

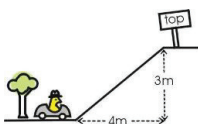
3.4 Slope and Rate of Change

Learning Objectives

- Find positive and negative slopes.
- Recognize and find slopes for horizontal and vertical lines.
- Understand rates of change.
- Interpret graphs and compare rates of change.

Introduction

We come across many examples of slope in everyday life. For example, a slope is in the pitch of a roof, the grade or incline of a road, and the slant of a ladder leaning on a wall. In math, we use the word **slope** to define steepness in a particular way.



$$\text{Slope} = \frac{\text{distance moved vertically}}{\text{distance moved horizontally}}$$

This is often reworded to be easier to remember:

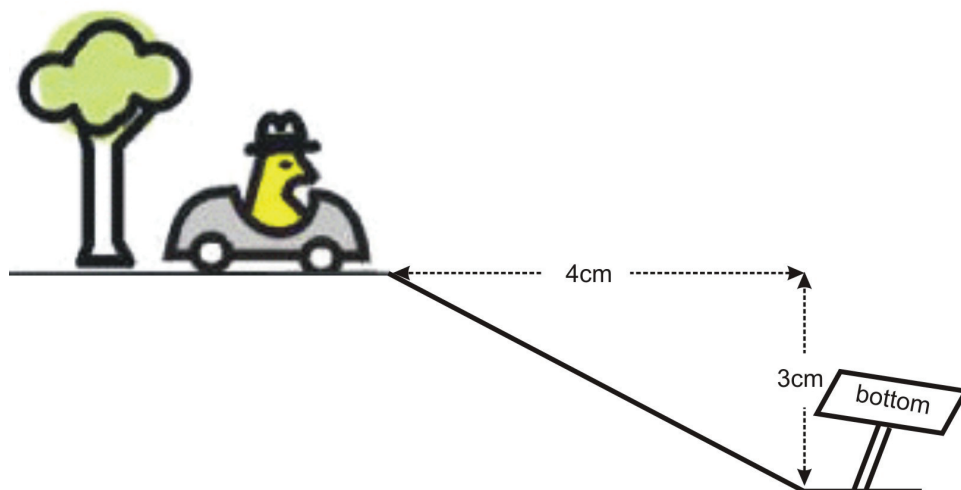
$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$

Essentially, slope is the change in y if x increases by 1.

In the picture to the right, the slope would be the ratio of the **height** of the hill (the **rise**) to the horizontal **length** of the hill (the **run**).

$$\text{Slope} = \frac{3}{4} = 0.75$$

If the car were driving to the **right** it would **climb** the hill. We say this is a positive slope. Anytime you see the graph of a line that goes up as you move to the right, the slope is **positive**.



If the car were to keep driving after it reached the top of the hill, it may come down again. If the car is driving to the **right** and **descending**, then we would say that the slope is **negative**. The picture at right has a **negative slope** of -0.75 .

Do not get confused! If the car turns around and drives back down the hill shown, we would still classify the slope as positive. This is because the rise would be -3 , but the run would be -4 (think of the x -axis if you move from right to left you are moving in the negative x -direction). Our ratio for moving **left** is:

$$\text{Slope} = \frac{-3}{-4} = 0.75$$

A negative divided by a negative is a positive.

So as we move from left to right, positive slopes increase while negative slopes decrease.

Find a Positive Slope

We have seen that a function with a positive slope increases in y as we increase x . A simple way to find a value for the slope is to draw a right angled triangle whose hypotenuse runs along the line. It is then a simple matter of measuring the distances on the triangle that correspond to the rise (the vertical dimension) and the run (the horizontal dimension).

Example 1

Find the slopes for the three graphs shown right.

There are already right-triangles drawn for each of the lines. In practice, you would have to do this yourself. Note that it is easiest to make triangles whose vertices are **lattice points** (i.e. the coordinates are all integers).

- a. The rise shown in this triangle is 4 units, the run is 2 units.

$$\text{Slope} = \frac{4}{2} = 2$$

- b. The rise shown in this triangle is 4 units, the run is also 4 units.

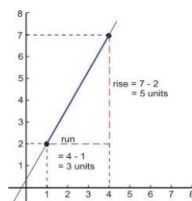
$$\text{Slope} = \frac{4}{4} = 1$$

c. The rise shown in this triangle is 2 units, the run is 4 units.

$$\text{Slope} = \frac{2}{4} = \frac{1}{2}$$

Example 2

Find the slope of the line that passes through the points $(1, 2)$ and $(4, 7)$.



We already know how to graph a line if we are given two points. We simply plot the points and connect them with a line. Look at the graph shown at right.

Since we already have coordinates for our right triangle, we can quickly work out that the rise would be 5 and the run would be 3 (see diagram). Here is our slope.

$$\text{Slope} = \frac{7 - 2}{4 - 1} = \frac{5}{3}$$

If you look closely at the calculations for the slope you will notice that the 7 and 2 are the y -coordinates of the two points and the 4 and 1 are the x -coordinates. This suggests a pattern we can follow to get a general formula for the slope between two points (x_1, y_1) and (x_2, y_2) .

Slope between (x_1, y_1) and $(x_2, y_2) = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{\Delta y}{\Delta x}$

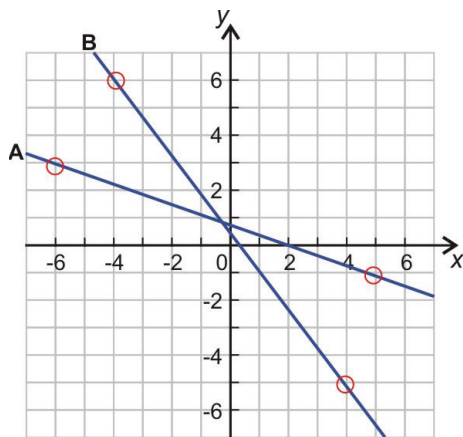
In the second equation, the letter m denotes the slope (you will see this a lot in this chapter) and the Greek letter delta (Δ) means **change**. So another way to express slope is *change in y* divided by *change in x* . In the next section, you will see that it does not matter which point you choose as point 1 and which you choose as point 2.

Find a Negative Slope

Any function with a negative slope is simply a function that decreases as we increase x . If you think of the function as the incline of a road a negative slope is a road that goes **downhill** as you drive to the **right**.

Example 3

Find the slopes of the lines on the graph to the right.



Look at the lines. Both functions fall (or decrease) as we move from left to right. Both of these lines have a **negative slope**.

Neither line passes through a great number of lattice points, but by looking carefully you can see a few points that look to have integer coordinates. These points have been identified (with rings) and we will use these to determine the slope. We will also do our calculations twice, to show that we get the same slope whichever way we choose point 1 and point 2.

For line A:

$$(x_1, y_1) = (-6, 3) \quad (x_2, y_2) = (5, -1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - (3)}{(5) - (-6)} = \frac{-4}{11} \approx -0.364$$

$$(x_1, y_1) = (5, -1) \quad (x_2, y_2) = (-6, 3)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (-1)}{(-6) - (5)} = \frac{-4}{-11} \approx -0.364$$

For line B:

$$(x_1, y_1) = (-4, 6) \quad (x_2, y_2) = (4, -5)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-5) - (6)}{(4) - (-4)} = \frac{-11}{8} = -1.375$$

$$(x_1, y_1) = (4, -5) \quad (x_2, y_2) = (-4, 6)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(6) - (-5)}{(-4) - (4)} = \frac{11}{-8} = -1.375$$

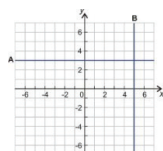
You can see that whichever way you select the points, the answers are the same!

Solution

Line A has slope -0.364 . Line B has slope -1.375 .

Find the Slopes of Horizontal and Vertical lines

Example 4



Determine the slopes of the two lines on the graph at the right.

There are two lines on the graph. A ($y = 3$) and B ($x = 5$).

Let's pick two points on line A . say, $(x_1, y_1) = (-4, 3)$ and $(x_2, y_2) = (5, 3)$ and use our equation for slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (3)}{(5) - (-4)} = \frac{0}{9} = 0$$

If you think about it, this makes sense. If there is no change in y as we increase x then there is no slope, or to be correct, a slope of zero. You can see that this must be true for all horizontal lines.

Horizontal lines ($y = \text{constant}$) all have a slope of 0.

Now consider line B . Pick two distinct points on this line and plug them in to the slope equation.

$(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (5, 4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(4) - (-3)}{(5) - (5)} = \frac{7}{0} \quad \text{A division by zero!}$$

Divisions by zero lead to infinities. In math we often use the term **undefined** for any division by zero.

Vertical lines ($x = \text{constant}$) all have an infinite (or undefined) slope.

Find a Rate of Change

The slope of a function that describes real, measurable quantities is often called a **rate of change**. In that case, the slope refers to a change in one quantity (y) **per** unit change in another quantity (x).

Example 5

*Andrea has a part time job at the local grocery store. She saves for her vacation at a rate of \$15 every week. Express this rate as money saved **per day** and money saved **per year**.*

Converting rates of change is fairly straight forward so long as you remember the equations for rate (i.e. the equations for slope) and know the conversions. In this case 1 week = 7 days and 52 weeks = 1 year.

$$\begin{aligned} \text{rate} &= \frac{\$15}{1 \text{ week}} \cdot \frac{1 \text{ week}}{7 \text{ days}} = \frac{\$15}{7 \text{ days}} = \frac{15}{7} \text{ dollars per day} \approx \$2.14 \text{ per day} \\ \text{rate} &= \frac{\$15}{1 \text{ week}} \cdot \frac{52 \text{ week}}{1 \text{ year}} = \$15 \cdot \frac{52}{\text{year}} = \$780 \text{ per year} \end{aligned}$$

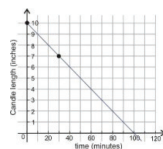
Example 6

A candle has a starting length of 10 inches. Thirty minutes after lighting it, the length is 7 inches. Determine the rate of change in length of the candle as it burns. Determine how long the candle takes to completely burn to nothing.

In this case, we will graph the function to visualize what is happening.

We have two points to start with. We know that at the moment the candle is lit (time = 0) the length of the candle is 10 inches. After thirty minutes (time = 30) the length is 7 inches. Since the candle length is a function of time we will plot time on the horizontal axis, and candle length on the vertical axis. Here is a graph showing this information.

Candle Length by Burning Time



The rate of change of the candle is simply the slope. Since we have our two points $(x_1, y_1) = (0, 10)$ and $(x_2, y_2) = (30, 7)$ we can move straight to the formula.

$$\text{Rate of change} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(7 \text{ inches}) - (10 \text{ inches})}{(30 \text{ minutes}) - (0 \text{ minutes})} = \frac{-3 \text{ inches}}{30 \text{ minutes}} = -0.1 \text{ inches per minute}$$

The slope is negative. A negative rate of change means that the quantity is decreasing with time.

We can also convert our rate to inches per hour.

$$\text{rate} = \frac{-0.1 \text{ inches}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} = \frac{-6 \text{ inches}}{1 \text{ hour}} = -6 \text{ inches per hour}$$

To find the point when the candle reaches zero length we can simply read off the graph (100 minutes). We can use the rate equation to verify this algebraically.

$$\text{Length burned} = \text{rate} \times \text{time}$$

$$0.1 \times 100 = 10$$

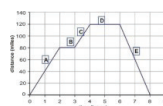
Since the candle length was originally 10 inches this confirms that 100 minutes is the correct amount of time.

Interpret a Graph to Compare Rates of Change

Example 7

Examine the graph below. It represents a journey made by a large delivery truck on a particular day. During the day, the truck made two deliveries, each one taking one hour. The driver also took a one hour break for lunch. Identify what is happening at each stage of the journey (stages A through E)

Trucks Distance from Home by Time



Here is the driver's journey.

A. The truck sets off and travels 80 miles in 2 hours .

B. The truck covers no distance for 1 hours .

C. The truck covers $(120 - 80) = 40$ miles in 1 hours

D. the truck covers no distance for 2 hours .

E. The truck covers 120 miles in 2 hours .

Lets look at the rates of change for each section.

A. Rate of change = $\frac{\Delta y}{\Delta x} = \frac{80 \text{ miles}}{2 \text{ hours}} = 40 \text{ miles per hour}$

- The rate of change is a **velocity**! This is a very important concept and one that deserves a special note!

The slope (or rate of change) of a distance-time graph is a velocity.

You may be more familiar with calling **miles per hour** a **speed**. **Speed** is the **magnitude** of a **velocity**, or, put another way, velocity has a direction, speed does not. This is best illustrated by working through the example.

On the first part of the journey sees the truck travel at a constant velocity of 40 mph for 2 hours covering a distance of 80 miles .

B. **Slope** = 0 so **rate of change** = 0 mph. The truck is stationary for one hour. This could be a lunch break, but as it is only 2 hours since the truck set off it is likely to be the first delivery stop.

C. Rate of change = $\frac{\Delta y}{\Delta x} = \frac{(120-80) \text{ miles}}{(4-3) \text{ hours}} = 40 \text{ miles per hour}$. The truck is traveling at 40 mph.

D. **Slope** = 0 so **rate of change** = 0 mph . The truck is stationary for two hours. It is likely that the driver used these 2 hours for a lunch break plus the second delivery stop. At this point the truck is 120 miles from the start position.

E. Rate of change = $\frac{\Delta y}{\Delta x} = \frac{(0-120) \text{ miles}}{(8-6) \text{ hours}} = \frac{-120 \text{ miles}}{2 \text{ hours}} = -60 \text{ miles per hour}$. The truck is traveling at **negative** 60 mph .

Wait, a negative velocity? Does this mean that the truck is reversing? Well, probably not. What it means is that the distance (and dont forget that is the distance measured from the starting position) is decreasing with time. The truck is simply driving in the opposite direction. In this case, back to where it started from. So, the speed of the truck would be 60 mph, but the velocity (which includes direction) is negative because the truck is getting closer to where it started from. The fact that it no longer has two heavy loads means that it travels faster (60 mph as opposed to 40 mph) covering the 120 mile return trip in 2 hours .

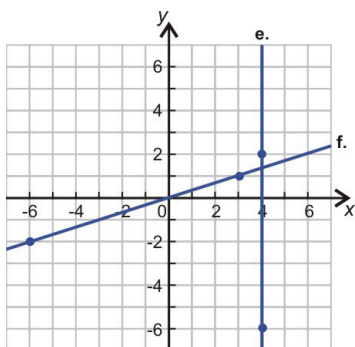
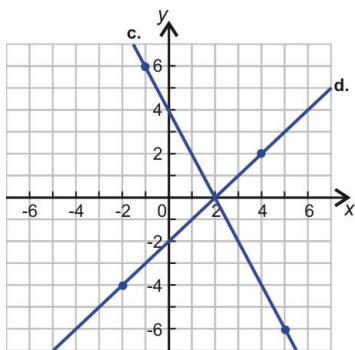
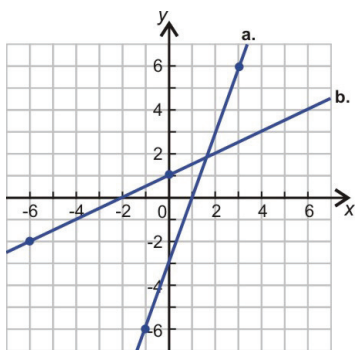
Lesson Summary

- **Slope** is a measure of change in the vertical direction for each step in the horizontal direction. Slope is often represented as "*m*".
- $\text{Slope} = \frac{\text{rise}}{\text{run}}$ or $m = \frac{\Delta y}{\Delta x}$
- The slope between two points (x_1, y_1) and $(x_2, y_2) = \frac{y_2 - y_1}{x_2 - x_1}$
- **Horizontal lines** ($y = \text{constant}$) all have a slope of 0.
- **Vertical lines** ($x = \text{constant}$) all have an infinite (or undefined) slope.
- The slope (or **rate of change**) of a distance-time graph is a **velocity**.

Review Questions

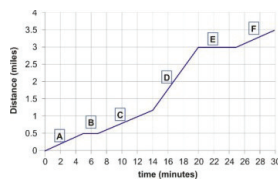
1. Use the slope formula to find the slope of the line that passes through each pair of points.
 - (a) $(-5, 7)$ and $(0, 0)$
 - (b) $(-3, -5)$ and $(3, 11)$
 - (c) $(3, -5)$ and $(-2, 9)$
 - (d) $(-5, 7)$ and $(-5, 11)$
 - (e) $(9, 9)$ and $(-9, -9)$
 - (f) $(3, 5)$ and $(-2, 7)$

2. Use the points indicated on each line of the graphs to determine the slopes of the following lines.



3. The graph below is a distance-time graph for Marks three and a half mile cycle ride to school. During this ride, he rode on cycle paths but the terrain was hilly. He rode slower up hills and faster down them. He stopped once at a traffic light and at one point he stopped to mend a tire puncture. Identify each section of the graph accordingly.

Andrews Distance from Home by Time



Review Answers

- 1.4
 - 2.67
 - 2.8
 - undefined

- (e) 1
 - (f) -0.4
- 2.
- (a) 3
 - (b) 0.5
 - (c) -2
 - (d) 1
 - (e) undefined
 - (f) $\frac{1}{3}$
- 3.
- (a) A. uphill
 - (b) B. stopped (traffic light)
 - (c) C. uphill
 - (d) D. downhill
 - (e) E. stopped (puncture)
 - (f) F. uphill