

## 8.1 Exponent Properties Involving Products

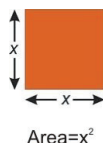
### Learning Objectives

- Use the product of a power property.
- Use the power of a product property.
- Simplify expressions involving product properties of exponents.

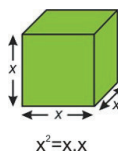
### Introduction

In this chapter, we will discuss exponents and exponential functions. In Lessons 8.1, 8.2 and 8.3, we will be learning about the rules governing exponents. We will start with what the word exponent means.

Consider the area of the square shown right. We know that the area is given by:



But we also know that for any rectangle, Area = (width)  $\cdot$  (height), so we can see that:



Similarly, the volume of the cube is given by:

$$\text{Volume} = \text{width} \cdot \text{depth} \cdot \text{height} = x \cdot x \cdot x$$

But we also know that the volume of the cube is given by Volume =  $x^3$  so clearly

$$x^3 = x \cdot x \cdot x$$

You probably know that the **power** (the small number to the top right of the  $x$ ) tells you how many  $x$ 's to multiply together. In these examples the  $x$  is called the **base** and the **power** (or **exponent**) tells us how many **factors** of the **base** there are in the full expression.

$$x^2 = \underbrace{x \cdot x}_{2 \text{ factors of } x}$$

$$x^3 = \underbrace{x \cdot x \cdot x}_{3 \text{ factors of } x}$$

$$x^7 = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{7 \text{ factors of } x}$$

$$x^n = \underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ factors of } x}$$

**Example 1**

Write in exponential form.

(a)  $2 \cdot 2$

(b)  $(-3)(-3)(-3)$

(c)  $y \cdot y \cdot y \cdot y \cdot y$

(d)  $(3a)(3a)(3a)(3a)$

**Solution**

(a)  $2 \cdot 2 = 2^2$  because we have 2 factors of 2

(b)  $(-3)(-3)(-3) = (-3)^3$  because we have 3 factors of  $(-3)$

(c)  $y \cdot y \cdot y \cdot y \cdot y = y^5$  because we have 5 factors of  $y$

(d)  $(3a)(3a)(3a)(3a) = (3a)^4$  because we have 4 factors of  $3a$

When we deal with numbers, we usually just simplify. We'd rather deal with 16 than with  $2^4$ . However, with variables, we need the exponents, because we'd rather deal with  $x^7$  than with  $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$ .

Lets simplify Example 1 by evaluating the numbers.

**Example 2**

Simplify.

(a)  $2 \cdot 2$

(b)  $(-3)(-3)(-3)$

(c)  $y \cdot y \cdot y \cdot y \cdot y$

(d)  $(3a)(3a)(3a)(3a)$

**Solution**

(a)  $2 \cdot 2 = 2^2 = 4$

(b)  $(-3)(-3)(-3) = (-3)^3 = -27$

(c)  $y \cdot y \cdot y \cdot y \cdot y = y^5$

(d)  $(3a)(3a)(3a)(3a) = (3a)^4 = 3^4 \cdot a^4 = 81a^4$

**Note:** You must be careful when taking powers of negative numbers. Remember these rules.

(negative number) (positive number) = negative number

(negative number) (negative number) = positive number

For **even powers of negative numbers**, the answer is always positive. Since we have an even number of factors, we make pairs of negative numbers and all the negatives cancel out.

$$(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} = +64$$

For **odd powers of negative numbers**, the answer is always negative. Since we have an odd number of factors, we can make pairs of negative numbers to get positive numbers but there is always an unpaired negative factor, so the answer is negative:

$$\text{Ex: } (-2)^5 = (-2)(-2)(-2)(-2)(-2) = \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)}_{-2} = -32$$

### Use the Product of Powers Property

What happens when we multiply one power of  $x$  by another? See what happens when we multiply  $x$  *to the power 5* by  $x$  *cubed*. To illustrate better we will use the full factored form for each:

$$\underbrace{(x \cdot x \cdot x \cdot x \cdot x)}_{x^5} \cdot \underbrace{(x \cdot x \cdot x)}_{x^3} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^8}$$

So  $x^5 \cdot x^3 = x^8$ . You may already see the pattern to multiplying powers, but let's confirm it with another example. We will multiply  $x$  *squared* by  $x$  *to the power 4*:

$$\underbrace{(x \cdot x)}_{x^2} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^6}$$

So  $x^2 \cdot x^4 = x^6$ . Look carefully at the powers and how many factors there are in each calculation. 5 factors of  $x$  times 3 factors of  $x$  equals  $(5 + 3) = 8$  factors of  $x$ . 2 factors of  $x$  times 4 factors of  $x$  equals  $(2 + 4) = 6$  factors of  $x$ .

You should see that when we take the product of two powers of  $x$ , the number of factors of  $x$  in the answer is the sum of factors in the terms you are multiplying. In other words the exponent of  $x$  in the answer is the sum of the exponents in the product.

Product rule for exponents:  $x^n \cdot x^m = x^{n+m}$

#### Example 3

Multiply  $x^4 \cdot x^5$ .

#### Solution

$$x^4 \cdot x^5 = x^{4+5} = x^9$$

When multiplying exponents of the same base, it is a simple case of adding the exponents. It is important that when you use the product rule you avoid easy-to-make mistakes. Consider the following.

#### Example 4

Multiply  $2^2 \cdot 2^3$ .

#### Solution

$$2^2 \cdot 2^3 = 2^5 = 32$$

Note that when you use the product rule you **DO NOT MULTIPLY BASES**. In other words, you must avoid the common error of writing  ~~$2^2 \cdot 2^3 = 4^5$~~ . Try it with your calculator and check which is right!

#### Example 5

Multiply  $2^2 \cdot 3^3$ .

#### Solution

$$2^2 \cdot 3^3 = 4 \cdot 27 = 108$$

In this case, the bases are different. The product rule for powers **ONLY APPLIES TO TERMS THAT HAVE THE SAME BASE**. Common mistakes with problems like this include  ~~$2^2 \cdot 3^3 = 6^5$~~ .

### Use the Power of a Product Property

We will now look at what happens when we raise a whole expression to a power. Lets take *to the power 4* and *cube it*. Again we will use the full factored form for each.

$$(x^4)^3 = x^4 \cdot x^4 \cdot x^4 \qquad 3 \text{ factors of } x \text{ to the power 4.}$$

$$\underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^{12}}$$

So  $(x^4)^3 = x^{12}$ . It is clear that when we raise a power of  $x$  to a new power, the powers multiply.

When we take an expression and raise it to a power, we are multiplying the existing powers of  $x$  by the power above the parenthesis.

Power rule for exponents:  $(x^n)^m = x^{n \cdot m}$

#### Power of a product

If we have a product inside the parenthesis and a power on the parenthesis, then the power goes on each element inside. So that, for example,  $(x^2y)^4 = (x^2)^4 \cdot (y)^4 = x^8y^4$ . Watch how it works the long way.

$$\underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y)}_{x^8y^4}$$

Power rule for exponents:  $(x^n)^m = x^{nm}$  and  $(x^n y^m)^p = x^{np} y^{mp}$

**WATCH OUT!** This does NOT work if you have a sum or difference inside the parenthesis. For example,  $(x + y)^2 \neq x^2 + y^2$ . This is a commonly made mistake. It is easily avoidable if you remember what an exponent means  $(x + y)^2 = (x + y)(x + y)$ . We will learn how to simplify this expression in a later chapter.

Lets apply the rules we just learned to a few examples.

When we have numbers, we just evaluate and most of the time it is not really important to use the product rule and the power rule.

#### Example 6

*Simplify the following expressions.*

(a)  $3^4 \cdot 3^7$

(b)  $2^6 \cdot 2$

(c)  $(4^2)^3$

#### Solution

In each of the examples, we want to evaluate the numbers.

(a) Use the product rule first:  $3^5 \cdot 3^7 = 3^{12}$

Then evaluate the result:  $3^{12} = 531,441$

OR

We can evaluate each part separately and then multiply them.  $3^5 \cdot 3^7 = 243 \cdot 2,187 = 531,441$ .

Use the product rule first.  $2^6 \cdot 2 = 2^7$

Then evaluate the result.  $2^7 = 128$

OR

We can evaluate each part separately and then multiply them.  $2^6 \cdot 2 = 64 \cdot 2 = 128$

(c) Use the power rule first.  $(4^2)^3 = 4^6$

Then evaluate the result.  $4^6 = 4096$

OR

We evaluate inside the parenthesis first.  $(4^2)^3 = (16)^3$

Then apply the power outside the parenthesis.  $(16)^3 = 4096$

When we have just one variable in the expression then we just apply the rules.

### Example 7

*Simplify the following expressions.*

(a)  $x^2 \cdot x^7$

(b)  $(y^3)^5$

### Solution

(a) Use the product rule.  $x^2 \cdot x^7 = x^{2+7} = x^9$

(b) Use the power rule.  $(y^3)^5 = y^{3 \cdot 5} = y^{15}$

When we have a mix of numbers and variables, we apply the rules to the numbers or to each variable separately.

### Example 8

*Simplify the following expressions.*

(a)  $(3x^2y^3) \cdot (4xy^2)$

(b)  $(4xyz) \cdot (x^2y^3) \cdot (2yz^4)$

(c)  $(2a^3b^3)^2$

### Solution

(a) We group like terms together.

$$(3x^2y^3) \cdot (4xy^2) = (3 \cdot 4) \cdot (x^2 \cdot x) \cdot (y^3 \cdot y^2)$$

We multiply the numbers and apply the product rule on each grouping.

$$12x^3y^5$$

(b) We group like terms together.

$$(4xyz) \cdot (x^2y^3) \cdot (2yz^4) = (4 \cdot 2) \cdot (x \cdot x^2) \cdot (y \cdot y^3 \cdot y) \cdot (z \cdot z^4)$$

We multiply the numbers and apply the product rule on each grouping.

$$8x^3y^5z^5$$

(c) We apply the power rule for each separate term in the parenthesis.

$$(2a^3b^3)^2 = 2^2 \cdot (a^3)^2 \cdot (b^3)^2$$

We evaluate the numbers and apply the power rule for each term.

$$4a^6b^6$$

In problems that we need to apply the product and power rules together, we must keep in mind the order of operation. Exponent operations take precedence over multiplication.

### Example 9

*Simplify the following expressions.*

(a)  $(x^2)^2 \cdot x^3$

(b)  $(2x^2y) \cdot (3xy^2)^3$

(c)  $(4a^2b^3)^2 \cdot (2ab^4)^3$

### Solution

(a)  $(x^2)^2 \cdot x^3$

We apply the power rule first on the first parenthesis.

$$(x^2)^2 \cdot x^3 = x^4 \cdot x^3$$

Then apply the product rule to combine the two terms.

$$x^4 \cdot x^3 = x^7$$

(b)  $(2x^2y) \cdot (3xy^2)^3$

We must apply the power rule on the second parenthesis first.

$$(2x^2y) \cdot (3xy^2)^3 = (2x^2y) \cdot (27x^3y^6)$$

Then we can apply the product rule to combine the two parentheses.

$$(2x^2y) \cdot (27x^3y^6) = 54x^5y^7$$

(c)  $(4a^2b^3)^2 \cdot (2ab^4)^3$

We apply the power rule on each of the parentheses separately.

$$(4a^2b^3)^2 \cdot (2ab^4)^3 = (16a^4b^6) \cdot (8a^3b^{12})$$

Then we can apply the product rule to combine the two parentheses.

$$(16a^4b^6) \cdot (8a^3b^{12}) = 128a^7b^{18}$$

### Review Questions

Write in exponential notation.

- $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
- $3x \cdot 3x \cdot 3x$
- $(-2a)(-2a)(-2a)(-2a)$
- $6 \cdot 6 \cdot 6 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$

Find each number:

- $5^4$
- $(-2)^6$
- $(0.1)^5$
- $(-0.6)^3$

Multiply and simplify.

- $6^3 \cdot 6^6$
- $2^2 \cdot 2^4 \cdot 2^6$
- $3^2 \cdot 4^3$
- $x^2 \cdot x^4$
- $(-2y^4)(-3y)$
- $(4a^2)(-3a)(-5a^4)$

Simplify.

- $(a^3)^4$
- $(xy)^2$
- $(3a^2b^3)^4$
- $(-2xy^4z^2)^5$
- $(-8x)^3(5x)^2$
- $(4a^2)(-2a^3)^4$
- $(12xy)(12xy)^2$
- $(2xy^2)(-x^2y)^2(3x^2y^2)$

### Review Answers

- $4^5$
- $(3x)^3$

3.  $(-2a)^4$
4.  $6^3x^2y^4$
5. 625
6. 64
7. 0.00001
8.  $-0.216$
9. 10077696
10. 4096
11. 576
12.  $x^6$
13.  $6y^5$
14.  $60a^7$
15.  $a^{12}$
16.  $x^2y^2$
17.  $81a^8b^{12}$
18.  $-32x^5y^{20}z^{10}$
19.  $12800x^5$
20.  $64a^{14}$
21.  $1728x^3y^3$
22.  $6x^7y^6$