

$$\begin{aligned} \dot{r} &= r(3-r-2s) = 3r - r^2 - 2rs \\ \dot{s} &= s(2-r-s) = 2s - sr - s^2 \end{aligned}$$

Problem Definition

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Rabbits and Sheep compete for food (grass)

- Rabbit pop. grows faster than Sheep pop.
- Sheep push rabbits out of the way - get more grass  
→ conflict affects rabbits more than sheep

$r = \text{rabbit pop.} > 0$ ,  $s = \text{sheep pop.} > 0$

Changing Parameters:

$$\dot{r} = r(3-2r-s) = 3r - 2r^2 - rs$$

$$\dot{s} = s(2-r-s) = 2s - sr - s^2$$

→ rabbits and sheep are equally good at getting grass  
and rabbits ~~die~~ die faster (lower population capacity)

What do we want to know about the system?

# 1. Fixed Points

To find fixed points, solve

$$\dot{r} = 3r - 2r^2 - rs = 0$$

$$\dot{s} = 2s - sr - s^2 = 0$$

simultaneously

$$\dot{r} = 0 \text{ when } r = 0, \quad 3 - 2r - s = 0$$

$$\dot{s} = 0 \text{ when } s = 0, \quad 2 - r - s = 0$$

$$\Rightarrow \text{f.p. at } r = 0, s = 0$$

$$r = 0, \quad 2 - r - s = 0 \Rightarrow 2 - s = 0, \quad s = 2$$

$$3 - 2r - s = 0, \quad s = 0 \Rightarrow 3 - 2r = 0, \quad r = \frac{3}{2}$$

$$3 - 2r - s = 0, \quad 2 - r - s = 0$$

$$\begin{aligned} 3 - 2r - s &= 0 \\ -2 + r + s &= 0 \end{aligned} \Rightarrow 1 - r = 0 \quad \underline{r = 1}$$

$$\rightarrow 2 - 1 - s = 0 \quad \underline{s = 1}$$

F.P. s
(0, 0)
(0, 2)
( $\frac{3}{2}$ , 0)
(1, 1)

Need to know the nature of each F.P.

## 2. Classify Fixed Points

Linearization:  $\dot{r} = f(r, s)$ ,  ~~$\dot{s} = f(r, s)$~~   $s = g(r, s)$

To linearize about  $r^*$ , use a small disturbance  $u$

$$\rightarrow r = r^* + u \rightarrow u = r - r^* ; v = s - s^*$$

$$\dot{u} = \frac{d}{dt}(r - r^*) ; r^* \text{ is a constant} \rightarrow \dot{u} = \dot{r}$$

$$\dot{u} = f(r^* + u, s^* + v) ; \text{ Recall } r = r^* + u \text{ etc.}$$

by Taylor expansion:

$$\dot{u} = f(r^*, s^*) + u \left. \frac{\partial f}{\partial r} \right|_{(r^*, s^*)} + v \left. \frac{\partial f}{\partial s} \right|_{(r^*, s^*)} + O(u^2, v^2, uv)$$

Neglect higher order terms as very small.

$$\text{for } s: \rightarrow \dot{v} = g(r^*, s^*) + u \left. \frac{\partial g}{\partial r} \right|_{(r^*, s^*)} + v \left. \frac{\partial g}{\partial s} \right|_{(r^*, s^*)}$$

since  $(r^*, s^*)$  is a fixed point,

$$f(r^*, s^*) \text{ and } g(r^*, s^*) = 0$$

$$\text{so } \begin{cases} \dot{u} = \left( u \frac{\partial f}{\partial r} + v \frac{\partial f}{\partial s} \right) \\ \dot{v} = \left( u \frac{\partial g}{\partial r} + v \frac{\partial g}{\partial s} \right) \end{cases}_{(r^*, s^*)}$$

In matrix notation:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial s} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial s} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{(r^*, s^*)}$$

where the matrix

$$A = \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial s} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial s} \end{pmatrix}_{(x^*, y^*)}$$

is the  
Jacobian  
matrix

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \text{ is our linearized system about } (r^*, s^*)$$

## 2. CONT Classify the Fixed Points

Returning to  $\dot{r} = 3r - 2r^2 - rs = 0$   
 $\dot{s} = 2s - sr - s^2 = 0$  ;

$$\frac{\partial \dot{r}}{\partial r} = 3 - 4r - s$$

$$\frac{\partial \dot{r}}{\partial s} = -r$$

$$\frac{\partial \dot{s}}{\partial r} = -s$$

$$\frac{\partial \dot{s}}{\partial s} = 2 - r - 2s$$

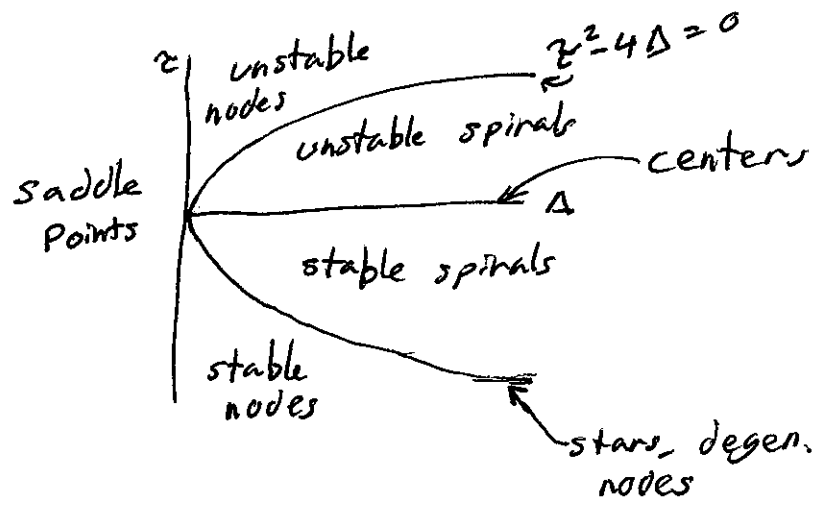
$$A = \begin{pmatrix} 3-4r-s & -r \\ -s & 2-r-2s \end{pmatrix}_{(r^*, s^*)}$$

(i) for  $(r^*, s^*) = (0, 0)$ ;

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$\rightarrow \tau = 5 \rightarrow > 0$   
 $\Delta = 6 \rightarrow > 0$   
 $\tau^2 - 4\Delta = 25 - 24 = 1 \rightarrow > 0$   
 $\lambda_1 = 3, \lambda_2 = 2$   
 $\Rightarrow$  unstable node

Classification Plot  
 for fixed Point p. 137



(ii) for  $(r^*, s^*) = (0, 2)$

$$A = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$$

$\rightarrow \tau = -1 \rightarrow < 0$   
 $\Delta = -2 \rightarrow < 0$   
 $\tau^2 - 4\Delta = 9 \rightarrow > 0$   
 $\lambda_1 = 1, \lambda_2 = -2$   
Saddle Point

(iii) for  $(r^*, s^*) = (3/2, 0)$

$$A = \begin{pmatrix} -3 & -3/2 \\ 0 & 1/2 \end{pmatrix}$$

$\rightarrow \tau = -2\frac{1}{2} \rightarrow < 0$   
 $\Delta = -1\frac{1}{2} \rightarrow < 0$   
 $\tau^2 - 4\Delta = 12\frac{1}{4} \rightarrow > 0$   
 $\lambda_1 = -3, \lambda_2 = \frac{1}{2}$   
Saddle Point

2. cont. Classify the Fixed Points

(iv) for  $(r^*, s^*) = (1, 1)$

$$A = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$$

$$\tau = -3 \rightarrow < 0$$

$$\Delta = 1 \rightarrow > 0$$

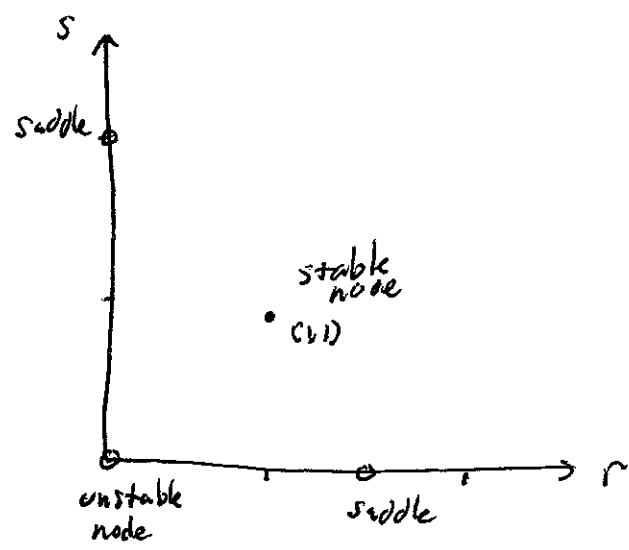
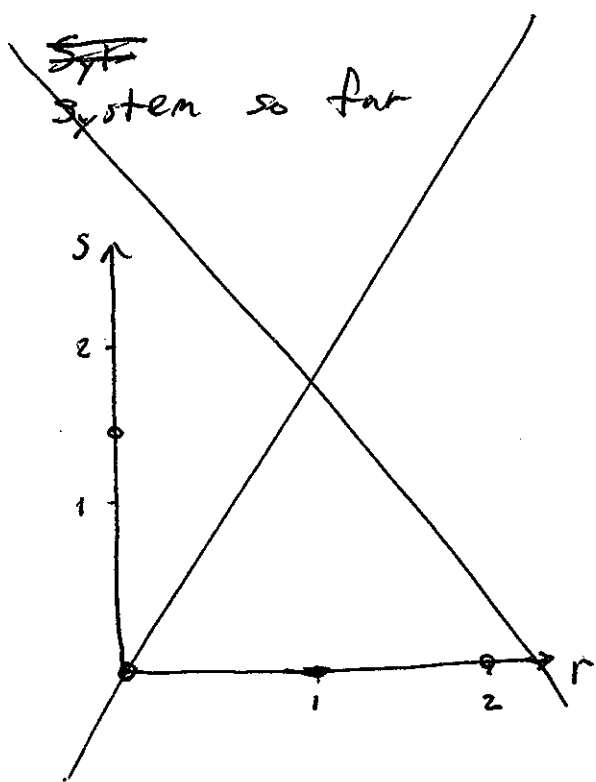
$$\tau^2 - 4\Delta = 5 \rightarrow > 0$$

~~$\lambda = \tau$~~

$$\lambda_1 = \frac{-3 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{-3 - \sqrt{5}}{2}$$

Stable Node

for eigenvalues, recall  
characteristic eqn:  
 $\lambda^2 - \tau\lambda + \Delta = 0$   
 $\rightarrow \lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$



Want more information to make the picture clearer.

### 3. Nullclines

Nullclines: line (trajectory) along which the rate of change of one component is zero, ( $\dot{r} = 0$ , OR  $\dot{s} = 0$ )

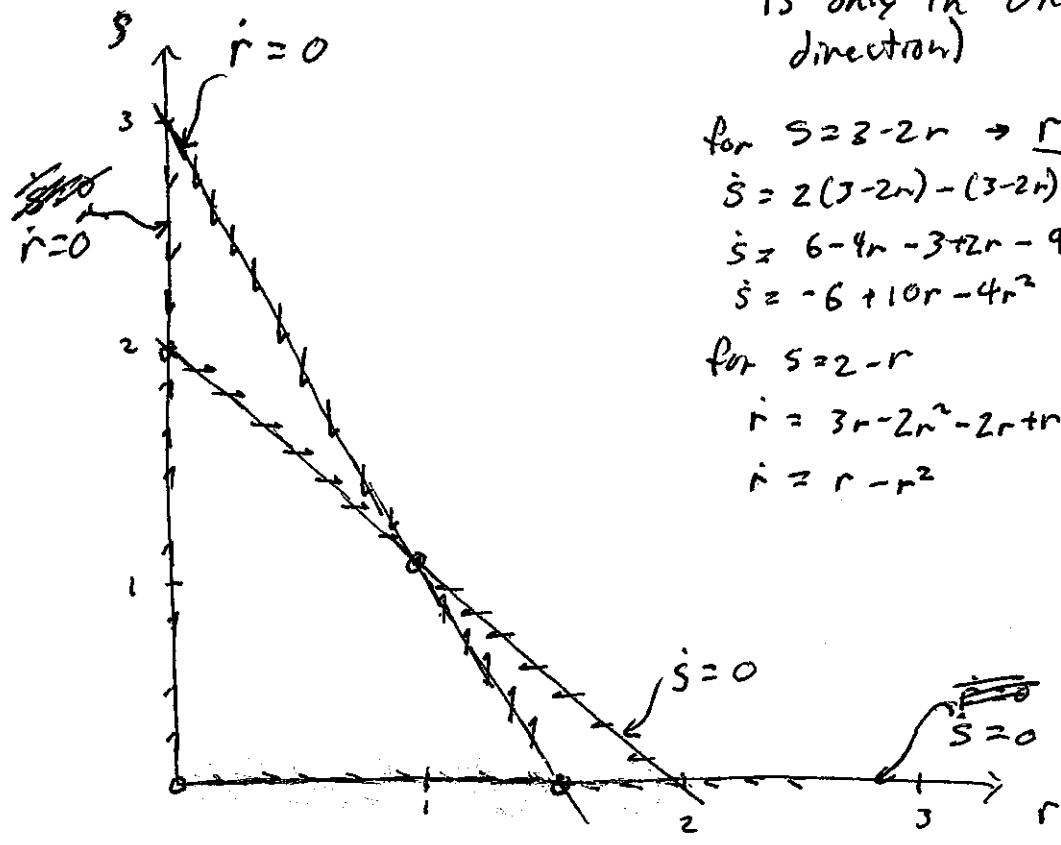
Find the Nullclines:

r-nullclines where  $\dot{r} = 0 = 3r - 2r^2 - rs$   
 $r = 0, s = 3 - 2r$  (written in this form for convenience on our plot)

s-nullclines where  $\dot{s} = 0 = 2s - 3r - s^2$   
 $s = 0, s = 2 - r$

On a nullcline, the vector field is only in one direction (r or s direction)

for  $s = 2 - 2r \rightarrow r = \frac{3-s}{2} \xrightarrow{\dot{s} = \frac{1}{2}s - \frac{s^2}{2}}$   
 $\dot{s} = 2(3-2r) - (3-2r) - (3-2r)^2$  Use  
 $\dot{s} = 6 - 4r - 3 + 2r - 9 + 12r - 4r^2$   
 $\dot{s} = -6 + 10r - 4r^2$   
for  $s = 2 - r$   
 $\dot{r} = 3r - 2r^2 - 2r + r^2$   
 $\dot{r} = r - r^2$



## 4. Eigenvectors

Goal: Define the straight-line trajectories about each fixed point.

These are defined by  $\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{v}_1 + c_2 e^{\lambda_2 t} \underline{v}_2$   
Eigenvalues indicate the stability (growth vs. decay) and speed.  
Eigenvectors indicate the direction of the straight-lines.

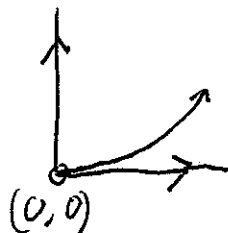
(i)  $(x^*, y^*) = (0, 0)$ ,  $\lambda_1 = 3$ ,  $\lambda_2 = 2$        $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

To find eigenvectors  
 $(A - \lambda I) \underline{v} = 0$

for  $\lambda_1$ ,  
$$0 = \begin{pmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0$$
  
$$\begin{pmatrix} 3-3 & 0 \\ 0 & 2-3 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0$$
  
$$\rightarrow 0v_a + 0v_b = 0$$
  
$$0v_a - v_b = 0 \rightarrow \underline{v_b = 0}$$
  
$$\rightarrow \underline{v_1 = (1, 0)}$$

for  $\lambda_2$ ,  
$$\begin{pmatrix} 3-2 & 0 \\ 0 & 2-2 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0 \rightarrow \begin{matrix} v_a + 0v_b = 0 \\ 0v_a + 0v_b = 0 \end{matrix} \rightarrow \underline{v_2 = (0, 1)}$$

Because  $|\lambda_1| > |\lambda_2|$ ,  $v_1$  is the fast eigenspace,  
 $v_2$  is the slow eigenspace.



#### 4. CONT. Eigenvectors

(ii)  $(r^*, s^*) = (0, 2)$       $A = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$       $\lambda_1 = 1, \lambda_2 = -2$

for  $\lambda_1$ ,  $\begin{pmatrix} 1-1 & 0 \\ -2 & -2-1 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0 \rightarrow \begin{matrix} 0v_a + 0v_b = 0 \\ -2v_a - 3v_b = 0 \end{matrix} \rightarrow \begin{matrix} -2v_a = 3v_b \\ -2v_a - 3v_b = 0 \end{matrix}$

$\rightarrow \underline{v_1 = (3, -2)}$

for  $\lambda_2$ ,  $\begin{pmatrix} 1+2 & 0 \\ -2 & -2+2 \end{pmatrix} = 0 \rightarrow \begin{matrix} 3v_a + 0v_b = 0 \\ -2v_a + 0v_b = 0 \end{matrix} \rightarrow v_a = 0$

$\rightarrow \underline{v_2 = (0, 1)}$

Because  $\lambda_1 > 0$ ,  $v_1$  is the unstable manifold  
 $\lambda_2 < 0$ ,  $v_2$  is the stable manifold

(iii)  $(r^*, s^*) = (3/2, 0)$       $A = \begin{pmatrix} -3 & -3/2 \\ 0 & 1/2 \end{pmatrix}$       $\lambda_1 = -3, \lambda_2 = 1/2$

for  $\lambda_1$ ,  $\begin{pmatrix} -3+3 & -3/2 \\ 0 & 1/2+3 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0 \rightarrow \begin{matrix} 0v_a - 3/2 v_b = 0 \\ 0v_a + 7/2 v_b = 0 \end{matrix} \rightarrow v_b = 0$

$\rightarrow \underline{v_1 = (1, 0)}$

for  $\lambda_2$ ,  $\begin{pmatrix} -3-1/2 & -3/2 \\ 0 & 1/2-1/2 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0 \rightarrow \begin{matrix} -7/2 v_a - 3/2 v_b = 0 \\ 0v_a + 0v_b = 0 \end{matrix} \rightarrow \begin{matrix} -7v_a = 3v_b \\ -7v_a = 3v_b \end{matrix}$

$\rightarrow \underline{v_2 = (3, -7)}$

Because  $\lambda_1 < 0$ ,  $v_1$  is the stable manifold  
 $\lambda_2 > 0$ ,  $v_2$  is the unstable manifold



#### 4. CONT. Eigenvectors

$$(iV) \quad (r^*, s^*) = (1, 1) \quad A = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \quad \lambda_1 = \frac{-3 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{-3 - \sqrt{5}}{2}$$

$$\text{for } \lambda_1, \begin{pmatrix} -2 + \frac{3 - \sqrt{5}}{2} & -1 \\ -1 & -1 + \frac{3 - \sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0 \Rightarrow \begin{aligned} \frac{-1 - \sqrt{5}}{2} v_a &= v_b \\ v_a &= \frac{1 - \sqrt{5}}{2} v_b \end{aligned}$$

$$\rightarrow v_1 = \left( 1, \frac{-1 - \sqrt{5}}{2} \right) \approx (1, -1.62)$$

$$\text{for } \lambda_2, \begin{pmatrix} -2 + \frac{3 + \sqrt{5}}{2} & -1 \\ -1 & -1 + \frac{3 + \sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0 \Rightarrow \begin{aligned} \frac{-1 + \sqrt{5}}{2} v_a &= v_b \\ v_a &= \frac{1 + \sqrt{5}}{2} v_b \end{aligned}$$

$$\rightarrow v_2 = \left( 1, \frac{-1 + \sqrt{5}}{2} \right) \approx (1, 0.62)$$

Because  $|\lambda_2| > |\lambda_1|$ ,  $v_2$  is the fast eigenspace  
 $v_1$  is the slow eigenspace