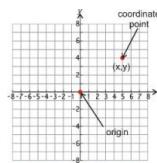


## 5.2 Functions as Graphs

### Learning Objectives

- Graph a function from a rule or table.
- Write a function rule from a graph.
- Analyze the graph of a real world situation.
- Determine whether a relation is a function.

### Introduction



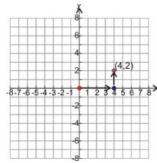
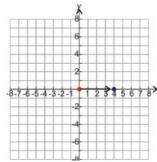
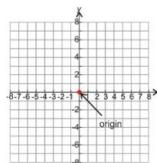
We represent functions graphically by plotting points on a **Coordinate Plane** (this is also sometimes called the **Cartesian plane**). The coordinate plane is a grid formed by a horizontal number line and a vertical number line that cross at a point called the **origin**. The origin is point  $(0,0)$  and it is the starting location. In order to plot points on the grid, you are told how many units you go right or left and how many units you go up or down from the origin. The horizontal line is called the  **$x$ -axis** and the vertical line is called the  **$y$** -axis. The arrows at the end of each axis indicate that the plane continues past the end of the drawing.

From a function, we can gather information in terms of pairs of points. For each value of the independent variable in the domain, the function is used to calculate the value of the dependent variable. We call these pairs of points **coordinate points** or  $x, y$  values and they are written as  $(x,y)$ .

To graph a coordinate point such as  $(4,2)$  we start at the origin.

Then we move 4 units to the right.

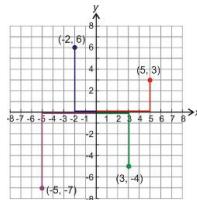
And then we move 2 units up from the last location.



**Example 1**

*Plot the following coordinate points on the Cartesian plane.*

- (a)  $(5, 3)$
- (b)  $(-2, 6)$
- (c)  $(3, -4)$
- (d)  $(-5, -7)$

**Solution**

We show all the coordinate points on the same plot.

Notice that:

For a positive  $x$  value we move to the right.

For a negative  $x$  value we move to the left.

For a positive  $y$  value we move up.

For a negative  $y$  value we move down.

The  $x$ -axis and  $y$ -axis divide the coordinate plane into four **quadrants**. The quadrants are numbered counter-clockwise starting from the upper right. The plotted point for (a) is in the **First** quadrant, (b) is in the **Second** quadrant, (c) is in the **Fourth** quadrant, and (d) is in the **Third** quadrant.

**Graph a Function From a Rule or Table**

Once a rule is known or if we have a table of values that describes a function, we can draw a graph of the function. A table of values gives us coordinate points that can be plotted on the Cartesian plane.

**Example 2**

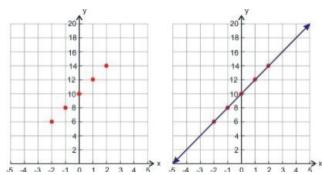
*Graph the function that has the following table of values.*

$x$	-2	-1	0	1	2
$y$	6	8	10	12	14

**Solution**

The table gives us five sets of coordinate points  $(-2, 6), (-1, 8), (0, 10), (1, 12), (2, 14)$ .

To graph the function, we plot all the coordinate points. Since we are not told the domain of the function or the context where it appears we can assume that the domain is the set of all real numbers. To show that the function holds for all values in the domain, we connect the points with a smooth line. Also, we must realize that the line continues infinitely in both directions.

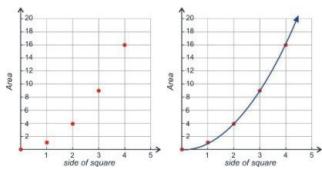
**Example 3**

Graph the function that has the following table of values.

Side of the Square	0	1	2	3	4
Area of the Square	0	1	4	9	16

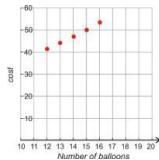
The table gives us five sets of coordinate points:  $(0,0), (1,1), (2,4), (3,9), (4,16)$ .

To graph the function, we plot all the coordinate points. Since we are not told the domain of the function, we can assume that the domain is the set of all non-negative real numbers. To show that the function holds for all values in the domain, we connect the points with a smooth curve. The curve does not make sense for negative values of the independent variable so it stops at  $x = 0$  but it continues infinitely in the positive direction.

**Example 4**

Graph the function that has the following table of values.

Number of Balloons	12	13	14	15	16
Cost	41	44	47	50	53



This function represents the total cost of the balloons delivered to your house. Each balloon is \$3 and the store delivers if you buy a dozen balloons or more. The delivery charge is a \$5 flat fee.

**Solution**

The table gives us five sets of coordinate points  $(12,-41), (13,-44), (14,-47), (15,-50), (16,-53)$ .

To graph the function, we plot all the coordinate points. Since the  $x$ -values represent the number of balloons for 12 balloons or more, the domain of this function is all integers greater than or equal to 12. In this problem, the points are not connected by a line or curve because it does not make sense to have non-integer values of balloons.

In order to draw a graph of a function given the function rule, we must first make a table of values. This will give us a set of coordinate points that we can plot on the Cartesian plane. Choosing the values of the independent variables for the table of values is a skill you will develop throughout this course. When you pick values here are some of the things you should keep in mind.

- Pick only values from the domain of the function.
- If the domain is the set of real numbers or a subset of the real numbers, the graph will be a continuous curve.
- If the domain is the set of integers or a subset of the integers, the graph will be a set of points not connected by a curve.
- Picking integers is best because it makes calculations easier, but sometimes we need to pick other values to capture all the details of the function.
- Often we start with a set of values. Then after drawing the graph, we realize that we need to pick different values and redraw the graph.

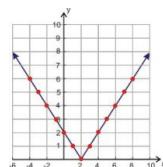
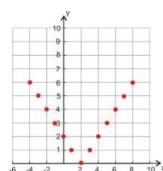
**Example 5**

Graph the following function  $f(x) = |x - 2|$

**Solution**

Make a table of values. Pick a variety of negative and positive integer values for the independent variable. Use the function rule to find the value of the dependent variable for each value of the independent variable. Then, graph each of the coordinate points.

$x$	$y = f(x) =  x - 2 $
-4	$ -4 - 2  =  -6  = 6$
-3	$ -3 - 2  =  -5  = 5$
-2	$ -2 - 2  =  -4  = 4$
-1	$ -1 - 2  =  -3  = 3$
0	$ 0 - 2  =  -2  = 2$
1	$ 1 - 2  =  -1  = 1$
2	$ 2 - 2  =  0  = 0$
3	$ 3 - 2  =  1  = 1$
4	$ 4 - 2  =  2  = 2$
5	$ 5 - 2  =  3  = 3$
6	$ 6 - 2  =  4  = 4$
7	$ 7 - 2  =  5  = 5$
8	$ 8 - 2  =  6  = 6$



It is wise to work with a lot of values when you begin graphing. As you learn about different types of functions, you will find that you will only need a few points in the table of values to create an accurate graph.

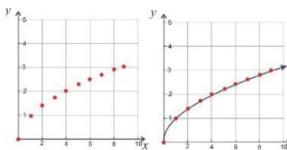
**Example 6**

Graph the following function:  $f(x) = \sqrt{x}$

### Solution

Make a table of values. We cannot use negative numbers for the independent variable because we can't take the square root of a negative number. The square root doesn't give real answers for negative inputs. The domain is all positive real numbers, so we pick a variety of positive integer values for the independent variable. Use the function rule to find the value of the dependent variable for each value of the independent variable.

$x$	$y = f(x) = \sqrt{x}$
0	$\sqrt{0} = 0$
1	$\sqrt{1} = 1$
2	$\sqrt{2} \approx 1.41$
3	$\sqrt{3} \approx 1.73$
4	$\sqrt{4} = 2$
5	$\sqrt{5} \approx 2.24$
6	$\sqrt{6} \approx 2.45$
7	$\sqrt{7} \approx 2.65$
8	$\sqrt{8} \approx 2.83$
9	$\sqrt{9} = 3$



**Note that the range is all positive real numbers.**

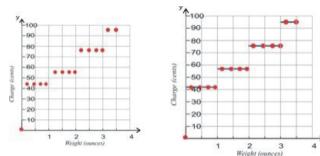
### Example 7

The post office charges 41 cents to send a letter that is one ounce or less and an extra 17 cents for any amount up to and including an additional ounce. This rate applies to letters up to 3.5 ounces.

### Solution

Make a table of values. We cannot use negative numbers for the independent variable because it does not make sense to have negative weight. We pick a variety of positive integer values for the independent variable but we also need to pick some decimal values because prices can be decimals too. This will give us a clear picture of the function. Use the function rule to find the value of the dependent variable for each value of the independent variable.

$x$	$y$
0	0
0.2	41
0.5	41
0.8	41
1	41
1.2	58
1.5	58
1.8	58
2	58
2.2	75
2.5	75
2.8	75
3.0	75
3.2	92
3.5	92

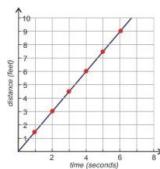


### Write a Function Rule from a Graph

Sometimes you will need to find the equation or rule of the function by looking at the graph of the function. From a graph, you can read pairs of coordinate points that are on the curve of the function. The coordinate points give values of dependent and independent variables that are related to each other by the rule. However, we must make sure that this rule works for all the points on the curve. In this course you will learn to recognize different kinds of functions. There will be specific methods that you can use for each type of function that will help you find the function rule. For now we will look at some simple examples and find patterns that will help us figure out how the dependent and independent variables are related.

#### Example 8

The graph to the right shows the distance that an ant covers over time. Find the function rule that shows how distance and time are related to each other.



#### Solution

We make a table of values of several coordinate points to see if we can identify a pattern of how they are related to each other.

Time	0	1	2	3	4	5	6
Distance	0	1.5	3	4.5	6	7.5	9

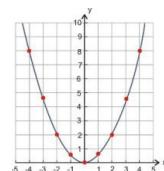
We can see that for every second the distance increases by 1.5 feet. We can write the function rule as:

$$\text{Distance} = 1.5 \times \text{time}$$

The equation of the function is  $f(x) = 1.5x$

### Example 9

Find the function rule that describes the function shown in the graph.



### Solution:

We make a table of values of several coordinate points to see if we can identify a pattern of how they are related to each other.

$x$	-4	-3	-2	-1	0	1	2	3	4
$y$	8	4.5	2	.5	0	.5	2	4.5	8

We notice that the values of  $y$  are half of perfect squares. Re-write the table of values as:

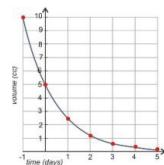
$x$	-4	-3	-2	-1	0	1	2	3	4
$y$	$\frac{16}{2}$	$\frac{9}{2}$	$\frac{4}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\frac{9}{2}$	$\frac{16}{2}$

We can see that to obtain  $y$ , we square  $x$  and divide by 2.

The function rule is  $y = \frac{1}{2}x^2$  and the equation of the function is  $f(x) = \frac{1}{2}x^2$ .

### Example 10

Find the function rule that shows what is the volume of a balloon at different times.



### Solution

We make table of values of several coordinate points to see if we can identify a pattern of how they are related to each other.

Time	-1	0	1	2	3	4	5
Volume	10	5	2.5	1.25	0.625	0.3125	0.15625

We can see that for every day the volume of the balloon is cut in half. Notice that the graph shows negative time. The negative time can represent what happened on days before you started measuring the volume.

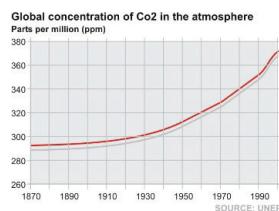
$$\begin{aligned} \text{Day0 : Volume} &= 5 \\ \text{Day1 : Volume} &= 5 \cdot \frac{1}{2} \\ \text{Day2 : Volume} &= 5 \cdot \frac{1}{2} \cdot \frac{1}{2} \\ \text{Day3 : Volume} &= 5 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \end{aligned}$$

The equation of the function is  $f(x) = 5 \left(\frac{1}{2}\right)^x$

### Analyze the Graph of a Real-World Situation

Graphs are used to represent data in all areas of life. You can find graphs in newspapers, political campaigns, science journals and business presentations.

Here is an example of a graph you might see reported in the news. Most mainstream scientists believe that increased emissions of greenhouse gases, particularly carbon dioxide, are contributing to the warming of the planet. This graph shows how carbon dioxide levels have increased as the world has industrialized.



From this graph, we can find the concentration of carbon dioxide found in the atmosphere in different years.

1900	285 part per million
1930	300 part per million
1950	310 parts per million
1990	350 parts per million

We can find approximate function rules for these types of graphs using methods that you learn in more advanced math classes. The function  $f(x) = 0.0066x^2 - 24.9x + 23765$  approximates this graph very well.

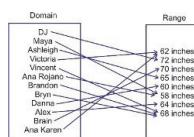
## Determine Whether a Relation is a Function

You saw that a function is a relation between the independent and the dependent variables. It is a rule that uses the values of the independent variable to give the values of the dependent variable. A function rule can be expressed in words, as an equation, as a table of values and as a graph. All representations are useful and necessary in understanding the relation between the variables. Mathematically, a function is a special kind of relation.

**In a function, for each input there is exactly one output.**

This usually means that each  $x$ -value has only one  $y$ -value assigned to it. But, not all functions involve  $x$  and  $y$ .

Consider the relation that shows the heights of all students in a class. The domain is the set of people in the class and the range is the set of heights. Each person in the class cannot be more than one height at the same time. This relation is a function because for each person there is exactly one height that belongs to him or her.



Notice that in a function, a value in the range can belong to more than one element in the domain, so more than one person in the class can have the same height. The opposite is not possible, one person cannot have multiple heights.

### Example 11

*Determine if the relation is a function.*

- $(1, 3), (-1, -2), (3, 5), (2, 5), (3, 4)$
- $(-3, 20), (-5, 25), (-1, 5), (7, 12), (9, 2)$
- c)

$x$	2	4	6	8	10
$y$	41	44	47	50	53

d)

$x$	2	1	0	1	2
$y$	12	10	8	6	4

### Solution

The easiest way to figure out if a relation is a function is to look at all the  $x$ -values in the list or the table. If a value of  $x$  appears more than once and the  $y$ -values are different then the relation is not a function.

- $(1, 3), (-1, -2), (3, 5), (2, 5), (3, 4)$

You can see that in this relation there are two different  $y$ -values that belong to the  $x$ -value of 3. This means that this relation is **not** a function.

- $(-3, 20), (-5, 25), (-1, 5), (7, 12), (9, 2)$

Each value of  $x$  has exactly one  $y$ -value. The relation is a function.

c)

$x$	2	4	6	8	10
$y$	4	4	4	4	4

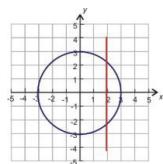
Each value of  $x$  appears only once. The relation is a function.

d)

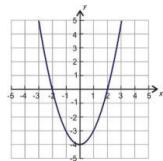
$x$	2	1	0	1	2
$y$	12	10	8	6	4

In this relation there are two  $y$ -values that belong to the  $x$ -value of 2 and two  $y$ -values that belong to the  $x$ -value of 1. The relation is **not** a function.

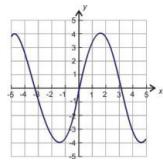
When a relation is represented graphically, we can determine if it is a function by using the **vertical line test**. If you can draw a vertical line that crosses the graph in more than one place, then the relation is not a function. Here are some examples.



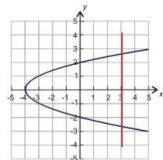
Not a function. It fails the vertical line test.



A function. No vertical line will cross more than one point on the graph.



A function. No vertical line will cross more than one point on the graph.



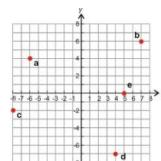
Not a function. It fails the vertical line test.

## Review Questions

1. Plot the coordinate points on the Cartesian plane.

- (a)  $(4, -4)$
- (b)  $(2, 7)$
- (c)  $(-3, -5)$
- (d)  $(6, 3)$
- (e)  $(-4, 3)$

2. Give the coordinates for each point in the Cartesian plane.



3. Graph the function that has the following table of values.

(a)

$x$	-10	-5	0	5	10
$y$	-3	-0.5	2	4.5	7

(b)

side of cube (in)	0	1	2	3
volume( $\text{in}^3$ )	0	1	8	27

(c)

time (hours)	-2	-1	0	1	2
distance from town center (miles)	50	25	0	25	50

4. Graph the following functions.

- (a) Brandon is a member of a movie club. He pays a \$50 annual membership and \$8 per movie.
- (b)  $f(x) = (x - 2)^2$
- (c)  $f(x) = 3.2^x$

5. Determine whether each relation is a function:

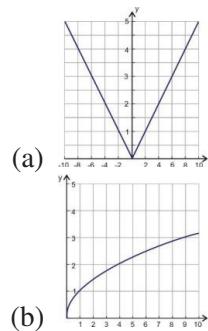
- (a)  $(1, 7), (2, 7), (3, 8), (4, 8), (5, 9)$
- (b)  $(1, 1), (1, -1), (4, 2), (4, -2), (9, 3), (9, -3)$
- (c)

$x$	-4	-3	-2	-1	0
$y$	16	9	4	1	0

(d)

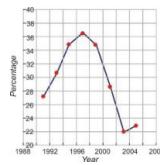
Age	20	25	25	30	35
Number of jobs by that age	3	4	7	4	2

6. Write the function rule for each graph.



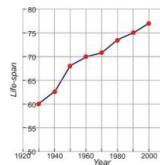
7. The students at a local high school took The Youth Risk Behavior Survey. The graph below shows the percentage of high school students who reported that they were current smokers. A person qualifies as a current smoker if he/she has smoked one or more cigarettes in the past 30 days. What percentage of high-school students were current smokers in the following years?

- (a) 1991
- (b) 1996
- (c) 2004
- (d) 2005



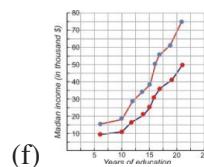
8. The graph below shows the average life-span of people based on the year in which they were born. This information comes from the National Vital Statistics Report from the Center for Disease Control. What is the average life-span of a person born in the following years?

- (a) 1940
- (b) 1955
- (c) 1980
- (d) 1995

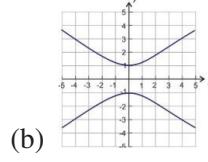
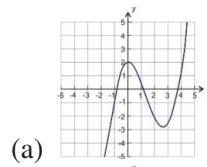


9. The graph below shows the median income of an individual based on his/her number of years of education. The top curve shows the median income for males and the bottom curve shows the median income for females. (Source: US Census, 2003.) What is the median income of a male that has the following years of education?

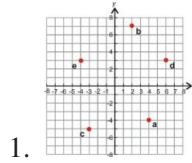
- (a) 10 years of education
- (b) 17 years of education
- (c) What is the median income of a female that has the same years of education?
- (d) 10 years of education
- (e) 17 years of education



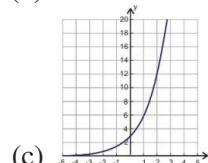
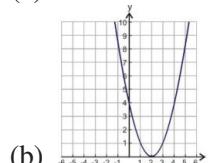
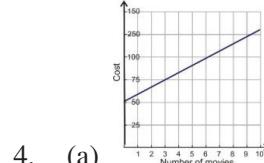
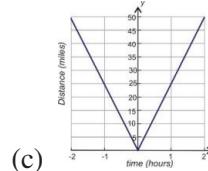
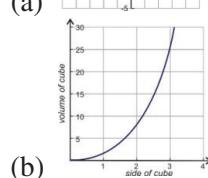
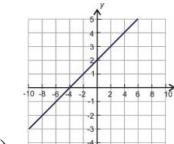
10. Use the vertical line test to determine whether each relation is a function.



### Review Answers



2. (a)  $(-6, 4)$ ;  
 (b)  $(7, 6)$ ;  
 (c)  $(-8, -2)$ ;  
 (d)  $(4, -7)$ ;  
 (e)  $(5, 0)$



5. (a) function

- (b) not a function
  - (c) function
  - (d) not a function
6. (a)  $f(x) = \frac{1}{2}|x|$
- (b)  $f(x) = \sqrt{x}$
7. (a) 27.5%
- (b) 35.6%
  - (c) 22.2%
  - (d) 23%
8. (a) 63 years
- (b) 69 years
  - (c) 74 years
  - (d) 76 years
9. (a) \$19,500
- (b) \$56,000
  - (c) \$10,000
  - (d) \$35,000
10. (a) function
- (b) not a function