

8.3 Zero, Negative, and Fractional Exponents

Learning Objectives

- Simplify expressions with zero exponents.
- Simplify expressions with negative exponents.
- Simplify expression with fractional exponents.
- Evaluate exponential expressions.

Introduction

There are many interesting concepts that arise when contemplating the product and quotient rule for exponents. You may have already been wondering about different values for the exponents. For example, so far we have only considered positive, whole numbers for the exponent. So called **natural numbers** (or **counting numbers**) are easy to consider, but even with the everyday things around us we think about questions such as is it possible to have a negative amount of money? or what would one and a half pairs of shoes look like? In this lesson, we consider what happens when the exponent is not a natural number. We will start with What happens when the exponent is zero?

Simplify Expressions with Exponents of Zero

Let us look again at the quotient rule for exponents (that $\frac{x^n}{x^m} = x^{n-m}$) and consider what happens when $n = m$. Lets take the example of x^4 divided by x^4 .

$$\frac{x^4}{x^4} = x^{(4-4)} = x^0$$

Now we arrived at the quotient rule by considering how the factors of x cancel in such a fraction. Lets do that again with our example of x^4 divided by x^4 .

$$\frac{x^4}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = 1$$

So $x^0 = 1$.

This works for any value of the exponent, not just 4.

$$\frac{x^n}{x^n} = x^{n-n} = x^0$$

Since there is the same number of factors in the numerator as in the denominator, they cancel each other out and we obtain $x^0 = 1$. The zero exponent rule says that any number raised to the power zero is one.

Zero Rule for Exponents: $x^0 = 1$, $x \neq 0$

Simplify Expressions With Negative Exponents

Again we will look at the quotient rule for exponents (that $\frac{x^n}{x^m} = x^{n-m}$) and this time consider what happens when $m > n$. Lets take the example of x^4 divided by x^6 .

$$\frac{x^4}{x^6} = x^{(4-6)} = x^{-2} \text{ for } x \neq 0.$$

By the quotient rule our exponent for x is -2 . But what does a negative exponent really mean? Lets do the same calculation long-hand by dividing the factors of x^4 by the factors of x^6 .

$$\frac{x^4}{x^6} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x} = \frac{1}{x \cdot x} = \frac{1}{x^2}$$

So we see that x to the power -2 is the same as one divided by x to the power $+2$. Here is the negative power rule for exponents.

Negative Power Rule for Exponents $\frac{1}{x^n} = x^{-n}$ $x \neq 0$

You will also see negative powers applied to products and fractions. For example, here it is applied to a product.

$$\begin{aligned} (x^3y)^{-2} &= x^{-6}y^{-2} && \text{using the power rule} \\ x^{-6}y^{-2} &= \frac{1}{x^6} \cdot \frac{1}{y^2} = \frac{1}{x^6y^2} && \text{using the negative power rule separately on each variable} \end{aligned}$$

Here is an example of a negative power applied to a quotient.

$$\begin{aligned} \left(\frac{a}{b}\right)^{-3} &= \frac{a^{-3}}{b^{-3}} && \text{using the power rule for quotients} \\ \frac{a^{-3}}{b^{-3}} &= \frac{a^{-3}}{1} \cdot \frac{1}{b^{-3}} = \frac{1}{a^3} \cdot \frac{b^3}{1} && \text{using the negative power rule on each variable separately} \\ \frac{1}{a^3} \cdot \frac{b^3}{1} &= \frac{b^3}{a^3} && \text{simplifying the division of fractions} \\ \frac{b^3}{a^3} &= \left(\frac{b}{a}\right)^3 && \text{using the power rule for quotients in reverse.} \end{aligned}$$

The last step is not necessary but it helps define another rule that will save us time. A fraction to a negative power is flipped.

Negative Power Rule for Fractions $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$, $x \neq 0, y \neq 0$

In some instances, it is more useful to write expressions without fractions and that makes use of negative powers.

Example 1

Write the following expressions without fractions.

- (a) $\frac{1}{x}$
- (b) $\frac{2}{x^2}$
- (c) $\frac{x^2}{y^3}$
- (d) $\frac{3}{xy}$

Solution

We apply the negative rule for exponents $\frac{1}{x^n} = x^{-n}$ on all the terms in the denominator of the fractions.

$$(a) \frac{1}{x} = x^{-1}$$

$$(b) \frac{2}{x^2} = 2x^{-2}$$

$$(c) \frac{x^2}{y^3} = x^2y^{-3}$$

$$(d) \frac{3}{xy} = 3x^{-1}y^{-1}$$

Sometimes, it is more useful to write expressions without negative exponents.

Example 2

Write the following expressions without negative exponents.

$$(a) 3x^{-3}$$

$$(b) a^2b^{-3}c^{-1}$$

$$(c) 4x^{-1}y^3$$

$$(d) \frac{2x^{-2}}{y^{-3}}$$

Solution

We apply the negative rule for exponents $\frac{1}{x^n} = x^{-n}$ on all the terms that have negative exponents.

$$(a) 3x^{-3} = \frac{3}{x^3}$$

$$(b) a^2b^{-3}c^{-1} = \frac{a^2}{b^3c}$$

$$(c) 4x^{-1}y^3 = \frac{4y^3}{x}$$

$$(d) \frac{2x^{-2}}{y^{-3}} = \frac{2y^3}{x^2}$$

Example 3

Simplify the following expressions and write them without fractions.

$$(a) \frac{4a^2b^3}{2a^5b}$$

$$(b) \left(\frac{x}{3y^2}\right)^3 \cdot \frac{x^2y}{4}$$

Solution

(a) Reduce the numbers and apply quotient rule on each variable separately.

$$\frac{4a^2b^3}{6a^5b} = 2 \cdot a^{2-5} \cdot b^{3-1} = 2a^{-3}b^2$$

(b) Apply the power rule for quotients first.

$$\left(\frac{2x}{y^2}\right)^3 \cdot \frac{x^2y}{4} = \frac{8x^3}{y^6} \cdot \frac{x^2y}{4}$$

Then simplify the numbers, use product rule on the x 's and the quotient rule on the y 's.

$$\frac{8x^3}{y^6} \cdot \frac{x^2y}{4} = 2 \cdot x^{3+2} \cdot y^{1-6} = 2x^5y^{-5}$$

Example 4

Simplify the following expressions and write the answers without negative powers.

(a) $\left(\frac{ab^{-2}}{b^3}\right)^2$

(b) $\frac{x^{-3}y^2}{x^2y^{-2}}$

Solution

(a) Apply the quotient rule inside the parenthesis.

$$\left(\frac{ab^{-2}}{b^3}\right)^2 = (ab^{-5})^2$$

Apply the power rule.

$$(ab^{-5})^2 = a^2b^{-10} = \frac{a^2}{b^{10}}$$

(b) Apply the quotient rule on each variable separately.

$$\frac{x^{-3}y^2}{x^2y^{-2}} = x^{-3-2}y^{2-(-2)} = x^{-5}y^4 = \frac{y^4}{x^5}$$

Simplify Expressions With Fractional Exponents

The exponent rules you learned in the last three sections apply to all powers. So far we have only looked at positive and negative integers. The rules work exactly the same if the powers are fractions or irrational numbers. Fractional exponents are used to express the taking of roots and radicals of something (square roots, cube roots, etc.). Here is an example.

$$\sqrt{a} = a^{1/2} \text{ and } \sqrt[3]{a} = a^{1/3} \text{ and } \sqrt[5]{a^2} = (a^2)^{\frac{1}{5}} = a^{\frac{2}{5}} = a^{2/5}$$

$$\text{Roots as Fractional Exponents } \sqrt[m]{a^n} = a^{n/m}$$

We will examine roots and radicals in detail in a later chapter. In this section, we will examine how exponent rules apply to fractional exponents.

Evaluate Exponential Expressions

When evaluating expressions we must keep in mind the order of operations. You must remember **PEMDAS**.

Evaluate inside the **P**arenthesis.

Evaluate **E**xponents.

Perform **M**ultiplication and **D**ivision operations from left to right.

Perform **A**ddition and **S**ubtraction operations from left to right.

Example 6

Evaluate the following expressions to a single number.

(a) 5^0

(b) 7^2

(c) $\left(\frac{2}{3}\right)^3$

(d) 3^{-3}

Solution(a) $5^0 = 1$ Remember that a number raised to the power 0 is always 1.

(b) $7^2 = 7 \cdot 7 = 49$

(c) $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$

(d) $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

Example 7*Evaluate the following expressions to a single number.*

(a) $3 \cdot 5^5 - 10 \cdot 5 + 1$

(b) $\frac{2 \cdot 4^2 - 3 \cdot 5^2}{3^2}$

(c) $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4}$

Solution

(a) Evaluate the exponent.

$$3 \cdot 5^2 - 10 \cdot 5 + 1 = 3 \cdot 25 - 10 \cdot 5 + 1$$

Perform multiplications from left to right.

$$3 \cdot 25 - 10 \cdot 5 + 1 = 75 - 50 + 1$$

Perform additions and subtractions from left to right.

$$75 - 50 + 1 = 26$$

(b) Treat the expressions in the numerator and denominator of the fraction like they are in parenthesis.

$$\frac{(2 \cdot 4^2 - 3 \cdot 5^2)}{(3^2 - 2^2)} = \frac{(2 \cdot 16 - 3 \cdot 25)}{(9 - 4)} = \frac{(32 - 75)}{5} = \frac{-43}{5}$$

(c) $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4} = \left(\frac{2^2}{3^3}\right)^2 \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{2^2} = \frac{2^2}{3^5} = \frac{4}{243}$

Example 8*Evaluate the following expressions for $x = 2, y = -1, z = 3$.*

(a) $2x^2 - 3y^3 + 4z$

(b) $(x^2 - y^2)^2$

(c) $\left(\frac{3x^2y^5}{4z}\right)^{-2}$

Solution

(a) $2x^2 - 3y + 4z = 2 \cdot 2^2 - 3 \cdot (-1)^3 + 4 \cdot 3 = 2 \cdot 4 - 3 \cdot (-1) + 4 \cdot 3 = 8 + 3 + 12 = 23$

(b) $(x^2 - y^2)^2 = (2^2 - (-1)^2)^2 = (4 - 1)^2 = 3^2 = 9$

(c) $\left(\frac{3x^2 - y^5}{4z}\right)^{-2} = \left(\frac{3 \cdot 2^2 \cdot (-1)^5}{4 \cdot 3}\right)^{-2} = \left(\frac{3 \cdot 4 \cdot (-1)}{12}\right)^{-2} = \left(\frac{-12}{12}\right)^{-2} = \left(\frac{-1}{1}\right)^{-2} = \left(\frac{1}{-1}\right)^2 = (-1)^2 = 1$

Review Questions

Simplify the following expressions, be sure that there aren't any negative exponents in the answer.

- $x^{-1} \cdot y^2$
- x^{-4}
- $\frac{x^{-3}}{x^{-7}}$
- $\frac{x^{-3}y^{-5}}{z^{-7}}$
- $(x^{\frac{1}{2}}y^{-\frac{2}{3}})(x^2y^{\frac{1}{3}})$
- $\left(\frac{a}{b}\right)^{-2}$
- $(3a^{-2}b^2c^3)^3$
- $x^{-3} \cdot x^3$

Simplify the following expressions so that there aren't any fractions in the answer.

- $\frac{a^{-3}(a^5)}{a^{-6}}$
- $\frac{5x^6y^2}{x^8y}$
- $\frac{(4ab^6)^3}{(ab)^5}$
- $\left(\frac{3x}{y^{1/3}}\right)^3$
- $\frac{3x^2y^{3/2}}{xy^{1/2}}$
- $\frac{(3x^3)(4x^4)}{(2y)^2}$
- $\frac{a^{-2}b^{-3}}{c^{-1}}$
- $\frac{x^{1/2}y^{5/2}}{x^{3/2}y^{3/2}}$

Evaluate the following expressions to a single number.

- 3^{-2}
- $(6.2)^0$
- $8^{-4} \cdot 8^6$
- $(16^{\frac{1}{2}})^3$
- $x^24x^3y^44y^2$ if $x = 2$ and $y = -1$
- $a^4(b^2)^3 + 2ab$ if $a = -2$ and $b = 1$
- $5x^2 - 2y^3 + 3z$ if $x = 3$, $y = 2$, and $z = 4$
- $\left(\frac{a^2}{b^3}\right)^{-2}$ if $a = 5$ and $b = 3$

Review Answers

- $\frac{y^2}{x}$
- $\frac{1}{x^4}$
- x^4

4. $\frac{z^7}{x^3y^5}$
5. $\frac{x^{5/2}}{y^{1/3}}$
6. $\left(\frac{b}{a}\right)^2$ or $\frac{b^2}{a^2}$
7. $\frac{27b^6c^9}{a^6}$
8. 1
9. a^8
10. $5x^{-2}y$
11. $64a^{-2}b^{\frac{1}{3}}$
12. $27x^2y^{-1}$
13. $3xy$
14. $6x^7y^{-2}$
15. $a^{-2}b^{-3}c$
16. $x^{-1}y$
17. 0.111
18. 1
19. 64
20. 64
21. 512
22. 12
23. 41
24. 1.1664