

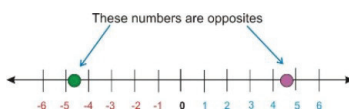
1.3 Subtraction of Rational Numbers

Learning objectives

- Find additive inverses.
- Subtract rational numbers.
- Evaluate change using a variable expression.
- Solve real world problems using fractions.

Find Additive Inverses

The **additive inverse** of a number is simply **opposite** of the number. (see section 2.1.4). Here are opposites on a number line.



When we think of additive inverses we are really talking about the **opposite process** (or **inverse process**) of **addition**. In other words, the process of **subtracting** a number is the same as adding the **additive inverse** of that number. When we add a number to its additive inverse, we get zero as an answer.

$$(6) + (-6) = 0$$

-6 is the additive inverse of 6 .

$$(279) + (-279) = 0$$

-279 is the additive inverse of 279 .

$$(x) + (-x) = 0$$

$-x$ is the additive inverse of x .

For any real numbers a and b ,

$$\mathbf{a - b = a + (-b) \text{ and } a - (-b) = a + b}$$

(To subtract, add the opposite, or additive inverse, of the number being subtracted.)

In general, this rule states we can convert every subtraction problem to an addition problem. We can then follow the rules for adding integers. Evaluate change using a variable expression.

Example 1

Rewrite each subtraction problem as an addition problem. Then determine the sum.

a. $5 - 3$ b. $-6 - 7$ c. $8 - (-2)$ d. $-9 - (-4)$

Solution

a. $5 + (-3) = 2$ b. $-6 + (-7) = -13$ c. $8 + 2 = 10$ d. $-9 + 4 = -5$

The method for subtracting fractions (as you should have assumed) is just the same as addition. We can use the idea of an additive inverse to relate the two processes. Just like in addition, we are going to need to write each of the rational numbers over a common denominator.

Example 2Simplify $\frac{1}{3} - \frac{1}{9}$

The lowest common multiple of 9 and 3 is 9. Our common denominator will be nine. We will not alter the second fraction because the denominator is already nine.

3 divides into 9 three times $\frac{1}{3} = \frac{1 \cdot 3}{3 \cdot 3} = \frac{3}{9}$. In other words $\frac{3}{9}$ is an **equivalent fraction** to $\frac{1}{3}$.

Our sum becomes $\frac{3}{9} - \frac{1}{9}$

Remember that when we add fractions with a common denominator, we **add** the **numerators** and the **denominator is unchanged**. A similar relationship holds for subtraction, only that we subtract the numerators.

When subtracting fractions $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$

Solution

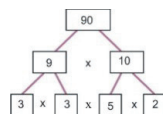
$$\frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

Two-ninths is the simplest form for our answer. So far we have only dealt with examples where it is easy to find the least common multiple of the denominators. With larger numbers, it is not so easy to be certain that we have the **least common denominator** (LCD). We need a more systematic method. In the next example, we will use the method of **prime factors** to find the least common denominator.

Example 3Simplify $\frac{29}{90} - \frac{13}{126}$

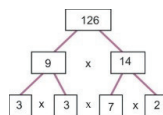
This time we need to find the lowest common multiple (LCM) of 90 and 126. To find the LCM, we first find the prime factors of 90 and 126. We do this by continually dividing the number by factors until we cannot divide any further. You may have seen a factor tree before:

The factor tree for 90 looks like this:



$$\begin{aligned} 90 &= 9 \cdot 10 \\ 9 &= 3 \cdot 3 \\ 10 &= 5 \cdot 2 \\ 90 &= 3 \cdot 3 \cdot 5 \cdot 2 \end{aligned}$$

The factor tree for 126 looks like this:



$$\begin{aligned}
 126 &= 9 \cdot 14 \\
 9 &= 3 \cdot 3 \\
 14 &= 7 \cdot 2 \\
 126 &= 3 \cdot 3 \cdot 7 \cdot 2
 \end{aligned}$$

The LCM for 90 and 126 is made from the **smallest possible collection of primes** that enables us to construct either of the two numbers. We take only enough of each prime to make the number with the highest number of factors of that prime in its factor tree.

TABLE 1.1:

Prime	Factors in 90	Factors in 126	We Take
2	1	1	1
3	2	2	2
5	1	0	1
7	0	1	1

So we need: one 2, two 3's, one 5 and one 7. In other words: $2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 630$

- The lowest common multiple of 90 and 126 is 630. The LCD for our calculation is 630.

90 divides into 630 seven times (notice that 7 is the only factor in 630 that is missing from 90) $\frac{29}{90} = \frac{7 \cdot 29}{7 \cdot 90} = \frac{203}{630}$

126 divides into 630 five times (notice that 5 is the only factor in 630 that is missing from 126) $\frac{13}{126} = \frac{5 \cdot 13}{5 \cdot 126} = \frac{65}{630}$

Now we complete the problem.

$$\frac{29}{90} - \frac{13}{126} = \frac{203}{630} - \frac{65}{630} = \frac{(203 - 65)}{630} = \frac{138}{630} \left\{ \text{remember, } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c} \right\}$$

This fraction **simplifies**. To be sure of finding the **simplest form** for $\frac{138}{630}$ we write out the numerator and denominator as **prime factors**. We already know the prime factors of 630. The prime factors of 138 are $138 = 2 \cdot 3 \cdot 23$.

$$\frac{138}{630} = \frac{2 \cdot 3 \cdot 23}{2 \cdot 3 \cdot 3 \cdot 5 \cdot 7} \quad \text{one factor of 2 and one factor of 3 cancels}$$

Solution

$$\frac{27}{90} - \frac{13}{126} = \frac{23}{105}$$

Example 4

Simplify $\frac{2}{7} - \frac{-3}{4}$

Solution

The least common denominator of 4 and 7 is 28.

$$\frac{2 \cdot 4}{7 \cdot 4} + \frac{-3 \cdot 7}{4 \cdot 7} = \frac{-8}{28} + \frac{-21}{28} = \frac{8 + (-21)}{28} = -\frac{13}{28}$$

Example 5

$$-\frac{3}{12} - \frac{5}{24}$$

The least common denominator of 12 and 24 is 24.

$$\frac{-3 \cdot 2}{12 \cdot 2} + \frac{-5}{24} = \frac{-6 + -5}{24} = -\frac{11}{24}$$

Example 6

A property management firm is acquiring parcels of land in order to build a small community of condominiums. It has bought three adjacent plots of land. The first is four-fifths of an acre, the second is five-twelfths of an acre, and the third is nineteen-twentieths of an acre. The firm knows that it must allow one-sixth of an acre for utilities and a small access road. How much of the remaining land is available for development?

The first thing we need to do is extract the relevant information. Here are the relevant fractions.

$$\frac{4}{5}, \frac{5}{12}, \frac{1}{6}$$

$$\text{and } \frac{19}{20}$$

The plots of land that the firm has acquired.

The amount of land that the firm has to give up.

This sum will determine the amount of land available for development.

$$\frac{4}{5} + \frac{5}{12} + \frac{19}{20} - \frac{1}{6}$$

$$5 = 5$$

$$12 = 2 \cdot 2 \cdot 3$$

$$20 = 2 \cdot 2 \cdot 5$$

$$6 = 2 \cdot 3$$

We need to find the LCM of 5, 12, 20 and 6.

one 5

two 2's, one 3

two 2's, one 5

one 3

The smallest set of primes that encompasses all of these is 2, 2, 3, 5. Our LCD is thus $2 \cdot 2 \cdot 3 \cdot 5 = 60$

Now we can convert all fractions to a common denominator of 60. To do this, we multiply by the factors of 60 that are missing in the denominator we are converting. For example, 5 is missing two 2's and a 3. This results in $2 \cdot 2 \cdot 3 = 12$.

$$\begin{aligned} \frac{4}{5} &= \frac{12 \cdot 4}{12 \cdot 5} = \frac{48}{60} \\ \frac{5}{12} &= \frac{5 \cdot 5}{5 \cdot 12} = \frac{25}{60} \\ \frac{19}{20} &= \frac{3 \cdot 19}{3 \cdot 20} = \frac{57}{60} \\ \frac{1}{6} &= \frac{10 \cdot 1}{10 \cdot 6} = \frac{10}{60} \end{aligned}$$

Our converted sum can be rewritten as: $\frac{48}{60} + \frac{25}{60} + \frac{57}{60} - \frac{10}{60} = \frac{(48+25+57-10)}{60} = \frac{120}{60}$

Next, we need to reduce this fraction. We can see immediately that the numerator is twice the denominator. This fraction reduces to $\frac{2}{1}$ or simply two. One is sometimes called the *invisible denominator*, as every whole number can be thought of as a rational number whose denominator is one.

Solution

The property firm has two acres available for development.

Evaluate Change Using a Variable Expression

When we write algebraic expressions to represent a real quantity, the difference between two values is the **change** in that quantity.

Example 7

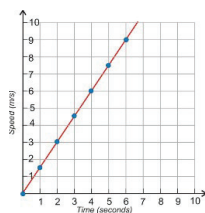
The speed of a train increases according to the expression $\text{speed} = 1.5t$ where speed is measured in meters per second, and “ t ” is the time measured in seconds. Find the change in the speed between $t = 2$ seconds and $t = 6$ seconds.

This function represents a train that is stopped when time equals zero ($\text{speed} = 0 \times 0.25$). As the stopwatch ticks, the train’s speed increases in a linear pattern. We can make a table of what the train’s speed is at every second.

TABLE 1.2:

Time (seconds)	Speed (m/s)
0	0
1	1.5
2	3
3	4.5
4	6
5	7.5
6	9

We can even graph this function. The graph of speed vs. time is shown here.



We wish to find the change in speed between $t = 2$ seconds and $t = 6$ seconds. There are several ways to do this. We could look at the table, and read off the speeds at 2 seconds (3 m/s) and 6 seconds (9 m/s). Or we could determine the speeds at those times by using the graph.

Another way to find the change would be to substitute the two values for t into our expression for speed. First, we will substitute $t = 2$ into our expression. To indicate that the speed we get is the speed at time = 2 seconds, we denote it as $\text{speed}(2)$.

$$\text{speed}(2) = 1.5(2) = 3$$

Next, we will substitute $t = 6$ into our expression. This is the speed at 6 seconds, so we denote it as $\text{speed}(6)$

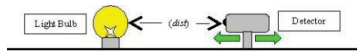
$$\text{speed}(6) = 1.5(6) = 9$$

The change between $t = 2$ and $t = 6$ is $\text{speed}(6) - \text{speed}(2) = 9 - 3 = 6$ m/s.

The speed change is **positive**, so the change is an **increase**.

Solution

Between the two and six seconds the train's speed increases by 6 m/s.

Example 8

The intensity of light hitting a detector when it is held a certain distance from a bulb is given by this equation.

$$\text{Intensity} = 3/(\text{dist})^2$$

Where (dist) is the distance measured in **meters**, and intensity is measured in **lumens**. Calculate the change in intensity when the detector is moved from two meters to three meters away.

We first find the values of the intensity at distances of two and three meters.

$$\begin{aligned}\text{Intensity}(2) &= \frac{3}{(2)^2} = \frac{3}{4} \\ \text{Intensity}(3) &= \frac{3}{(3)^2} = \frac{3}{9} = \frac{1}{3}\end{aligned}$$

The **difference** in the two values will give the **change** in the intensity. We move **from** two meters **to** three meters away.

$$\text{Change} = \text{Intensity}(3) - \text{Intensity}(2) = \frac{1}{3} - \frac{3}{4}$$

To find the answer, we will need to write these fractions over a common denominator.

The LCM of 3 and 4 is 12, so we need to rewrite each fraction with a denominator of 12:

$$\begin{aligned}\frac{1}{3} &= \frac{4 \cdot 1}{4 \cdot 3} = \frac{4}{12} \\ \frac{3}{4} &= \frac{3 \cdot 3}{3 \cdot 4} = \frac{9}{12}\end{aligned}$$

Our change is given by this equation.

$$\frac{4}{12} - \frac{9}{12} = \frac{(4-9)}{12} = -\frac{5}{12}$$

A negative indicates that the intensity is reduced.

Solution

When moving the detector from two meters to three meters the intensity falls by $\frac{5}{12}$ lumens.

Lesson Summary

- **Subtracting** a number is the same as adding the **opposite** (or **additive inverse**) of the number.
- When **subtracting fractions**: $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
- The number one is sometimes called the **invisible denominator**, as every whole number can be thought of as a rational number whose denominator is one.
- The **difference** between two values is the **change** in that quantity.

Review Questions

1. Subtract the following rational numbers. Be sure that your answer is in the **simplest form**.
 - (a) $\frac{5}{12} - \frac{9}{18}$
 - (b) $\frac{2}{3} - \frac{1}{4}$
 - (c) $\frac{3}{4} - \frac{1}{3}$
 - (d) $\frac{15}{11} - \frac{9}{7}$
 - (e) $\frac{2}{13} - \frac{-1}{11}$
 - (f) $\frac{7}{27} - \frac{19}{39}$
 - (g) $\frac{6}{11} - \frac{3}{27}$
 - (h) $\frac{13}{64} - \frac{7}{40}$
 - (i) $\frac{11}{70} - \frac{11}{30}$
2. Consider the equation $y = 3x + 2$. Determine the change in y between $x = 3$ and $x = 7$.
3. Consider the equation $y = \frac{2}{3}x + \frac{1}{2}$. Determine the change in y between $x = 1$ and $x = 2$.
4. The time taken to commute from San Diego to Los Angeles is given by the equation $\text{time} = \frac{120}{\text{speed}}$ where *time* is measured in **hours** and *speed* is measured in **miles per hour** (mph). Calculate the change in time that a rush hour commuter would see when switching from traveling by bus to train. The bus averages 40 mph to a new high speed train which averages 90 mph.

Review Answers

1.
 - (a) $\frac{-1}{12}$
 - (b) $\frac{5}{12}$
 - (c) $\frac{5}{12}$
 - (d) $\frac{6}{77}$
 - (e) $\frac{35}{143}$
 - (f) $-\frac{80}{351}$
 - (g) $\frac{9}{22}$
 - (h) $\frac{320}{9}$
 - (i) $\frac{-22}{105}$
2. Change = +12
3. Change = $-\frac{1}{3}$
4. The journey time would decrease by $1\frac{2}{3}$ hours.