

7.5 Special Types of Linear Systems

Learning Objectives

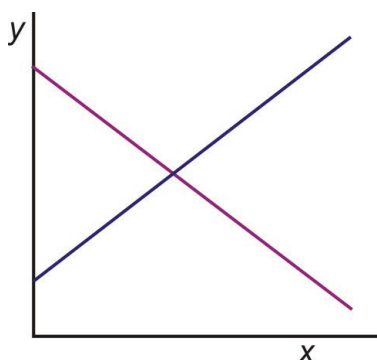
- Identify and understand what is meant by an **inconsistent linear system**.
- Identify and understand what is meant by a **consistent linear system**.
- Identify and understand what is meant by a **dependent linear system**.

Introduction

As we saw in Section 7.1, a system of linear equations is a set of linear equations which must be solved together. The lines in the system can be graphed together on the same coordinate graph and the solution to the system is the point at which the two lines intersect.

Determining the Type of System Graphically

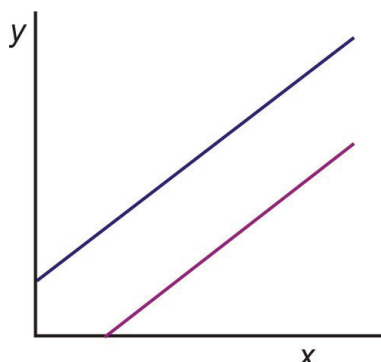
If we graph two lines on the same coordinate plane, three different situations may occur.



Case 1: The two lines intersect at a single point; hence the lines are not parallel.

If these lines were to represent a system of equations, the system would have exactly one solution, where the lines cross.

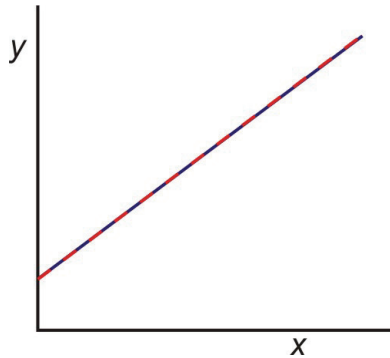
A system with exactly one solution is called a **consistent system**.



Case 2: The two lines do not intersect. The two lines are parallel.

If the lines represent a system of equations, then the system has no solutions.

A system with no solutions is called an **inconsistent system**.



Case 3: The two lines are identical. They intersect at all points on the line.

If this were a system of equations it would have an **infinite number** of solutions. Reason being, the two equations are really the same.

Such a system is called a **dependent system**.

To identify a system as **consistent**, **inconsistent**, or **dependent**, we can graph the two lines on the same graph and match the system with one of the three cases we discussed.

Another option is to write each line in slope-intercept form and compare the slopes and y -intercepts of the two lines. To do this we must remember that:

- Lines that intersect have different slopes.
- Lines that are parallel have the same slope but different y -intercepts.
- Lines that have the same slope and the same y -intercepts are identical.

Example 1

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned}y &= 3x + 2 \\ y &= -2x + 1\end{aligned}$$

Solution

The equations are already in slope-intercept form. The slope of the first equation is 3 and the slope of the second equation is -2 . Since the slopes are different, the lines must intersect at a single point. Therefore, the system has exactly one solution. This is a **consistent system**.

Example 2

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned}2x - 5y &= 2 \\ 4x + y &= 5\end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form.

$$\begin{array}{rcl}
 2x - 5y = 2 & & -5y = -2x + 2 & & y = \frac{2}{5}x - \frac{2}{5} \\
 & \Rightarrow & & \Rightarrow & \\
 4x + y = 5 & & y = -4x + 5 & & y = -4x + 5
 \end{array}$$

The slopes of the two equations are different. Therefore, the lines must cross at a single point, and the system has exactly one solution. This is a **consistent system**.

Example 3

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{array}{l}
 3x = 5 - 4y \\
 6x + 8y = 7
 \end{array}$$

Solution

We must rewrite the equations so they are in slope-intercept form.

$$\begin{array}{rcl}
 3x = 5 - 4y & & 4y = -3x + 5 & & y = \frac{-3}{4}x + \frac{5}{4} \\
 & \Rightarrow & & \Rightarrow & \\
 6x + 8y = 7 & & 8y = -6x + 7 & & y = \frac{-3}{4}x + \frac{7}{8}
 \end{array}$$

The slopes of the two equations are the same but the y -intercepts are different, therefore the lines never cross and the system has no solutions. This is an **inconsistent system**.

Example 4

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{array}{l}
 x + y = 3 \\
 3x + 3y = 9
 \end{array}$$

Solution

We must rewrite the equations so they are in slope-intercept form.

$$\begin{array}{rcl}
 x + y = 3 & & y = -x + 3 & & y = -x + 3 \\
 & \Rightarrow & & \Rightarrow & \\
 3x + 3y = 9 & & 3y = -3x + 9 & & y = -x + 3
 \end{array}$$

The lines are identical. Therefore the system has an infinite number of solutions. It is a **dependent system**.

Determining the Type of System Algebraically

A third option for identifying systems as **consistent**, inconsistent or dependent is to solve the system algebraically using any method and use the result as a guide.

Example 5

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned}10x - 3y &= 3 \\ 2x + y &= 9\end{aligned}$$

Solution

Lets solve this system using the substitution method.

Solve the second equation for the y variable.

$$2x + y = 9 \Rightarrow y = -2x + 9$$

Substitute for y in the first equation.

$$\begin{aligned}10x - 3y &= 3 \\ 10x - 3(-2x + 9) &= 3 \\ 10x + 6x - 27 &= 3 \\ 16x &= 30 \\ x &= \frac{15}{8}\end{aligned}$$

Substitute the value of x back into the second equation and solve for y.

$$2x + y = 9 \Rightarrow y = -2x + 9 \Rightarrow y = -2 \cdot \frac{15}{8} + 9 \Rightarrow y = \frac{21}{4}$$

Answer The solution to the system is $(\frac{15}{8}, \frac{21}{4})$. The system is consistent since it has only one solution.

Another method to determine if the system of equations is an inconsistent, consistent or dependent system is to solve them algebraically using the elimination or substitution method.

Example 6

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned}3x - 2y &= 4 \\ 9x - 6y &= 1\end{aligned}$$

Solution

Lets solve this system by the method of multiplication.

Multiply the first equation by 3.

$$\begin{array}{ccc}
 3(3x - 2y = 4) & & 9x - 6y = 12 \\
 & \Rightarrow & \\
 9x - 6y = 1 & & 9x - 6y = 1
 \end{array}$$

Add the two equations.

$$\begin{array}{r}
 9x - 6y = 12 \\
 9x - 6y = 1 \\
 \hline
 0 = 13 \quad \text{This Statement is not true}
 \end{array}$$

Answer If, by trying to obtain a solution to a system, we arrive at a statement that is not true, then the system is **inconsistent**.

Example 7

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{array}{l}
 4x + y = 3 \\
 12x + 3y = 9
 \end{array}$$

Solution

Lets solve this system by substitution.

Solve the first equation for y.

$$4x + y = 3 \Rightarrow y = -4x + 3$$

Substitute this expression for y in the second equation.

$$\begin{array}{l}
 12x + 3y = 9 \\
 12x + 3(-4x + 3) = 9 \\
 12x - 12x + 9 = 9 \\
 9 = 9
 \end{array}$$

This is always a **true** statement.

Answer If, by trying to obtain a solution to a system, we arrive at a statement that is always true, then the system is **dependent**.

A second glance at the system in this example reveals that the second equation is three times the first equation so the two lines are identical. The system has an infinite number of solutions because they are really the same equation and trace out the same line.

Lets clarify this statement. An infinite number of solutions does not mean that any ordered pair (x, y) satisfies the system of equations. Only ordered pairs that solve the equation in the system are also solutions to the system.

For example, $(1, 2)$ is not a solution to the system because when we plug it into the equations it does not check out.

$$\begin{aligned}4x + y &= 3 \\4(1) + 2 &\neq 3\end{aligned}$$

To find which ordered pair satisfies this system, we can pick any value for x and find the corresponding value for y .

$$\text{For } x = 1, 4(1) + y = 3 \Rightarrow y = -1$$

$$\text{For } x = 2, 4(2) + y = 3 \Rightarrow y = -3$$

Lets summarize our finding for determining the type of system algebraically.

- A **consistent system** will always give exactly one solution.
- An **inconsistent system** will always give a FALSE statement (for example $9 = 0$).
- A **dependent system** will always give a TRUE statement (such as $9 = 9$ or $0 = 0$).

Applications

In this section, we will look at a few application problems and see how consistent, inconsistent and dependent systems might arise in practice.

Example 8

A movie rental store CineStar offers customers two choices. Customers can pay a yearly membership of \$45 and then rent each movie for \$2 or they can choose not to pay the membership fee and rent each movie for \$3.50. How many movies would you have to rent before membership becomes the cheaper option?

Solution

Lets translate this problem into algebra. Since there are two different options to consider, we will write two different equations and form a system.

The choices are membership option and no membership option.

Our variables are

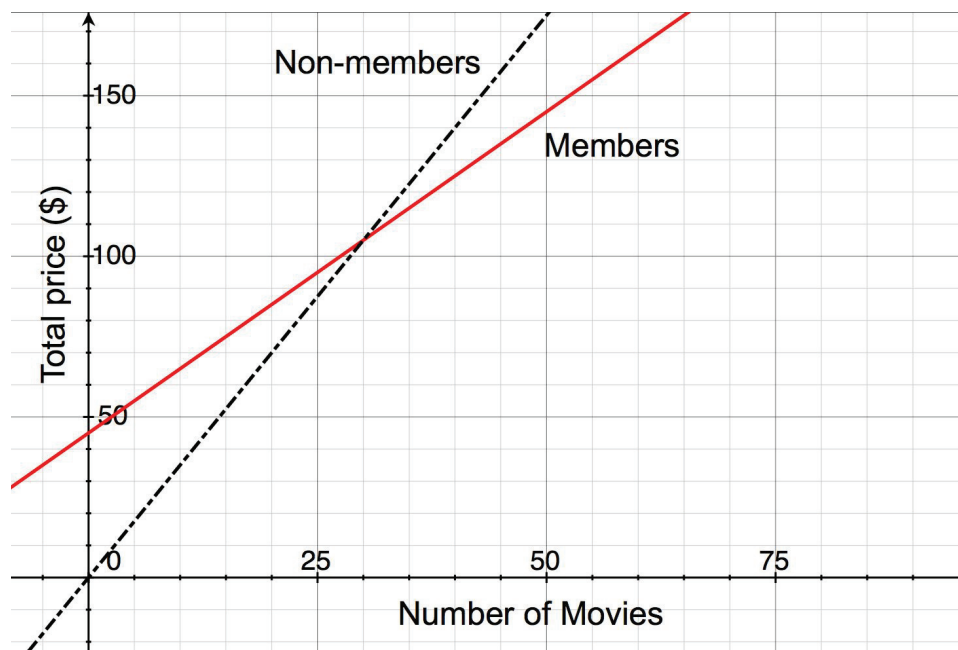
The number of movies you rent, lets call this x .

The total cost for renting movies, lets call this y .

TABLE 7.3:

	flat fee	rental fee	total
membership	\$45	$2x$	$y = 45 + 2x$
no membership	\$0	$3.50x$	$y = 3.5x$

The flat fee is the dollar amount you pay per year and the rental fee is the dollar amount you pay when you rent movies. For the membership option, the rental fee is $2x$ since you would pay \$2 for each movie you rented. For the no membership option, the rental fee is $3.50x$ since you would pay \$3.50 for each movie you rented.



Our system of equations is

$$y = 45 + 2x$$

$$y = 3.50x$$

The graph of our system of equations is shown to the right.

This system can be solved easily with the method of substitution since each equation is already solved for y . Substitute the second equation into the first one

$$y = 45 + 2x$$

$$\Rightarrow 3.50x = 45 + 2x \Rightarrow 1.50x = 45 \Rightarrow x = 30 \text{ movies}$$

$$y = 3.50x$$

Answer You would have to rent 30 movies per year before the membership becomes the better option.

This example shows a real situation where a consistent system of equations is useful in finding a solution. Remember that for a consistent system, the lines that make up the system intersect at single point. In other words, the lines are not parallel or the slopes are different.

In this case, the slopes of the lines represent the price of a rental per movie. The lines cross because the price of rental per movie is different for the two options in the problem

Lets examine a situation where the system is inconsistent. From the previous explanation, we can conclude that the lines will not intersect if the slopes are the same (but the y -intercept is different). Lets change the previous problem so that this is the case.

Example 9

Two movie rental stores are in competition. Movie House charges an annual membership of \$30 and charges \$3 per movie rental. Flicks for Cheap charges an annual membership of \$15 and charges \$3 per movie rental. After how many movie rentals would Movie House become the better option?

Solution

It should already be clear to see that Movie House will never become the better option, since its membership is more expensive and it charges the same amount per movie as Flicks for Cheap.

The lines that describe each option have different y -intercepts, namely 30 for Movie House and 15 for Flicks for Cheap. They have the same slope, three dollars per movie. This means that the lines are parallel and the system is inconsistent.

Lets see how this works algebraically:

Our variables are:

The number of movies you rent, lets call this x .

The total cost for renting movies, lets call this y .

TABLE 7.4:

	flat fee	rental fee	total
Movie House	\$30	$3x$	$y = 30 + 3x$
Flicks for Cheap	\$15	$3x$	$y = 15 + 3x$

The system of equations that describes this problem is

$$y = 30 + 3x$$

$$y = 15 + 3x$$

Lets solve this system by substituting the second equation into the first equation.

$$y = 30 + 3x$$

$$\Rightarrow 15 + 3x = 30 + 3x \Rightarrow 15 = 30$$

$$y = 15 + 3x$$

This statement is always false.

Answer This means that the system is inconsistent.

Example 10

Peter buys two apples and three bananas for \$4. Nadia buys four apples and six bananas for \$8 from the same store. How much does one banana and one apple costs?

Solution

We must write two equations, one for Peters purchase and one for Nadias purchase.

Lets define our variables as

a is the cost of one apple

b is the cost of one banana

TABLE 7.5:

	Cost of Apples	Cost of Bananas	Total Cost
Peter	$2a$	$3b$	$2a + 3b = 4$
Nadia	$4a$	$6b$	$4a + 6b = 8$

The system of equations that describes this problem is:

$$\begin{aligned}2a + 3b &= 4 \\4a + 6b &= 8\end{aligned}$$

Lets solve this system by multiplying the first equation by -2 and adding the two equations.

$$\begin{array}{rcl} -2(2a + 3b = 4) & & -4a - 6b = -8 \\ 4a + 6b = 8 & \Rightarrow & 4a + 6b = 8 \\ \hline & & 0 + 0 = 0 \end{array}$$

This statement is always true. This means that the system is **dependent**.

Looking at the problem again, we see that we were given exactly the same information in both statements. If Peter buys two apples and three bananas for \$4 it makes sense that if Nadia buys twice as many apples (four apples) and twice as many bananas (six bananas) she will pay twice the price (\$8). Since the second equation does not give any new information, it is not possible to find out the price of each piece of fruit.

Answer The two equations describe the same line. This means that the system is dependent.

Review Questions

- Express each equation in slope-intercept form. Without graphing, state whether the system of equations is consistent, inconsistent or dependent.
 - $3x - 4y = 13$
 $y = -3x - 7$
 - $\frac{3x}{5} + y = 3$
 $1.2x + 2y = 6$
 - $3x - 4y = 13$
 $y = -3x - 7$
 - $3x - 3y = 3$
 $x - y = 1$
 - $0.5x - y = 30$
 $0.5x - y = -30$
 - $4x - 2y = -2$
 $3x + 2y = -12$
- Find the solution of each system of equations using the method of your choice. Please state whether the system is inconsistent or dependent.
 - $3x + 2y = 4$
 $-2x + 2y = 24$
 - $5x - 2y = 3$
 $2x - 3y = 10$
 - $3x - 4y = 13$
 $y = -3x - y$
 - $5x - 4y = 1$
 $-10x + 8y = -30$

- (e) $4x + 5y = 0$
 $3x = 6y + 4.5$
 - (f) $-2y + 4x = 8$
 $y - 2x = -4$
 - (g) $x - \frac{y}{2} = \frac{3}{2}$
 $3x + y = 6$
 - (h) $0.05x + 0.25y = 6$
 $x + y = 24$
 - (i) $x + \frac{2y}{3} = 6$
 $3x + 2y = 2$
3. A movie house charges \$4.50 for children and \$8.00 for adults. On a certain day, 1200 people enter the movie house and \$8,375 is collected. How many children and how many adults attended?
4. Andrew placed two orders with an internet clothing store. The first order was for thirteen ties and four pairs of suspenders, and totaled \$487. The second order was for six ties and two pairs of suspenders, and totaled \$232. The bill does not list the per-item price but all ties have the same price and all suspenders have the same price. What is the cost of one tie and of one pair of suspenders?
5. An airplane took four hours to fly 2400 miles in the direction of the jet-stream. The return trip against the jet-stream took five hours. What were the airplanes speed in still air and the jet-stream's speed?
6. Nadia told Peter that she went to the farmers market and she bought two apples and one banana and that it cost her \$2.50. She thought that Peter might like some fruit so she went back to the seller and bought four more apples and two more bananas. Peter thanked Nadia, but told her that he did not like bananas, so will only pay her for four apples. Nadia told him that the second time she paid \$6.00 for fruit. Please help Peter figure out how much to pay Nadia paid for four apples.

Review Answers

- 1. (a) consistent
(b) dependent
(c) consistent
(d) dependent
(e) inconsistent
(f) consistent
- 2. (a) $x = -4, y = 8$
(b) $x = -1, y = -4$
(c) $x = -1, y = -4$
(d) inconsistent
(e) $x = -2.5, y = 2$
(f) dependent
(g) $x = \frac{9}{5}, y = \frac{3}{5}$
(h) $x = 0, y = 24$
(i) dependent
- 3. 350 children, 850 Adults
- 4. Ties = \$23, suspenders = \$47
- 5. Airplane speed = 540 mph, jet – stream speed = 60 mph
- 6. This represents an inconsistent system. Someone is trying to overcharge! It is not possible to determine the price of apples alone.