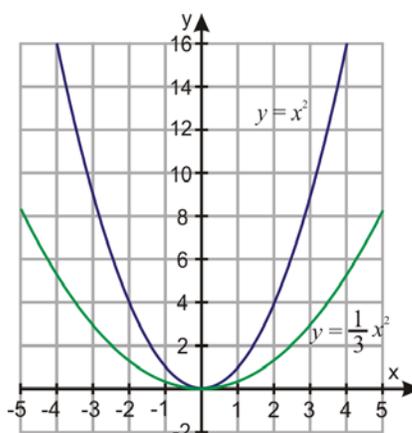
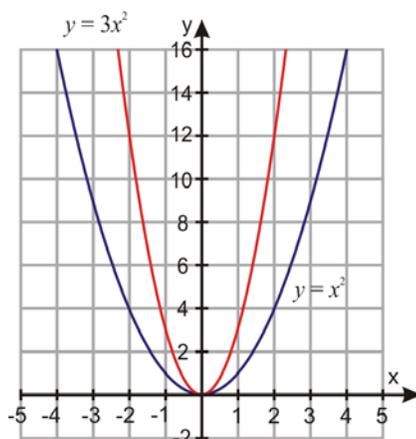
- If a is positive, the parabola opens up.
- If a is negative, the parabola opens down.

The following plots show the graphs of $y = x^2$ and $y = -x^2$. You see that the parabola has the same shape in both graphs, but the graph of $y = x^2$ opens up and the graph of $y = -x^2$ opens down.



Dilation

Changing the value of the coefficient a makes the graph “wider” or “narrower”. Let’s look at how graphs compare for different positive values of a . On the left shows the graphs of $y = -x^2$ and $y = 3x^2$. On the right shows the graphs of $y = -x^2$ and $y = (\frac{1}{3})x^2$.



Notice that the larger the value of a is, the narrower or skinnier the graph is. For example, in the left graph, $y = 3x^2$ is narrower or skinnier than the graph of $y = x^2$. Also, the smaller a is (i.e. the closer to 0), the wider or fatter the graph is. In the right graph $y = (\frac{1}{3})x^2$ is wider or fatter than the graph of $y = x^2$. This might seem counter-intuitive, but if you think about it, it should make sense. Let’s look at a table of values of these graphs and see if we can explain why this happens.

TABLE 3.6:

x	$y = x^2$	$y = 3x^2$	$y = \frac{1}{3}x^2$
-3	$(-3)^2 = 9$	$3(-3)^2 = 27$	$\frac{1}{3}(-3)^2 = 3$
-2	$(-2)^2 = 4$	$3(-2)^2 = 12$	$\frac{1}{3}(-2)^2 = \frac{4}{3}$

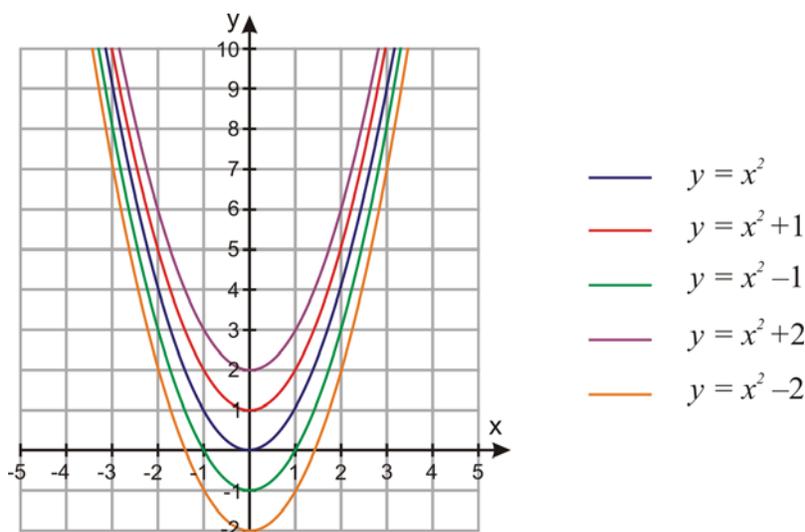
TABLE 3.6: (continued)

x	$y = x^2$	$y = 3x^2$	$y = \frac{1}{3}x^2$
-1	$(-1)^2 = 1$	$3(-1)^2 = 3$	$\frac{1}{3}(-1)^2 = \frac{1}{3}$
0	$(0)^2 = 0$	$3(0)^2 = 0$	$\frac{1}{3}(0)^2 = 0$
1	$(1)^2 = 1$	$3(1)^2 = 3$	$\frac{1}{3}(1)^2 = \frac{1}{3}$
2	$(2)^2 = 4$	$3(2)^2 = 12$	$\frac{1}{3}(2)^2 = \frac{4}{3}$
3	$(3)^2 = 9$	$3(3)^2 = 27$	$\frac{1}{3}(3)^2 = 3$

From the table, you can see that the values of $y = 3x^2$ are bigger than the values of $y = x^2$. This is because each value of y gets multiplied by 3. As a result, the parabola will be skinnier or narrower because the y -value or function value grows three times faster than $y = x^2$. On the other hand, you can see that the values of $y = \left(\frac{1}{3}\right)x^2$ are smaller than the values of $y = x^2$. This is because each value of y gets divided by 3. As a result, the parabola will be fatter or wider because the y -value or function value grows at one third the rate of $y = x^2$.

Vertical Shift

Changing the value of the coefficient c (called the constant term) has the effect of moving the parabola up and down. The following shows the graphs of $y = x^2$, $y = x^2 + 1$, $y = x^2 - 1$, $y = x^2 + 2$, $y = x^2 - 2$.



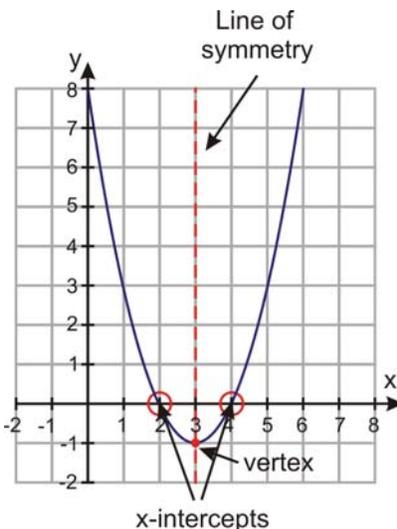
We see that if c is positive, the graph moves up by c units. If c is negative, the graph moves down by c units. In one of the later sections we will also talk about **horizontal shift (i.e. moving to the right or to the left)**. Before we can do that we need to learn how to rewrite the quadratic equations in different forms.

Graph Quadratic Functions in Intercept Form

As you saw, in order to get a good graph of a parabola, we sometimes need to pick a lot of points in our table of values. Now, we will talk about different properties of a parabola that will make the graphing process less tedious. Let's look at the graph of $y = x^2 - 6x + 8$.

There are several things that we notice.

- The parabola crosses the x axis at two points: $x = 2$ and $x = 4$.
 - These points are called the x -**intercepts** of the parabola.
- The lowest point of the parabola occurs at point $(3, -1)$.
 - This point is called the **vertex** of the parabola.
 - The vertex is the lowest point in a parabola that opens upward, and it is the highest point in a parabola that opens downward.
 - The vertex is exactly halfway between the two x -intercepts. This will always be the case and you can find the vertex using that rule.



3. A parabola has **symmetry**. If you draw a vertical line through the vertex, you can see that the two halves of the parabola are mirror images of each other. The vertical line is called the **line of symmetry**.

We said that the general form of a quadratic function is $y = ax^2 + bx + c$. If we can factor the quadratic expression, we can rewrite the function in **intercept form**

$$y = a(x - m)(x - n)$$

This form is very useful because it makes it easy for us to find the x -intercepts and the vertex of the parabola. The x -intercepts are the values of x where the graph crosses the x -axis. In other words, they are the values of x when $y = 0$. To find the x -intercepts from the quadratic function, we set $y = 0$ and solve.

$$0 = a(x - m)(x - n)$$

Since the equation is already factored, we use the zero-product property to set each factor equal to zero and solve the individual linear equations.

$$\begin{array}{ccc} x - m = 0 & & x - n = 0 \\ & \text{or} & \\ x = m & & x = n \end{array}$$

So the x -intercepts are at points $(m, 0)$ and $(n, 0)$.

Once we find the x -intercepts, it is simple to find the vertex. The x -coordinate of the vertex is halfway between the two x intercepts, so we can find it by taking the average of the two values $\frac{(m+n)}{2}$.

The y -value of the vertex can be found by substituting the average of the x -intercepts back into the equation of the function.

Let's do some examples that find the x -intercepts and the vertex:

Example 2

Find the x -intercepts and the vertex of the following quadratic function.

(a) $y = x^2 - 8x + 15$

(b) $y = 3x^2 + 6x - 24$

Solution

a) $y = x^2 - 8x + 15$

Write the quadratic function in intercept form by factoring the right hand side of the equation.

Remember, to factor the trinomial we need two numbers whose product is 15 and whose sum is -8. These numbers are -5 and -3.

The function in intercept form is $y = (x - 5)(x - 3)$

We find the x -intercepts by setting $y = 0$.

We have

$$\begin{array}{ccc} 0 = (x - 5)(x - 3) & & \\ x - 5 = 0 & & x - 3 = 0 \\ & \text{or} & \\ x = 5 & & x = 3 \end{array}$$

The x -intercepts are (5, 0) and (3, 0).

The vertex is halfway between the two x -intercepts. We find the x value by taking the average of the two x -intercepts, $x = \frac{(5+3)}{2} = 4$.

We find the y value by substituting the x value we just found back into the original equation.

$$y = x^2 - 8x + 15 \Rightarrow y = (4)^2 - 8(4) + 15 = 16 - 32 + 15 = -1$$

The vertex is (4, -1).

b) $y = 3x^2 + 6x - 24$

Rewrite the function in intercept form.

Factor out the common term of 3 first $y = 3(x^2 + 2x - 8)$.

Then factor completely $y = 3(x + 4)(x - 2)$

Set $y = 0$ and solve: $0 = 3(x + 4)(x - 2)$.

$$x + 4 = 0 \Rightarrow x = -4$$

or

$$x - 2 = 0 \Rightarrow x = 2$$

The x -intercepts are: $(-4, 0)$ and $(2, 0)$

For the vertex,

$$x = \frac{-4+2}{2} = -1 \text{ and } y = 3(-1)^2 + 6(-1) - 24 = 3 - 6 - 24 = -27$$

The vertex is: $(-1, -27)$.

When graphing, it is very useful to know the vertex and x -intercepts. Knowing the vertex, tells us the lowest or highest point of the parabola. The y -value or function value at this point is called the minimum or maximum. When making a table of values we pick the vertex as a point in the table. Then we choose a few smaller and larger values of x . In this way, we get an accurate graph of the quadratic function without having to have too many points in our table.

Example 3

Find the x -intercepts and vertex. Use these points to create a table of values and graph each function.

a) $y = x^2 - 4$

b) $y = -x^2 + 14x - 48$

Solution

a) $y = x^2 - 4$

Let's find the x -intercepts and the vertex.

Factor the right-hand-side of the function to put the equation in intercept form.

$$y = (x - 2)(x + 2)$$

Set $y = 0$ and solve.

$$0 = (x - 2)(x + 2)$$

$$x - 2 = 0$$

$$x = 2$$

$$x + 2 = 0$$

or

$$x = -2$$

x -intercepts are: $(2, 0)$ and $(-2, 0)$

Find the vertex.

$$x = \frac{2-2}{2} = 0$$

$$y = (0)^2 - 4 = -4$$

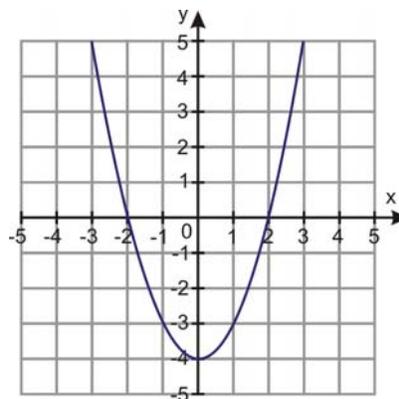
The vertex is (0, -4)

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 0$. Include the x -intercepts in the table.

$$y = x^2 - 4$$

TABLE 3.7:

	x	y
	-3	$y = (-3)^2 - 4 = 5$
x -intercept	-2	$y = (-2)^2 - 4 = 0$
	-1	$y = (-1)^2 - 4 = -3$
vertex	0	$y = (0)^2 - 4 = -4$
	1	$y = (1)^2 - 4 = -3$
x -intercept	2	$y = (2)^2 - 4 = 0$
	3	$y = (3)^2 - 4 = 5$



b) $y = -x^2 + 14x - 48$

Let's find the x -intercepts and the vertex.

Factor the right hand side of the function to put the equation in intercept form.

$$y = -(x^2 - 14x + 48) = -(x - 6)(x - 8)$$

Set $y = 0$ and solve.

$$0 = -(x - 6)(x - 8)$$

$$x - 6 = 0$$

$$x = 6$$

$$x - 8 = 0$$

or

$$x = 8$$

The x -intercepts are: (6, 0) and (8, 0)

Find the vertex

$$x = \frac{6+8}{2} = 7$$

$$y = (7)^2 + 14(7) - 48 = 1$$

The vertex is (7, 1).

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 7$. Include the x -intercepts in the table. Then graph the parabola.

$$y = -x^2 + 14x - 48$$

TABLE 3.8:

	x	$y = -x^2 + 14x - 48$
	4	$y = -(4)^2 + 14(4) - 48 = -8$
	5	$y = -(5)^2 + 14(5) - 48 = -3$
x - intercept	6	$y = -(6)^2 + 14(6) - 48 = 0$
x - vertex	7	$y = -(7)^2 + 14(7) - 48 = 1$
x - intercept	8	$y = -(8)^2 + 14(8) - 48 = 0$
	9	$y = -(9)^2 + 14(9) - 48 = -3$
	10	$y = -(10)^2 + 14(10) - 48 = -8$

As we mentioned at the beginning of this section, parabolic curves are common in real-world applications. Here we will look at a few graphs that represent some examples of real-life application of quadratic functions.

Example 4 Area

Andrew has 100 feet of fence to enclose a rectangular tomato patch. He wants to find the dimensions of the rectangle that encloses most area.

Solution

We can find an equation for the area of the rectangle by looking at a sketch of the situation.

Let x be the length of the rectangle and y be the width.

The perimeter of a rectangle is:

$$P = 2(\text{length}) + 2(\text{width})$$

in this case we have

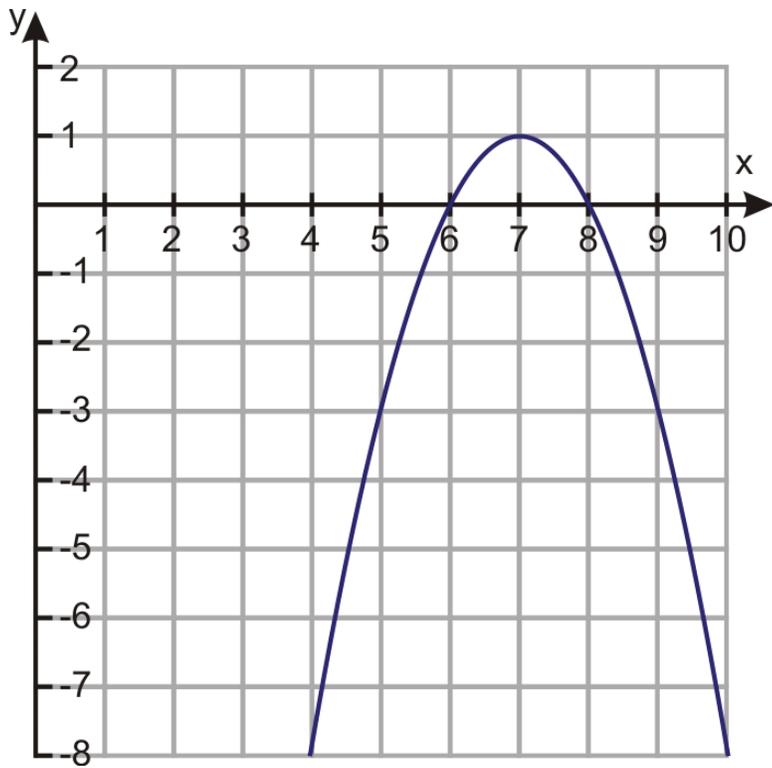


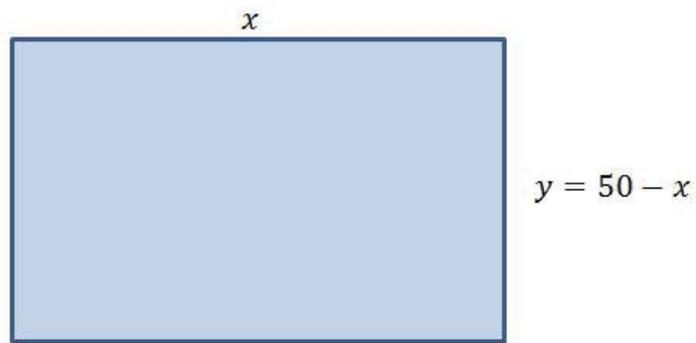
FIGURE 3.1

$$P = 2x + 2y$$

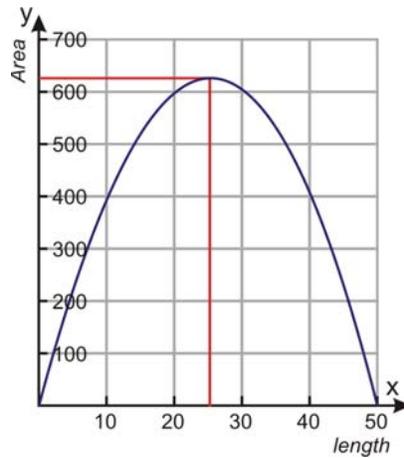
$$100 = 2x + 2y$$

dividing both sides by 2

$$50 = x + y \text{ or } y = 50 - x$$



The following graph shows the graph of $A = x(50 - x)$. Notice the vertex gives us the value of x that maximizes the area and the maximum area.



We can see from the graph that the highest value of the area occurs when the length of the rectangle is 25. The area of the rectangle for this side length equals 625. Notice that the width is also 25, which makes the shape a square with side length 25.

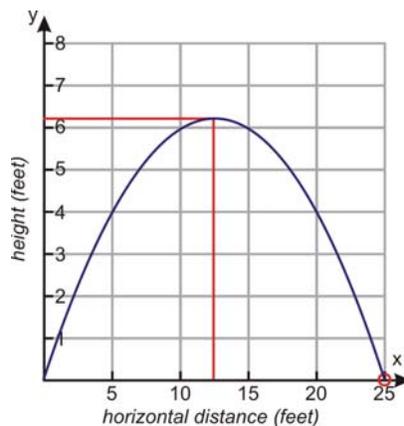
This is an example of an optimization problem.

Example 5 Projectile motion

Anne is playing golf. On the 4th tee, she hits a slow shot down the level fairway. The ball follows a parabolic path described by the equation, $y = x - 0.04x^2$. This relates the height of the ball y to the horizontal distance as the ball travels down the fairway. The distances are measured in feet. How far from the tee does the ball hit the ground? At what distance, x from the tee, does the ball attain its maximum height? What is the maximum height?

Solution

Let's graph the equation of the path of the ball: $y = x - 0.04x^2$.



From the graph, we see that the ball hits the ground 25 feet from the tee. We see that the maximum height is attained at 12.5 feet from the tee and the maximum height the ball reaches is 6.25 feet.

Review Questions

Rewrite the following functions in intercept form. Find the x -intercepts and the vertex.

1. $f(x) = x^2 - 2x - 8$
2. $f(x) = -x^2 + 10x - 21$
3. $f(x) = 2x^2 + 6x + 4$

Does the graph of the parabola turn up or down?

4. $f(x) = -2x^2 - 2x - 3$

5. $f(x) = 3x^2$

6. $f(x) = 16 - 4x^2$

The vertex of which parabola is higher?

7. $f(x) = x^2$ or $f(x) = 4x^2$

8. $f(x) = -2x^2$ or $f(x) = -2x^2 - 2$

9. $f(x) = 3x^2 - 3$ or $f(x) = 3x^2 - 6$

Which parabola is wider?

10. $f(x) = x^2$ or $f(x) = 4x^2$

11. $f(x) = 2x^2 + 4$ or $f(x) = \frac{1}{2}x^2 + 4$

12. $f(x) = -2x^2 - 2$ or $f(x) = -x^2 - 2$

Graph the following functions by making a table of values. Use the vertex and x -intercepts to help you pick values for the table.

13. $f(x) = 4x^2 - 4$

14. $f(x) = -x^2 + x + 12$

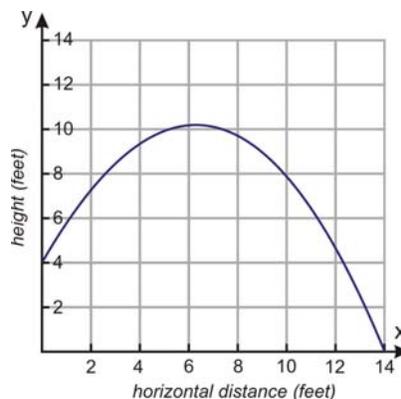
15. $f(x) = 2x^2 + 10x + 8$

16. $f(x) = \frac{1}{2}x^2 - 2x$

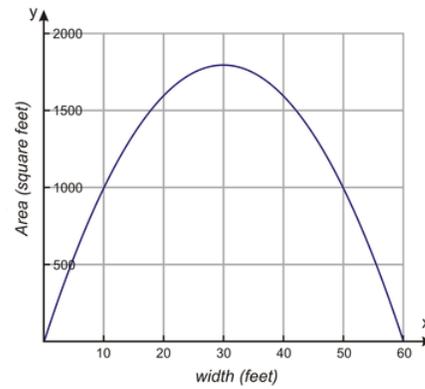
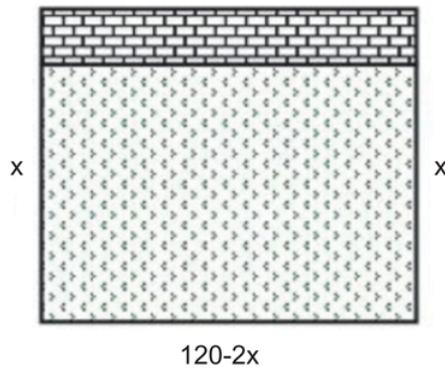
17. $f(x) = x - 2x^2$

18. $f(x) = 4x^2 - 8x + 4$

19. Nadia is throwing a ball to Peter. Peter does not catch the ball and it hits the ground. The graph shows the path of the ball as it flies through the air. The equation that describes the path of the ball is $y = 4 + 2x - 0.16x^2$. Here y is the height of the ball and x is the horizontal distance from Nadia. Both distances are measured in feet. How far from Nadia does the ball hit the ground? At what distance, x from Nadia, does the ball attain its maximum height? What is the maximum height?

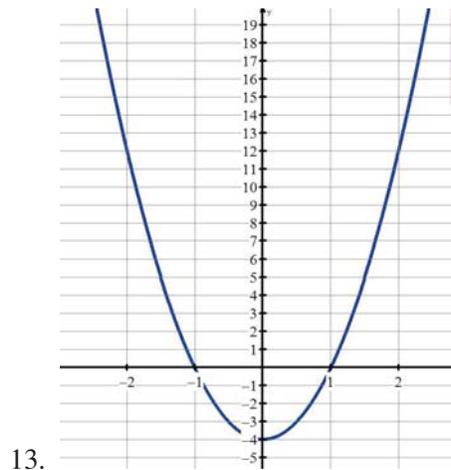


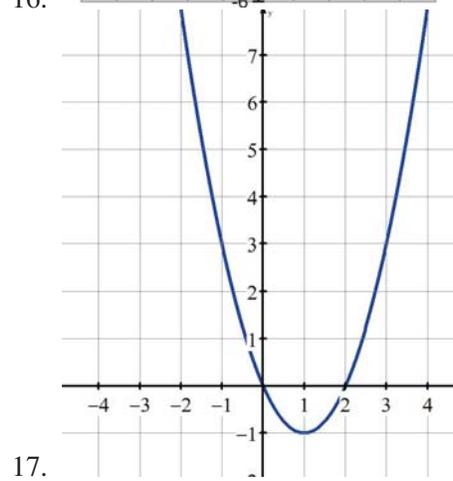
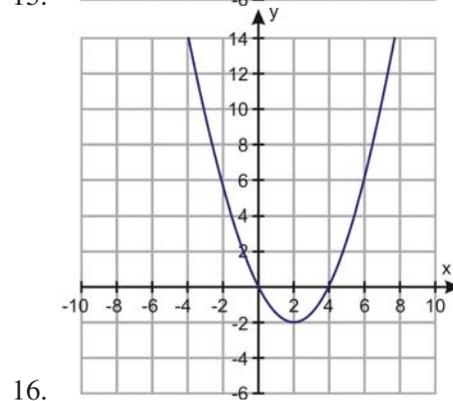
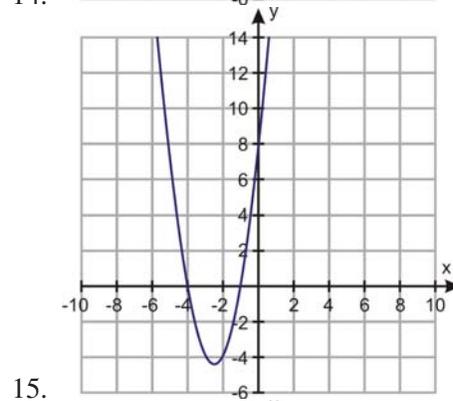
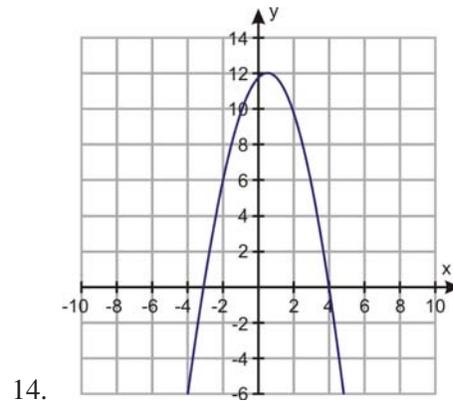
20. Peter wants to enclose a vegetable patch with 120 feet of fencing. He wants to put the vegetable against an existing wall, so he only needs fence for three of the sides. The equation for the area is given by $a = 120x - x^2$. From the graph find what dimensions of the rectangle would give him the greatest area.

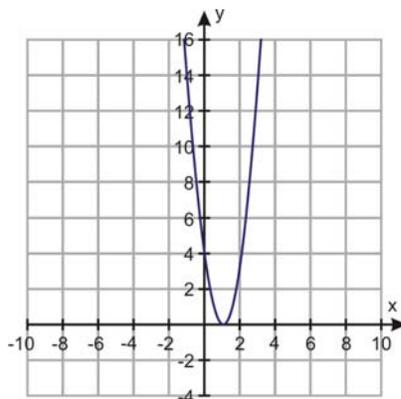


Review Answers

1. $x = -2, x = 4$ Vertex $(1, -9)$
2. $x = 3, x = 7$ Vertex $(5, 4)$
3. $x = -2, x = -1$ Vertex $(-3.5, 7.5)$
4. Down
5. Up
6. Down
7. $f(x) = x^2 + 4$
8. $f(x) = -2x^2$
9. $f(x) = 3x^2 - 3$
10. $f(x) = x^2$
11. $f(x) = \left(\frac{1}{2}\right)x^2 + 4$
12. $f(x) = -x^2 - 2$







18.

19. 14.25 feet, 6.25 feet, 10.25 feet

20. width = 30 feet, length = 60 feet