

7.6 Systems of Linear Inequalities

Learning Objectives

- Graph linear inequalities in two variables.
- Solve systems of linear inequalities.
- Solve optimization problems.

Introduction

In the last chapter, you learned how to graph a linear inequality in two variables. To do that you graphed the equation of the straight line on the coordinate plane. The line was solid for signs where the equal sign is included. The line was dashed for $<$ or $>$ where signs the equal sign is not included. Then you shaded above the line (if $y >$ or $y \geq$) or below the line (if $y <$ or $y \leq$).

In this section, we will learn how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph and the solution for the system is the common shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two **half-planes**. A **system** of two or more linear inequalities can divide the plane into more complex shapes. Lets start by solving a system of two inequalities.

Graph a System of Two Linear Inequalities

Example 1

Solve the following system.

$$\begin{aligned}2x + 3y &\leq 18 \\ x - 4y &\leq 12\end{aligned}$$

Solution

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection regions of the solution.

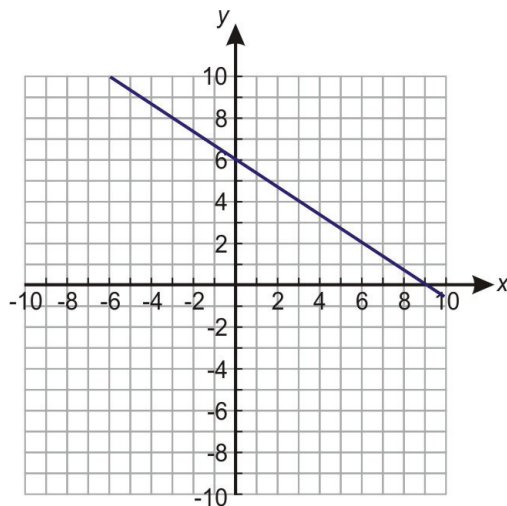
Lets rewrite each equation in slope-intercept form. This form is useful for graphing but also in deciding which region of the coordinate plane to shade. Our system becomes

$$\begin{array}{ccc}3y \leq -2x + 18 & & y \leq \frac{-2}{3}x + 6 \\ & \Rightarrow & \\ -4y \leq -x + 12 & & y \geq \frac{x}{4} - 3\end{array}$$

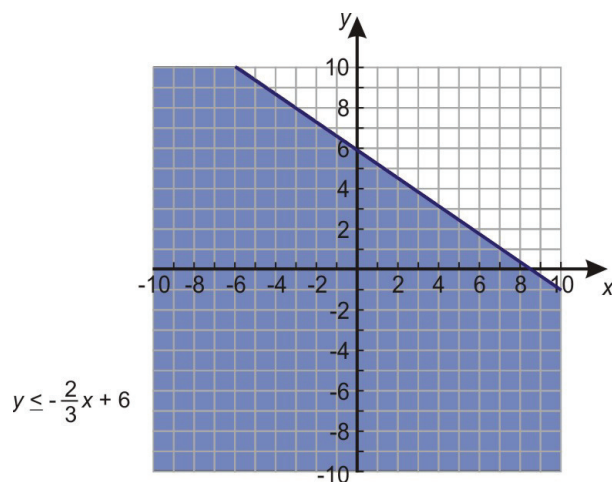
Notice that the inequality sign in the second equation changed because we divided by a negative number.

For this first example, we will graph each inequality separately and then combine the results.

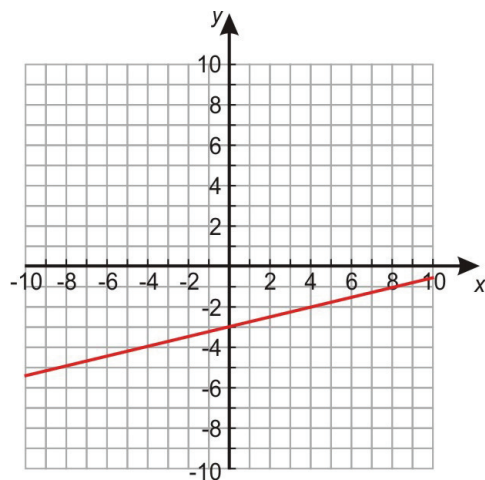
We graph the equation of the line in the first inequality and draw the following graph.



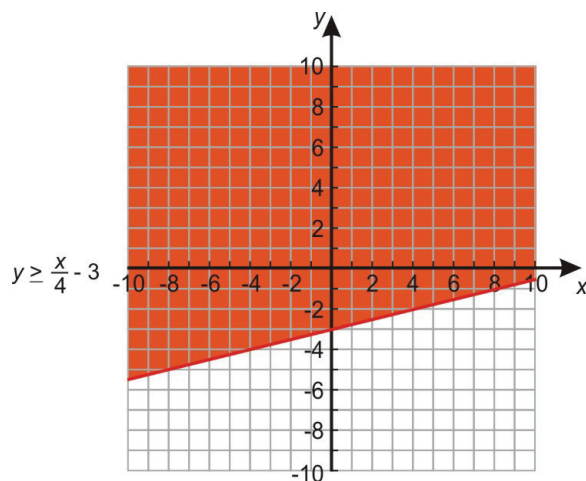
The line is solid because the equal sign is included in the inequality. Since the inequality is less than or equal to, we shade below the line.



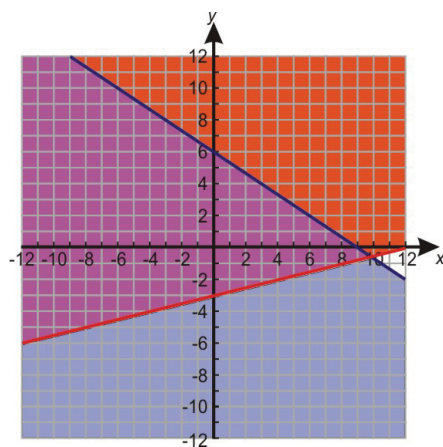
We graph the second equation in the inequality and obtain the following graph.



The line is solid again because the equal sign is included in the inequality. We now shade above because y is **greater** than or equal.



When we combine the graphs, we see that the blue and red shaded regions overlap. This overlap is where both inequalities work. Thus the purple region denotes the solution of the system.



The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

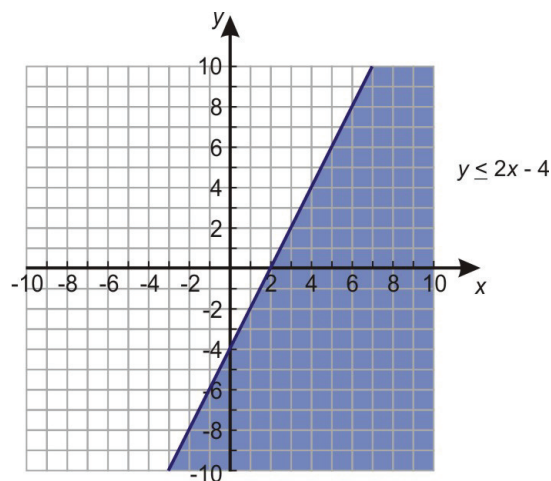
Example 2

There are also situations where a system of inequalities has no solution. For example, let's solve this system.

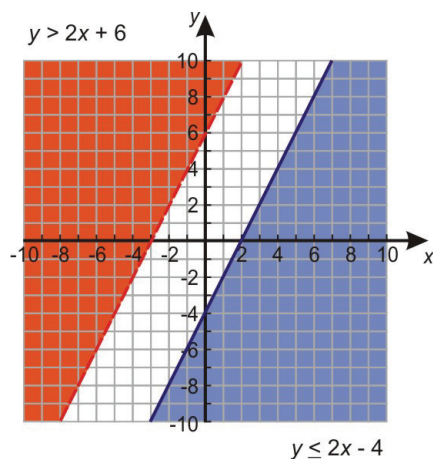
$$y \leq 2x - 4$$

$$y > 2x + 6$$

Solution: We start by graphing the first line. The line will be solid because the equal sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equal sign is not included in the inequality. We must shade upward because y is greater than.



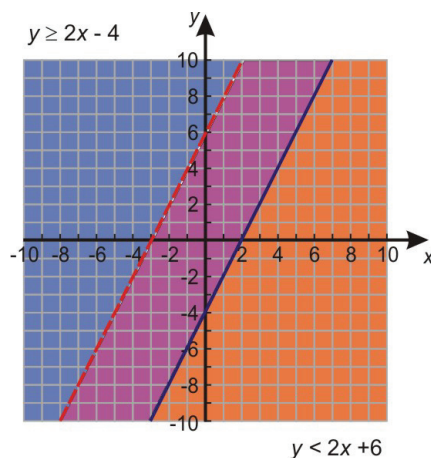
This graph shows no overlapping between the two shaded regions. We know that the lines will never intersect because they are parallel. The slope equals two for both lines. The regions will never overlap even if we extend the lines further.

This is an example of a system of inequalities with no solution.

For a system of inequalities, we can still obtain a solution even if the lines are parallel. Let's change the system of inequalities in Example 2 so the inequality signs for the two expressions are reversed.

$$y \geq 2x - 4$$

$$y < 2x + 6$$



The procedure for solving this system is almost identical to the previous one, except we shade upward for the first inequality and we shade downward for the second inequality. Here is the result.

In this case, the shaded regions do overlap and the system of inequalities has the solution denoted by the purple region.

Graph a System of More Than Two Linear Inequalities

In the previous section, we saw how to find the solution to a system of two linear inequalities. The solutions for these kinds of systems are always unbounded. In other words, the region where the shadings overlap continues infinitely in at least one direction. We can obtain **bounded** solutions by solving systems that contain more than two inequalities. In such cases the solution region will be bounded on four sides.

Lets examine such a solution by solving the following example.

Example 3

Find the solution to the following system of inequalities.

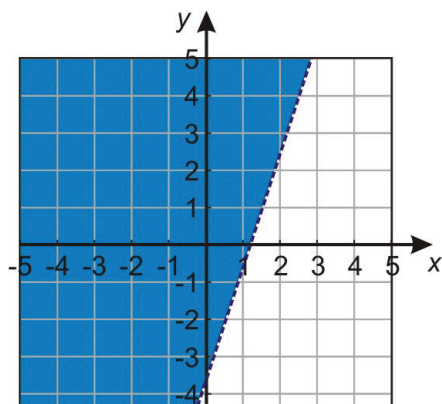
$$\begin{aligned} 3x - y &< 4 \\ 4y + 9x &< 8 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solution

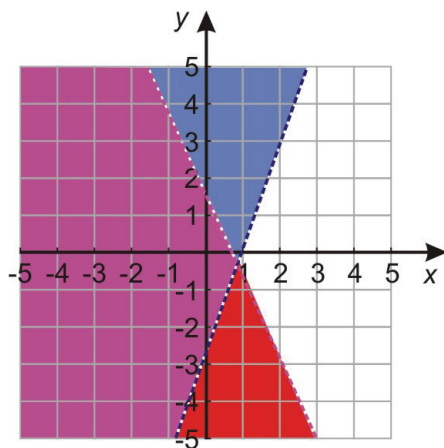
Lets start by writing our equation in slope-intercept form.

$$\begin{aligned} y &> 3x - 4 \\ y &< -\frac{9}{4}x + 2 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

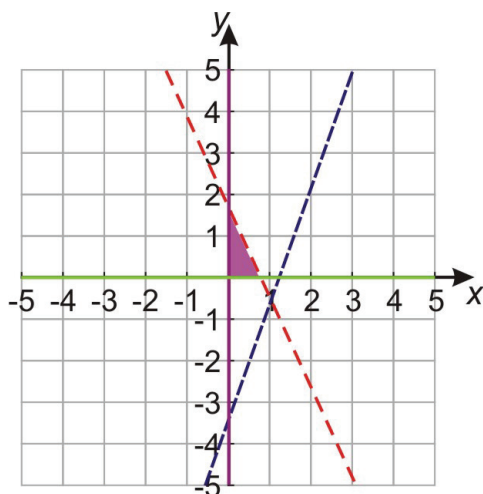
Now we can graph each line and shade appropriately. First we graph $y > 3x - 4$.



Next we graph $y < -\frac{9}{4}x + 2$



Finally we graph $x \geq 0$ and $y \geq 0$, and the intersecting region is shown in the following figure.



The solution is **bounded** because there are lines on all sides of the solution region. In other words the solution region is a bounded geometric figure, in this case a triangle.

Write a System of Linear Inequalities

There are many interesting application problems that involve the use of system of linear inequalities. However, before we fully solve application problems, lets see how we can translate some simple word problems into algebraic equations.

For example, you go to your favorite restaurant and you want to be served by your best friend who happens to work there. However, your friend works in a certain region of the restaurant. The restaurant is also known for its great views but you have to sit in a certain area of the restaurant that offers these view. Solving a system of linear inequalities will allow you to find the area in the restaurant where you can sit to get the best views and be served by your friend.

Typically, systems of linear inequalities deal with problems where you are trying to find the best possible situation given a set of constraints.

Example 4

Write a system of linear inequalities that represents the following conditions.

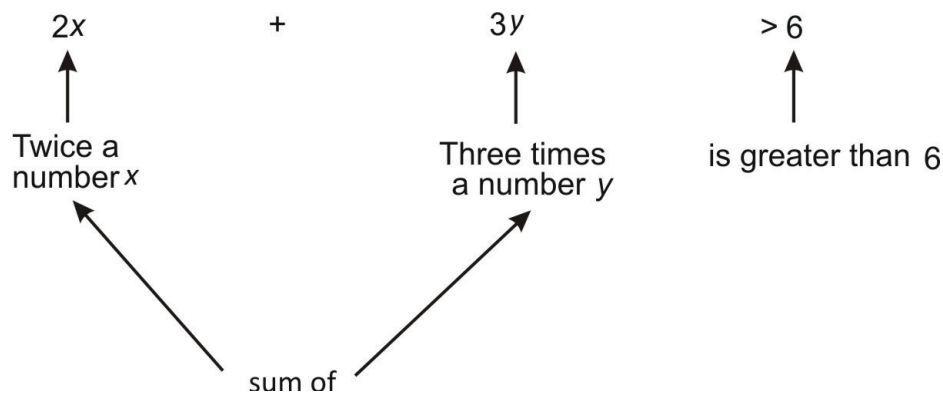
The sum of twice a number x and three times another number y is greater than 6, and y is less than three times x .

Solution

Lets take each statement in turn and write it algebraically:

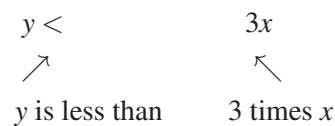
1. The sum of twice a number x and three times another number y is greater than 6.

This can be written as



2. y is less than three times x .

This can be written as



The system of inequalities arising from these statements is

$$\begin{aligned} 2x + 3y &> 6 \\ y &< 3x \end{aligned}$$

This system of inequalities can be solved using the method outlined earlier in this section. We will not solve this system because we want to concentrate on learning how to write a system of inequalities from a word problem.

Solve Real-World Problems Using Systems of Linear Inequalities

As we mentioned before, there are many interesting application problems that require the use of systems of linear inequalities. Most of these application problems fall in a category called **linear programming** problems.

Linear programming is the process of taking various linear inequalities relating to some situation, and finding the best possible value under those conditions. A typical example would be taking the limitations of materials and labor, then determining the best production levels for maximal profits under those conditions. These kinds of problems are used every day in the organization and allocation of resources. These real life systems can have dozens or hundreds of variables, or more. In this section, we will only work with the simple two-variable linear case.

The general process is to:

- Graph the inequalities (called **constraints**) to form a bounded area on the x,y -plane (called **the feasibility region**).
- Figure out the coordinates of the corners (or vertices) of this feasibility region by solving the systems of equations that give the solutions to each of the intersection points.
- Test these corner points in the formula (called the **optimization equation**) for which you're trying to find the **maximum** or **minimum** value.

Example 5

Find the maximum and minimum value of $z = 2x + 5y$ given the constraints.

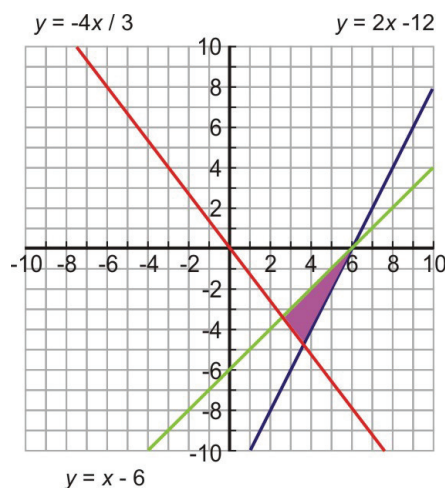
$$2x - y \leq 12$$

$$4x + 3y \geq 0$$

$$x - y \leq 6$$

Solution

Step 1: Find the solution to this system of linear inequalities by graphing and shading appropriately. To graph we must rewrite the equations in slope-intercept form.



$$y \geq 2x - 12$$

$$y \geq -\frac{4}{3}x$$

$$y \geq x - 6$$

These three linear inequalities are called the **constraints**.

The solution is the shaded region in the graph. This is called the **feasibility region**. That means all possible solutions occur in that region. However in order to find the optimal solution we must go to the next steps.

Step 2

Next, we want to find the corner points. In order to find them exactly, we must form three systems of linear equations and solve them algebraically.

System 1:

$$y = 2x - 12$$

$$y = -\frac{4}{3}x$$

Substitute the first equation into the second equation:

$$-\frac{4}{3}x = 2x - 12 \Rightarrow -4x = 6x - 36 \Rightarrow -10x = -36 \Rightarrow x = 3.6$$

$$y = 2x - 12 \Rightarrow y = 2(3.6) - 12 \Rightarrow y = -4.8$$

The intersection point of lines is $(3.6, -4.8)$

System 2:

$$y = 2x - 12$$

$$y = x - 6$$

Substitute the first equation into the second equation.

$$x - 6 = 2x - 12 \Rightarrow 6 = x \Rightarrow x = 6$$

$$y = x - 6 \Rightarrow y = 6 - 6 \Rightarrow y = 0$$

The intersection point of lines is $(6, 0)$.

System 3:

$$y = -\frac{4}{3}x$$

$$y = x - 6$$

Substitute the first equation into the second equation.

$$x - 6 = -\frac{4}{3}x \Rightarrow 3x - 18 = -4x \Rightarrow 7x = 18 \Rightarrow x = 2.57$$

$$y = x - 6 \Rightarrow y = 2.57 - 6 \Rightarrow y = -3.43$$

The intersection point of lines is $(2.57, -3.43)$.

So the corner points are $(3.6, -4.8)$, $(6, 0)$ and $(2.57, -3.43)$.

Step 3

Somebody really smart proved that, for linear systems like this, the maximum and minimum values of the optimization equation will always be on the corners of the feasibility region. So, to find the solution to this exercise, we need to plug these three points into $z = 2x + 5y$.

$(3.6, -4.8)$	$z = 2(3.6) + 5(-4.8) = -16.8$
$(6, 0)$	$z = 2(6) + 5(0) = 12$
$(2.57, -3.43)$	$z = 2(2.57) + 5(-3.43) = -12.01$

The highest value of 12 occurs at point $(6, 0)$ and the lowest value of -16.8 occurs at $(3.6, -4.8)$.

In the previous example, we learned how to apply the method of linear programming out of context of an application problem. In the next example, we will look at a real-life application.

Example 6

You have \$10,000 to invest, and three different funds from which to choose. The municipal bond fund has a 5% return, the local bank's CDs have a 7% return, and a high-risk account has an expected 10% return. To minimize risk, you decide not to invest any more than \$1,000 in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. Assuming the year-end yields are as expected, what are the optimal investment amounts?

Solution

Lets define some *variables*.

x is the amount of money invested in the municipal bond at 5% return.

y is the amount of money invested in the banks CD at 7% return.

$10000 - x - y$ is the amount of money invested in the high-risk account at 10% return.

z is the total interest returned from all the investments or $z = .05x + .07y + .1(10000 - x - y)$ or $z = 1000 - 0.05x - 0.03y$. This is the amount that we are trying to maximize. Our goal is to find the values of x and y that maximizes the value of z .

Now, lets write inequalities for the *constraints*.

You decide not to invest more than \$1000 in the high-risk account.

$$10000 - x - y \leq 1000$$

You need to invest at least three times as much in the municipal bonds as in the bank CDs.

$$3y \leq x$$

Also we write expressions for the fact that we invest more than zero dollars in each account.

$$x \geq 0$$

$$y \geq 0$$

$$10000 - x - y \geq 0$$

To summarize, we must maximize the expression $z = 1000 - .05x - .03y$.

Using the constraints,

$$10000 - x - y \leq 1000$$

$$3y \leq x$$

$$x \geq 0$$

$$y \geq 0$$

$$10000 - x - y \geq 0$$

$$y \geq 9000 - x$$

$$y \leq \frac{x}{3}$$

$$x \geq 0$$

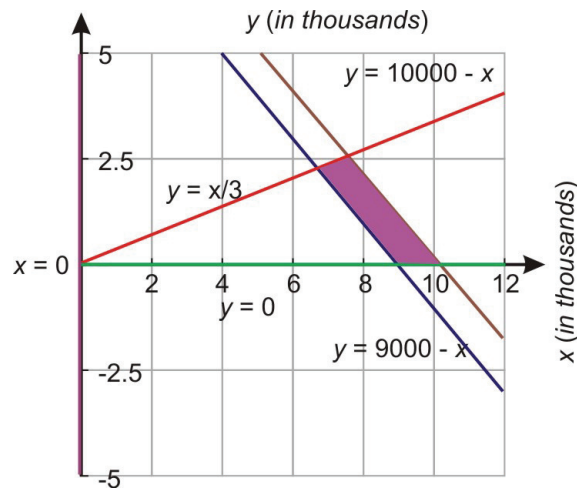
$$y \geq 0$$

$$y \leq 10000 - x$$

Let's rewrite each in slope-intercept form.

Step 1 Find the solution region to the set of inequalities by graphing each line and shading appropriately.

The following figure shows the overlapping region.



The purple region is the feasibility region where all the possible solutions can occur.

Step 2 Next, we need to find the corner points of the shaded solution region. Notice that there are four intersection points. To find them we must pair up the relevant equations and solve the resulting system.

System 1:

$$y = -\frac{x}{3}$$

$$y = 10000 - x$$

Substitute the first equation into the second equation.

$$\begin{aligned}\frac{x}{3} &= 10000 - x \Rightarrow x = 30000 - 3x \Rightarrow x = 7500 \\ y &= \frac{x}{3} \Rightarrow y = \frac{7500}{3} \Rightarrow y = 2500\end{aligned}$$

The intersection point is (7500, 2500).

System 2:

$$\begin{aligned}y &= -\frac{x}{3} \\ y &= 9000 - x\end{aligned}$$

Substitute the first equation into the second equation.

$$\frac{x}{3} = 9000 - x \Rightarrow x = 27000 - 3x \Rightarrow 4x = 27000 \Rightarrow x = 6750$$

$$\frac{x}{3} \Rightarrow y = \frac{6750}{3} \Rightarrow y = 2250$$

The intersection point is (6750, 2250).

System 3:

$$\begin{aligned}y &= 0 \\ y &= 10000 - x\end{aligned}$$

The intersection point is (10000, 0).

System 4:

$$\begin{aligned}y &= 0 \\ y &= 9000 - x\end{aligned}$$

The intersection point is (9000, 0).

Step 3: In order to find the maximum value for z , we need to plug all intersection points into z and take the largest number.

(7500, 2500)	$z = 1000 - 0.05(7500) - 0.03(2500) = 550$
(6750, 2250)	$z = 1000 - 0.05(6750) - 0.03(2250) = 595$
(10000, 0)	$z = 1000 - 0.05(10000) - 0.03(0) = 500$
(9000, 0)	$z = 1000 - 0.05(9000) - 0.03(0) = 550$

Answer

The maximum return on the investment of \$595 occurs at point (6750, 2250). This means that \$6,750 is invested in the municipal bonds.
\$2,250 is invested in the bank CDs.
\$1,000 is invested in the high-risk account.

Review Questions

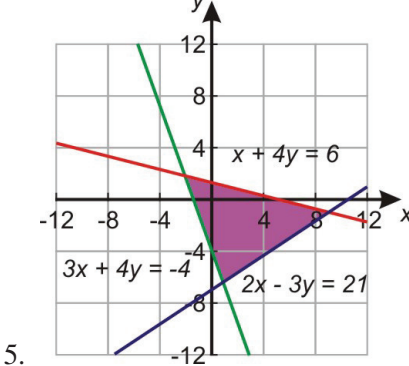
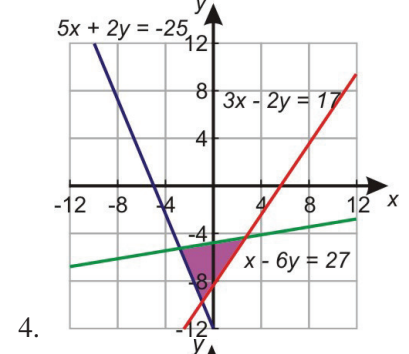
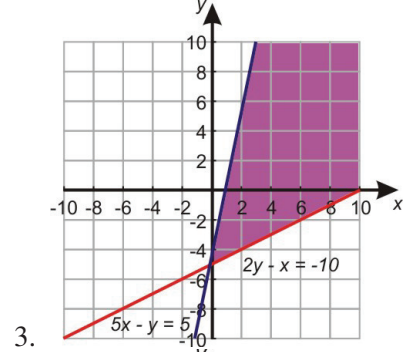
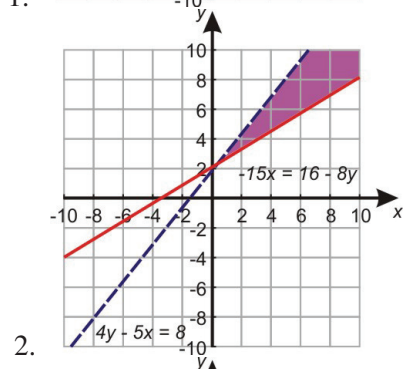
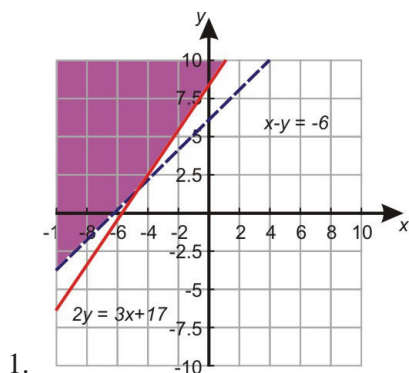
Find the solution region of the following systems of inequalities

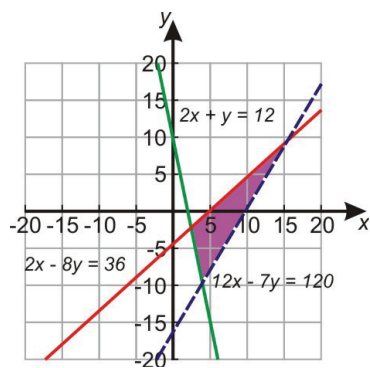
1. $x - y < -6$
 $2y \geq 3x + 17$
2. $4y - 5x < 8$
 $-5x \geq 16 - 8y$
3. $5x - y \geq 5$
 $2y - x \geq -10$
4. $5x + 2y \geq -25$
 $3x - 2y \leq 17$
 $x - 6y \geq 27$
5. $2x - 3y \leq 21$
 $x + 4y \leq 6$
 $3x + y \geq -4$
6. $12x - 7y < 120$
 $7x - 8y \geq 36$
 $5x + y \geq 12$

Solve the following linear programming problems:

7. Given the following constraints find the maximum and minimum values of $z = -x + 5y$
 $x + 3y \leq 0$
 $x - y \geq 0$
 $3x - 7y \leq 16$
8. In Andrews Furniture Shop, he assembles both bookcases and TV cabinets. Each type of furniture takes him about the same time to assemble. He figures he has time to make at most 18 pieces of furniture by this Saturday.
The materials for each bookcase cost him \$20 and the materials for each TV stand costs him \$45 . He has \$600 to spend on materials. Andrew makes a profit of \$60 on each bookcase and a profit of \$100 for each TV stand.
Find how many of each piece of furniture Andrew should make so that he maximizes his profit.

Review Answers





6.

7. Maximum of $z = 0$ at point $(0, 0)$, minimum of $z = -16$ at point $(-4, -4)$
8. Maximum profit of \$1,440 by making 9 bookcases and 9 TV stands.