

## 3.4 Solving Quadratic Equations by Completing the Square

### Learning objectives

- Complete the square of a quadratic expression.
- Solve quadratic equations by completing the square.
- Solve quadratic equations in standard form.
- Graph quadratic equations in vertex form.
- Solve real-world problems using functions by completing the square.

### Introduction

You saw in the last section that if you have a quadratic equation of the form

$$(x - 2)^2 = 5$$

We can easily solve it by taking the square root of each side.

$$x - 2 = \sqrt{5} \text{ and } x - 2 = -\sqrt{5}$$

Then simplify and solve.

$$x = 2 + \sqrt{5} \approx 4.24 \text{ and } x = 2 - \sqrt{5} \approx -0.24$$

Unfortunately, quadratic equations are not usually written in this nice form. In this section, you will learn the method of **completing the square** in which you take any quadratic equation and rewrite it in a form so that you can take the square root of both sides.

### Complete the Square of a Quadratic Expression

The purpose of the method of completing the square is to rewrite a quadratic expression so that it contains a perfect square trinomial that can be factored as the square of a binomial. Here is an example of squaring binomials.

$$(x + a)^2 = x^2 + 2ax + a^2$$

$$(x - a)^2 = x^2 - 2ax + a^2$$

In order to have a perfect square trinomial, we need two terms that are perfect squares ( $x^2$  and  $a^2$ ) and one term that is twice the product of the square roots of the other terms ( $2ax$  and  $-2ax$ ).

#### Example 1

Complete the square for the quadratic expression  $x^2 + 4x$ .

**Solution** To complete the square, we need a constant term that turns the expression into a perfect square trinomial. Since the middle term in a perfect square trinomial is always two times the product of the square roots of the other two terms, we rewrite our expression as

$$x^2 + 2(2)(x)$$

We see that the constant we are seeking must be  $2^2$ .

$$x^2 + 2(2)(x) + 2^2$$

**Answer** By adding 4, this can be factored as:  $(x + 2)^2$

BUT, we just changed the value of this expression  $x^2 + 4x \neq (x + 2)^2$ . Later we will show how to account for this problem.

Also, this was a relatively easy example because  $a$ , the coefficient of the  $x^2$  term was 1. If  $a \neq 1$ , we must factor  $a$  from the whole expression before completing the square.

**Example 2**

Complete the square for the quadratic expression  $4x^2 + 32x$

**Solution**

Factor the coefficient of the  $x^2$  term.

$$4(x^2 + 8x)$$

Now complete the square of the expression in parentheses then rewrite the expression.

$$4(x^2 + 2(4)(x))$$

We complete the square by adding the constant  $4^2$ .

$$4(x^2 + 2(4)(x) + 4^2)$$

Factor the perfect square trinomial inside the parenthesis.

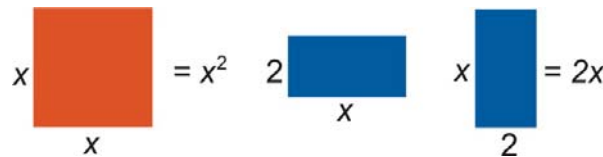
$$4(x + 4)^2$$

Our answer is  $4(x + 4)^2$ .

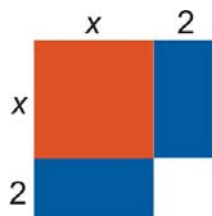
The expression “**completing the square**” comes from a geometric interpretation of this situation. Let’s revisit the quadratic expression in Example 1.

$$x^2 + 4x$$

We can think of this expression as the sum of three areas. The first term represents the area of a square of side  $x$ . The second expression represents the areas of two rectangles with a length of 2 and a width of  $x$ :

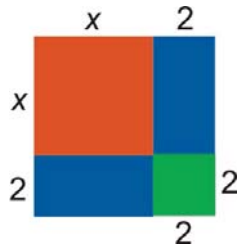


We can combine these shapes as follows



We obtain a square that is not quite complete.

In order to complete the square, we need a square of side 2.



We obtain a square of side  $x + 2$ .

The area of this square is:  $(x + 2)^2$ .

You can see that completing the square has a geometric interpretation.

Finally, here is the algebraic procedure for completing the square.

$$\begin{aligned}x^2 + bx + c &= 0 \\x^2 + bx &= -c \\x^2 + bx + \left(\frac{b}{2}\right)^2 &= -c + \left(\frac{b}{2}\right)^2 \\ \left(x + \frac{b}{2}\right)^2 &= -c + \left(\frac{b}{2}\right)^2\end{aligned}$$

Notice:  $\left(\frac{b}{2}\right)^2$  was added to **both** sides of the equation!

### Solve Quadratic Equations by Completing the Square

Let's demonstrate the method of **completing the square** with an example.

#### Example 3

Solve the following quadratic equation  $x^2 + 12x = 3$ .

#### Solution

The method of completing the square is as follows.

1. Rewrite as  $x^2 + 2(6)x = 3$
2. In order to have a perfect square trinomial on the right-hand-side we need to add the constant  $6^2$ . Add this constant to both sides of the equation.

$$x^2 + 2(6)(x) + 6^2 = 3 + 6^2$$

3. Factor the perfect square trinomial and simplify the right hand side of the equation.

$$(x + 6)^2 = 39$$

4. Take the square root of both sides.

$$x + 6 = \sqrt{39}$$

$$x = -6 + \sqrt{39} \approx 0.24$$

and  
and

$$x + 6 = -\sqrt{39}$$

$$x = -6 - \sqrt{39} \approx -12.24$$

**Answer**  $x = 0.24$  and  $x = -12.24$

If the coefficient of the  $x^2$  term is not one, we must divide that number from the whole expression before completing the square.

#### Example 4

Solve the following quadratic equation  $3x^2 - 10x = -1$ .

**Solution:**

1. Divide all terms by the coefficient of the  $x^2$  term.

$$x^2 - \frac{10}{3}x = -\frac{1}{3}$$

2. Rewrite as

$$x^2 - 2\left(\frac{5}{3}\right)(x) = -\frac{1}{3}$$

3. In order to have a perfect square trinomial on the right hand side we need to add the constant  $\left(\frac{5}{3}\right)^2$ . Add this constant to both sides of the equation.

$$x^2 - 2\left(\frac{5}{3}\right)(x) + \left(\frac{5}{3}\right)^2 = -\frac{1}{3} + \left(\frac{5}{3}\right)^2$$

4. Factor the perfect square trinomial and simplify.

$$\left(x - \frac{5}{3}\right)^2 = -\frac{1}{3} + \frac{25}{9}$$

$$\left(x - \frac{5}{3}\right)^2 = \frac{22}{9}$$

5. Take the square root of both sides.

$$x - \frac{5}{3} = \sqrt{\frac{22}{9}} \quad \text{and} \quad x - \frac{5}{3} = -\sqrt{\frac{22}{9}}$$

$$x = \frac{5}{3} + \sqrt{\frac{22}{9}} \approx 3.23 \quad \text{and} \quad x = \frac{5}{3} - \sqrt{\frac{22}{9}} \approx 0.1$$

**Answer**  $x = 3.23$  and  $x = 0.1$

### Solve Quadratic Equations in Standard Form

An equation in standard form is written as  $ax^2 + bx + c = 0$ . We solve an equation in this form by the method of completing the square. First we move the constant term to the right hand side of the equation.

#### Example 5

Solve the following quadratic equation  $x^2 + 15x + 12 = 0$ .

#### Solution

The method of completing the square is as follows:

1. Move the constant to the other side of the equation.

$$x^2 + 15x = -12$$

2. Rewrite as

$$x^2 + 2\left(\frac{15}{2}\right)(x) = -12$$

3. Add the constant  $\left(\frac{15}{2}\right)^2$  to both sides of the equation

$$x^2 + 2\left(\frac{15}{2}\right)(x) + \left(\frac{15}{2}\right)^2 = -12 + \left(\frac{15}{2}\right)^2$$

4. Factor the perfect square trinomial and simplify.

$$\begin{aligned}\left(x + \frac{15}{2}\right)^2 &= -12 + \frac{225}{4} \\ \left(x + \frac{15}{2}\right)^2 &= \frac{177}{4}\end{aligned}$$

5. Take the square root of both sides.

$$\begin{aligned}x + \frac{15}{2} &= \sqrt{\frac{177}{4}} && \text{and} && x + \frac{15}{2} &= -\sqrt{\frac{177}{4}} \\ x + \frac{15}{2} + \sqrt{\frac{177}{4}} &\approx -0.85 && \text{and} && x + \frac{15}{2} + \sqrt{\frac{177}{4}} &\approx -14.15\end{aligned}$$

**Answer**  $x = -0.85$  and  $x = -14.15$

### Graph Quadratic Functions in Vertex Form

Probably one of the best applications of the method of completing the square is using it to rewrite a quadratic function in vertex form.

The **vertex form** of a quadratic function is  $y - k = a(x - h)^2$ . Sometimes vertex form is also written as  $y = a(x - h)^2 + k$

This form is very useful for graphing because it gives the vertex of the parabola explicitly. The vertex is at point  $(h, k)$ .

It is also simple to find the  $x$ -intercepts from the vertex form by setting  $y = 0$  and taking the square root of both sides of the resulting equation.

The  $y$ -intercept can be found by setting  $x = 0$  and simplifying.

### Example 6

Find the vertex, the  $x$ -intercepts and the  $y$ -intercept of the following parabolas.

(a)  $y - 2 = (x - 1)^2$

(b)  $y + 8 = 2(x - 3)^2$

### Solution

a)  $y - 2 = (x - 1)^2$

Vertex is  $(1, 2)$

To find  $x$ -intercepts,

$$\begin{array}{llll} \text{Set } y = 0 & -2 = (x - 1)^2 & & \\ \text{Take the square root of both sides} & \sqrt{-2} = x - 1 & \text{and} & -\sqrt{-2} = x - 1 \end{array}$$

The solutions are not real (because you cannot take the square root of a negative number), so there are **no**  $x$ -intercepts.

To find  $y$ -intercept,

$$\begin{array}{ll} \text{Set } x = 0 & y - 2 = (-1)^2 \\ \text{Simplify} & y - 2 = 1 \Rightarrow y = 3 \end{array}$$

b)  $y + 8 = 2(x - 3)^2$

$$\begin{array}{ll} \text{Rewrite} & y - (-8) = 2(x - 3)^2 \\ \text{Vertex is} & (3, -8) \end{array}$$

To find  $x$ -intercepts,

$$\begin{array}{llll} \text{Set } y = 0 : & 8 = 2(x - 3)^2 & & \\ \text{Divide both sides by 2.} & 4 = (x - 3)^2 & & \\ \text{Take the square root of both sides :} & 2 = x - 3 & \text{and} & -2 = x - 3 \\ \text{Simplify :} & x = 5 & \text{and} & x = 1 \end{array}$$

To find the  $y$ -intercept,

Set  $x = 0$ .

$y + 8 = 2(-3)^2$

Simplify :

$y + 8 = 18 \Rightarrow y = 10$

To graph a parabola, we only need to know the following information.

- The coordinates of the vertex.
- The  $x$ -intercepts.
- The  $y$ -intercept.
- Whether the parabola turns up or down. Remember that if  $a > 0$ , the parabola turns up and if  $a < 0$  then the parabola turns down.

**Example 7**

Graph the parabola given by the function  $y + 1 = (x + 3)^2$ .

**Solution**

Rewrite.

$y - (-1) = (x - (-3))^2$

Vertex is

$(-3, -1)$

To find the  $x$ -intercepts

Set  $y = 0$

$1 = (x + 3)^2$

Take the square root of both sides

$1 = x + 3$

and

$-1 = x + 3$

Simplify

$x = -2$

and

$x = -4$

$x$ -intercepts:  $(-2, 0)$  and  $(-4, 0)$

To find the  $y$ -intercept

Set  $x = 0$

$y + 1 = (3)^2$

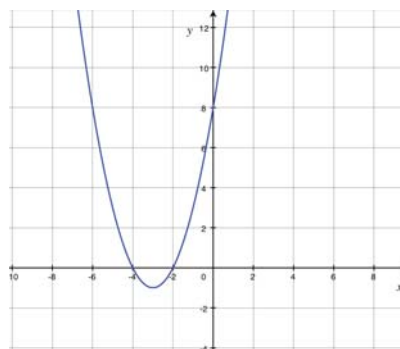
Simplify

$y = 8$

$y$ -intercept :  $(0, 8)$

Since  $a > 0$ , the parabola **turns up or opens up**.

Graph all the points and connect them with a smooth curve.



**Example 8**

Graph the parabola given by the function  $y = -\frac{1}{2}(x - 2)^2$

**Solution:**

Re-write

$$y - (0) = -\frac{1}{2}(x - 2)^2$$

Vertex is

$$(2, 0)$$

To find the  $x$ -intercepts,

Set  $y = 0$ .

$$0 = -\frac{1}{2}(x - 2)^2$$

Multiply both sides by  $-2$ .

$$0 = (x - 2)^2$$

Take the square root of both sides.

$$0 = x - 2$$

Simplify.

$$x = 2$$

$x$ -**intercept** (2, 0)

Note: there is only one  $x$ -intercept, indicating that the vertex is located at this point (2, 0).

To find the  $y$ -intercept

Set  $x = 0$

$$y = -\frac{1}{2}(-2)^2$$

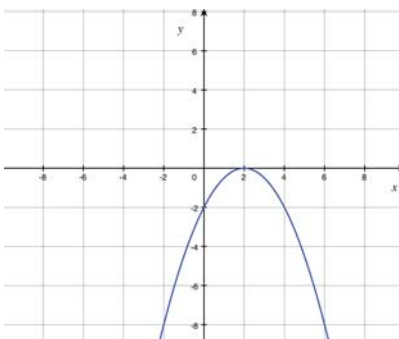
Simplify

$$y = -\frac{1}{2}(4) \Rightarrow y = -2$$

$y$ -**intercept** (0, -2)

Since  $a < 0$ , the parabola **turns down or opens down**.

Graph all the points and connect them with a smooth curve.



### Solve Real-World Problems Using Quadratic Functions by Completing the Square

#### Projectile motion with vertical velocity

In the last section you learned that an object that is dropped falls under the influence of gravity. The equation for its height ( $y$ ) with respect to time ( $t$ ) is given by



$$y = \frac{1}{2}gt^2 + y_0$$

The term  $y_0$  represents the initial height of the object and the coefficient of gravity on earth is given by  $g = -9.8 \text{ m/s}^2$  or  $g = -32 \text{ ft/s}^2$ .

On the other hand, if an object is thrown straight up or straight down in the air, it has an initial vertical velocity. This term is usually represented by the notation  $v_{0y}$ . Its value is positive if the object is thrown up in the air, and, it is negative if the object is thrown down. The equation for the height of the object in this case is given by the equation

$$y = \frac{1}{2}gt^2 + v_{0y}t + y_0$$

There are two choices for the equation to use in these problems.

$$y = -4.9t^2 + v_{0y}t + y_0$$

If you wish to have the height in meters.

$$y = -16t^2 + v_{0y}t + y_0$$

If you wish to have the height in feet.

### Example 9

An arrow is shot straight up from a height of 2 meters with a velocity of 50 m/s.

- How high will an arrow be four seconds after being shot? After eight seconds?
- At what time will the arrow hit the ground again?
- What is the maximum height that the arrow will reach and at what time will that happen?

### Solution

Since we are given the velocity in meters per second, use the equation  $y = -4.9t^2 + v_{0y}t + y_0$

We know  $v_{0y} = 50 \text{ m/s}$  and  $y_0 = 2 \text{ meters}$  so,  $y = -4.9t^2 + 50t + 2$

- a) To find how high the arrow will be 4 seconds after being shot we substitute 4 for  $t$

$$\begin{aligned} y &= -4.9(4)^2 + 50(4) + 2 \\ &= -4.9(16) + 200 + 2 = 123.6 \text{ meters} \end{aligned}$$

we substitute  $t = 8$

$$\begin{aligned} y &= -4.9(8)^2 + 50(8) + 2 \\ &= -4.9(64) + 400 + 2 = 88.4 \text{ meters} \end{aligned}$$

- b) The height of the arrow on the ground is  $y = 0$ , so  $0 = -4.9t^2 + 50t + 2$

Solve for $t$ by completing the square	$-4.9t^2 + 50t = -2$
Factor out $-4.9$	$-4.9(t^2 - 10.2t) = -2$
Divide both sides by	$t^2 - 10.2t = 0.41$
Add $5.1^2$ to both sides	$t^2 - 2(5.1)t + (5.1)^2 = 0.41 + (5.1)^2$
Factor	$(t - 5.1)^2 = 26.43$
Solve	$t - 5.1 \approx 5.14$ and $t - 5.1 \approx -5.14$
	$t \approx 10.2 \text{ sec}$ and $t \approx -0.04 \text{ sec}$

c) If we graph the height of the arrow with respect to time, we would get an upside down parabola ( $a < 0$ ). The maximum height and the time when this occurs is really the vertex of this parabola ( $t, h$ ).

We rewrite the equation in vertex form.	$y = -4.9t^2 + 50t + 2$
	$y - 2 = -4.9t^2 + 50t$
	$y - 2 = -4.9(t^2 - 10.2t)$
Complete the square inside the parenthesis.	$y - 2 - 4.9(5.1)^2 = -4.9(t^2 - 10.2t + (5.1)^2)$
	$y - 129.45 = -4.9(t - 5.1)^2$

The vertex is at  $(5.1, 129.45)$ . In other words, **when**  $t = 5.1 \text{ seconds}$ , **the height is**  $y = 129 \text{ meters}$ .

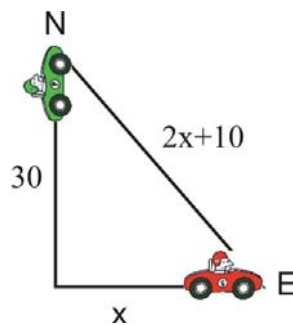
Another type of application problem that can be solved using quadratic equations is one where two objects are moving away in directions perpendicular from each other. Here is an example of this type of problem.

### Example 10

Two cars leave an intersection. One car travels north; the other travels east. When the car traveling north had gone 30 miles, the distance between the cars was 10 miles more than twice the distance traveled by the car heading east. Find the distance between the cars at that time.

### Solution

Let  $x$  = the distance traveled by the car heading east.



$2x + 10$  = the distance between the two cars

Let's make a sketch

We can use the Pythagorean Theorem ( $a^2 + b^2 = c^2$ ) to find an equation for  $x$ :

$$x^2 + 30^2 = (2x + 10)^2$$

Expand parentheses and simplify.

$$\begin{aligned}x^2 + 900 &= 4x^2 + 40x + 100 \\800 &= 3x^2 + 40x\end{aligned}$$

Solve by completing the square.

$$\begin{aligned}\frac{800}{3} &= x^2 + \frac{40}{3}x \\ \frac{800}{3} + \left(\frac{20}{3}\right)^2 &= x^2 + 2\left(\frac{20}{3}\right)x + \left(\frac{20}{3}\right)^2 \\ \frac{2800}{9} &= \left(x + \frac{20}{3}\right)^2 \\ x + \frac{20}{3} &\approx 17.6 \text{ and } x + \frac{20}{3} \approx -17.6 \\ x &\approx 11 \text{ and } x \approx -24.3\end{aligned}$$

Since only positive distances make sense here, the distance between the two cars is  $2(11) + 10 = 32$  miles.

**Answer** The distance between the two cars is 32 miles.

### Review Questions

Complete the square for each expression.

1.  $x^2 + 5x$
2.  $x^2 - 2x$
3.  $x^2 + 3x$
4.  $x^2 - 4x$
5.  $3x^2 + 18x$
6.  $2x^2 - 22x$
7.  $8x^2 - 10x$
8.  $5x^2 + 12x$

Solve each quadratic equation by completing the square.

9.  $x^2 - 4x = 5$
10.  $x^2 - 5x = 10$
11.  $x^2 + 10x + 15 = 0$
12.  $x^2 + 4x + 16 = 0$
13.  $2x^2 - 18x = 0$
14.  $4x^2 + 5x = -1$
15.  $10x^2 - 30x - 8 = 0$
16.  $5x^2 + 15x - 40 = 0$

Rewrite each quadratic function in vertex form.

17.  $y = x^2 - 6x$

18.  $y + 1 = -2x^2 - x$

19.  $y = 9x^2 + 3x - 10$

20.  $y = -32x^2 + 60x + 10$

For #21-24, find

- The vertex
- $x$ -intercepts
- $y$ -intercept
- If it turns up or down.
- The graph the parabola.

21.  $y - 4 = x^2 + 8x$

22.  $y = -4x^2 + 20x - 24$

23.  $y = 3x^2 + 15x$

24.  $y + 6 = -x^2 + x$

25. Sam throws an egg straight down from a height of 25 feet. The initial velocity of the egg is 16 ft/sec. How long does it take the egg to reach the ground?

26. Amanda and Dolvin leave their house at the same time. Amanda walks south and Dolvin bikes east. Half an hour later they are 5.5 miles away from each other and Dolvin has covered three miles more than the distance that Amanda covered. How far did Amanda walk and how far did Dolvin bike?

### Review Answers

1.  $x^2 + 5x + \frac{25}{4} = (x + \frac{5}{2})^2$

2.  $x^2 - 2x + 1 = (x - 1)^2$

3.  $x^2 + 3x + \frac{9}{4} = (x + \frac{3}{2})^2$

4.  $x^2 - 4x + 4 = (x - 2)^2$

5.  $3(x^2 + 6x + 9) = 3(x + 3)^2$

6.  $2(x^2 - 11x + \frac{121}{4}) = 2(x - \frac{11}{2})^2$

7.  $8(x^2 - \frac{5}{4}x + \frac{25}{64}) = 8(x - \frac{5}{8})^2$

8.  $5(x^2 + \frac{12}{5}x + \frac{36}{25}) = 5(x + \frac{6}{5})^2$

9. 5, -1

10.  $\frac{5}{2} \pm \frac{\sqrt{65}}{2}$

11.  $-5 \pm \sqrt{10}$

12.  $-2 \pm 2\sqrt{3}i$

13. 0, 9

14.  $-1, -\frac{1}{4}$

15.  $\frac{15 \pm \sqrt{305}}{10}$

16.  $-\frac{3 \pm \sqrt{41}}{2}$

17.  $y + 9 = (x - 3)^2$

18.  $y + \frac{7}{8} = -2(x + \frac{1}{4})^2$

19.  $y + \frac{41}{4} = 9(x + \frac{1}{6})^2$

20.  $y - \frac{305}{8} = -32(x - \frac{15}{16})^2$

21.  $y + 12 = (x + 4)^2$ ; vertex  $(-4, -12)$ ;  $x$ -intercepts  $(-7.46, 0)$ ,  $(-.54, 0)$ ;  $y$ -intercept  $(0, 4)$ ; turns up.
22.  $y - 1 = -4\left(x - \frac{5}{2}\right)^2$ ; vertex  $(2.5, 1)$ ;  $x$ -intercepts  $(2, 0)$ ,  $(3, 0)$ ;  $y$ -intercept  $(0, -24)$ ; turns down.
23.  $y + \frac{75}{4} = 3\left(x + \frac{5}{2}\right)^2$ ; vertex  $(-2.5, -18.75)$ ;  $x$ -intercepts  $(0, 0)$ ,  $(-5, 0)$ ;  $y$ -intercept  $(0, 0)$ ; turns up.
24.  $y + \frac{23}{4} = -\left(x - \frac{1}{2}\right)^2$ ; vertex  $(0.5, -5.75)$ ;  $x$ -intercepts none;  $y$ -intercept  $(0, -6)$ ; turns down.
25. 1.85 *seconds*
26. Amanda 2.1 *miles*, Dolvin 5.1 *miles*