

7.2 Solving Linear Systems by Substitution

Learning Objectives

- Solve systems of equations with two variables by substituting for either variable.
- Manipulate **standard form** equations to isolate a single variable.
- Solve real-world problems using systems of equations.
- Solve mixture problems using systems of equations.

Introduction

In this lesson, we will learn to solve a system of two equations using the method of substitution.

Solving Linear Systems Using Substitution of Variable Expressions

Lets look again at the problem involving Peter and Nadia racing.

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

In that example, we came up with two equations.

$$\begin{array}{ll} \text{Nadia's equation} & d = 6t \\ \text{Peter's equation} & d = 5t + 20 \end{array}$$

We have seen that each relationship produces its own line on a graph, but that to solve the system we find the point at which the lines intersect (Lesson 1). At that point the values for d and t satisfy **both** relationships.

In this simple example, this means that the d in Nadias equation is the same as the d in Peters. We can set the two equations equal to each other to solve for t .

$$\begin{array}{ll} 6t = 5t + 20 & \text{Subtract } 5t \text{ from both sides.} \\ t = 20 & \text{Substitute this value for } t \text{ into Nadia's equation.} \\ d = 6 \cdot 20 = 120 & \end{array}$$

Even if the equations are not so obvious, we can use simple algebraic manipulation to find an expression for one variable in terms of the other. We can rearrange Peters equation to isolate t .

$$\begin{array}{ll} d = 5t + 20 & \text{Subtract 20 from both sides.} \\ d - 20 = 5t & \text{Divide by 5.} \\ \frac{d - 20}{5} = t & \end{array}$$

We can now *substitute* this expression for t into Nadias equation ($d = 6t$) to solve it.

$$\begin{array}{ll}
 d = 6\left(\frac{d-20}{5}\right) & \text{Multiply both sides by 5.} \\
 5d = 6(d-20) & \text{Distribute the 6.} \\
 5d = 6d - 120 & \text{Subtract } 6d \text{ from both sides.} \\
 -d = -120 & \text{Divide by } -1. \\
 d = 120 & \text{Substitute value for } d \text{ into our expression for } t. \\
 t = \frac{120-20}{5} = \frac{100}{5} = 20 &
 \end{array}$$

We find that Nadia and Peter meet 20 seconds after they start racing, at a distance of 120 yards away.

The method we just used is called the **Substitution Method**. In this lesson, you will learn several techniques for isolating variables in a system of equations, and for using the expression you get for solving systems of equations that describe situations like this one.

Example 1

Let us look at an example where the equations are written in **standard form**.

Solve the system

$$\begin{array}{l}
 2x + 3y = 6 \\
 -4x + y = 2
 \end{array}$$

Again, we start by looking to isolate one variable in either equation. If you look at the second equation, you should see that the coefficient of y is 1. It makes sense to use this equation to solve for y .

Solve the second equation for the y variable:

$$\begin{array}{ll}
 -4x + y = 2 & \text{Add } 4x \text{ to both sides.} \\
 y = 2 + 4x &
 \end{array}$$

Substitute this expression into the second equation.

$$\begin{array}{ll}
 2x + 3(2 + 4x) = 6 & \text{Distribute the 3.} \\
 2x + 6 + 12x = 6 & \text{Collect like terms.} \\
 14x + 6 = 6 & \text{Subtract 6 from both sides.} \\
 14x = 0 & \\
 x = 0 &
 \end{array}$$

Substitute back into our expression for y .

$$y = 2 + 4 \cdot 0 = 2$$

As you can see, we end up with the same solution ($x = 0, y = 2$) that we found when we graphed these functions (Lesson 7.1). As long as you are careful with the algebra, the substitution method can be a very efficient way to solve systems.

Next, consider a more complicated example. In the following example the solution gives fractional answers for both x and y , and so would be very difficult to solve by graphing alone!

Example 2

Solve the system

$$\begin{aligned}2x + 3y &= 3 \\ 2x - 3y &= -1\end{aligned}$$

Again, we start by looking to isolate one variable in either equation. Right now it doesn't matter which equation we use or which variable we solve for.

Solve the first equation for x

$$\begin{aligned}2x + 3y &= 3 && \text{Subtract } 3y \text{ from both sides.} \\ 2x &= 3 - 3y && \text{Divide both sides by } 2. \\ x &= \frac{3 - 3y}{2}\end{aligned}$$

Substitute this expression into the second equation.

$$\begin{aligned}2 \cdot \frac{1}{2}(3 - 3y) - 3y &= -1 && \text{Cancel the fraction and rewrite terms.} \\ 3 - 3y - 3y &= -1 && \text{Collect like terms.} \\ 3 - 6y &= -1 && \text{Subtract } 3 \text{ from both sides.} \\ -6y &= -4 && \text{Divide by } -6. \\ y &= \frac{2}{3}\end{aligned}$$

Substitute into the expression and solve for x .

$$\begin{aligned}x &= \frac{1}{2} \left(3 - 3 \left(\frac{2}{3} \right) \right) \\ x &= \frac{1}{2}\end{aligned}$$

So our solution is, $x = \frac{1}{2}, y = \frac{2}{3}$. You can see why the graphical solution $(\frac{1}{2}, \frac{2}{3})$ might be difficult to read accurately.

Solving Real-World Problems Using Linear Systems

There are many situations where we can use simultaneous equations to help solve real-world problems. We may be considering a purchase. For example, trying to decide whether it is cheaper to buy an item online where you pay shipping or at the store where you do not. Or you may wish to join a CD music club, but do not know if you would

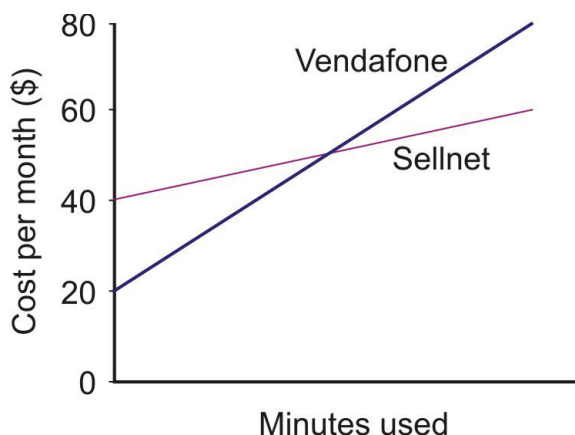
really save any money by buying a new CD every month in that way. One example with which we are all familiar is considering phone contracts. Lets look at an example of that now.

Example 3

Anne is trying to choose between two phone plans. The first plan, with Vendafone costs \$20 per month, with calls costing an additional 25 cents per minute. The second company, Sellnet, charges \$40 per month, but calls cost only 8 cents per minute. Which should she choose?

Annes choice will depend upon how many minutes of calls she expects to use each month. We start by writing two equations for the cost in dollars in terms of the minutes used. Since the number of minutes is the independent variable, it will be our x . Cost is dependent on minutes. The *cost per month* is the *dependent* variable and will be assigned y .

For Vendafone	$y = 0.25x + 20$
For Sellnet	$y = 0.08x + 40$



By writing the equations in slope-intercept form ($y = mx + b$) you can visualize the situation in a simple sketched graph, shown right. The line for Vendafone has an intercept of 20 and a slope of 0.25. The Sellnet line has an intercept of 40 and a slope of 0.08 (which is roughly a third of the Vendafone line). In order to help Anne decide which to choose, we will determine where the two lines cross, by solving the two equations as a system. Since equation one gives us an expression for $y(0.25x + 20)$, we can substitute this expression directly into equation two.

$0.25x + 20 = 0.08x + 40$	Subtract 20 from both sides.
$0.25x = 0.08x + 20$	Subtract $0.08x$ from both sides.
$0.17x = 20$	Divide both sides by 0.17.
$x = 117.65$ minutes	Rounded to two decimal places.

We can now use our sketch, plus this information to provide an answer:

If Anne will use 117 minutes or less every month, she should choose Vendafone. If she plans on using 118 or more minutes, she should choose Sellnet.

Mixture Problems

Systems of equations crop up frequently when considering chemicals in solutions, and can even be seen in things like mixing nuts and raisins or examining the change in your pocket! Lets look at some examples of these.

Example 4

Nadia empties her purse and finds that it contains only nickels (worth 5 cents each) and dimes (worth 10 cents each). If she has a total of 7 coins and they have a combined value of 55 cents, how many of each coin does she have?

Since we have two types of coins, let's call the number of nickels x and the number of dimes will be our y . We are given two key pieces of information to make our equations, the number of coins and their value.

Number of coins equation	$x + y = 7$	(number of nickels) + (number of dimes)
The value equation	$5x + 10y = 55$	Since nickels are worth five cents and dimes ten cents

We can quickly rearrange the first equation to isolate x .



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$$\begin{aligned}
 x &= 7 - y \\
 5(7 - y) + 10y &= 55 \\
 35 - 5y + 10y &= 55 \\
 35 + 5y &= 55 \\
 5y &= 20 \\
 y &= 4 \\
 +4 &= 7 \\
 x &= 3
 \end{aligned}$$

Now substitute into equation two.
 Distribute the 5.
 Collect like terms.
 Subtract 35 from both sides.
 Divide by 5.
 Substitute back into equation one.
 Subtract 4 from both sides.

Solution

Nadia has 3 nickels and 4 dimes.

Sometimes the question asks you to determine (from concentrations) how much of a particular substance to use. The substance in question could be something like coins as above, or it could be a chemical in solution, or even heat. In such a case, you need to know the amount of whatever substance is in each part. There are several common

situations where to get one equation you simply add two given quantities, but to get the second equation you need to use a **product**. Three examples are below.

TABLE 7.1:

Type of Mixture	First Equation	Second Equation
Coins (items with \$ value)	Total number of items ($n_1 n_2$)	Total value (item value \times no. of items)
Chemical solutions	Total solution volume ($V_1 + V_2$)	Amount of solute (vol \times concentration)
Density of two substances	Total amount or volume of mix	Total mass (volume \times density)

For example, when considering mixing chemical solutions, we will most likely need to consider the total amount of solute in the individual parts and in the final mixture. A solute is simply the chemical that is dissolved in a solution. An example of a solute is salt when added to water to make a brine. Even if the chemical is more exotic, we are still interested in the **total amount** of that chemical in each part. To find this, simply multiply the amount of the mixture by the **fractional concentration**. To illustrate, let's look at an example where you are given amounts relative to the whole.

Example 5

A chemist needs to prepare 500 ml of copper-sulfate solution with a 15% concentration. In order to do this, he wishes to use a high concentration solution (60%) and dilute it with a low concentration solution (5%). How much of each solution should he use?

To set this problem up, we first need to define our variables. Our unknowns are the amount of concentrated solution (x) and the amount of dilute solution (y). We will also convert the percentages (60%, 15% and 5%) into decimals (0.6, 0.15 and 0.05). The two pieces of critical information we need is the final volume (500 ml) and the final amount of solute (15% of 500 ml = 75 ml). Our equations will look like this.

$$\begin{array}{ll} \text{Volume equation} & x + y = 500 \\ \text{Solute equation} & 0.6x + 0.05y = 75 \end{array}$$

You should see that to isolate a variable for substitution it would be easier to start with equation one.



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$$\begin{aligned}
 x + y &= 500 \\
 x &= 500 - y \\
 0.6(500 - y) + 0.05y &= 75 \\
 300 - 0.6y + 0.05y &= 75 \\
 300 - 0.55y &= 75 \\
 -0.55y &= -225 \\
 y &= 409 \text{ ml} \\
 x &= 500 - 409 = 91 \text{ ml}
 \end{aligned}$$

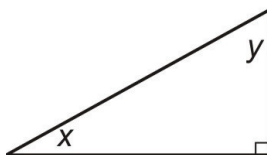
Subtract y from both sides.
 Now substitute into equation two.
 Distribute the 6.
 Collect like terms.
 Subtract 300 from both sides.
 Divide both sides by -0.55 .
 Substitute back into equation for x .

Solution

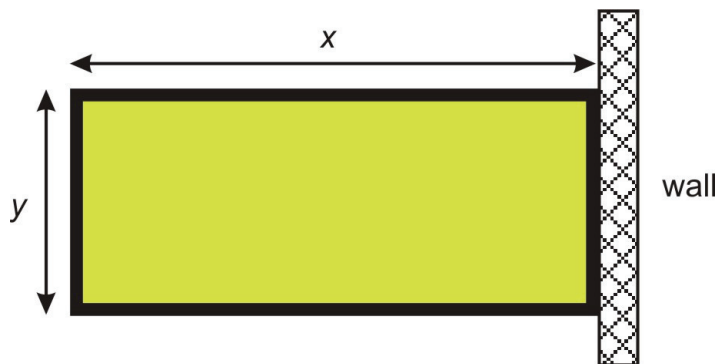
The chemist should mix 91 ml of the 60% solution with 409 ml of the 5% solution.

Review Questions

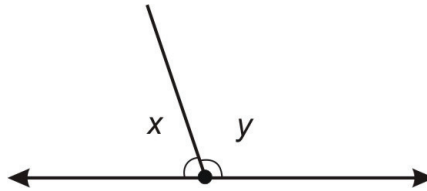
- Solve the system:
 $x + 2y = 9$
 $3x + 5y = 20$
- solve the system.
 $x - 3y = 10$
 $2x + y = 13$
- Of the two non-right angles in a right angled triangle, one measures twice that of the other. What are the angles?



- The sum of two numbers is 70. They differ by 11. What are the numbers?
- A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is 320 yards. If the field has a total perimeter of 400 yards, what are the dimensions of the field?



- A ray cuts a line forming two angles. The difference between the two angles is 18° . What does each angle measure?



7. I have \$15 and wish to buy five pounds of mixed nuts for a party. Peanuts cost \$2.20 per pound. Cashews cost \$4.70 per pound. How many pounds of each should I buy?
8. A chemistry experiment calls for one liter of sulfuric acid at a 15% concentration, but the supply room only stocks sulfuric acid in concentrations of 10% and in 35%. How many liters of each should be mixed to give the acid needed for the experiment?
9. Bachellet wants to know the density of her bracelet, which is a mix of gold and silver. Density is total mass divided by total volume. The density of gold is 19.3 g/cc and the density of silver is 10.5 g/cc . The jeweler told her that the volume of silver used was 10 cc and the volume of gold used was 20 cc . Find the combined density of her bracelet.

Review Answers

1. $x = -5, y = 7$
2. $x = 7, y = -1$
3. $x = 30^\circ, y = 60^\circ$
4. 29.5 and 40.5
5. $x = 120$ yards, $y = 80$ yards
6. $x = 81^\circ, y = 99^\circ$
7. 3.4 pounds of peanuts, 1.6 pounds of cashews
8. 0.8 liters of 10%, 0.2 liters of 35%
9. 16.4 g/cc