

Nonlinear Dynamics in Biological Systems Lecture Notes Set 1:

Reading Assignment: Chapter 1 & 2 of Strogatz

I. Introduction to Nonlinear Dynamics

Question to the Class: Why do we model things as engineers?

- Prediction: we want to predict the behavior, prediction allows for things like design or optimization
- Design: how should we design the system? If we change parameter, k , how will our system change? Can I design the system to have a particular k value? If so what should that value of k be to get the best performance?
- Understanding: how can we explain the foundation or mechanism for observed behavior?

Dynamical system:

- systems evolve with time in a way that the state at time t depends upon the state at a slightly earlier time τ where $\tau < t$
- they have a “memory”
- they are therefore inherently *deterministic*: state is determined by the earlier state

Dynamics is a common mathematical framework that has applications to biology, chemical kinetics, electrical systems, mechanics

Dynamical Systems are Represented by Two different equation types:

1) Differential Equations – continuous time

$$\frac{dx}{dt} = f(x)$$

- how x changes with time (the rate of change) depends upon the current state of x
- in other words whether or not x increases, decreases, or doesn't change depends upon what x is now.
- time is continuous and x changes smoothly with time

2) Difference Equations – discrete time

$$X_{t+1} = f(X_t)$$

Ordinary Differential Equations:

We will focus on differential equations in this class. In particular we will focus on ordinary differential equations (ODEs).

ODEs take the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \dots \frac{d^ny}{dx^n}\right) = 0$$

as opposed to partial differential equations (PDE's)

We will only look at systems changing in time in this course.

Deterministic Does NOT Always Mean Practically Predictable in the Real World!

This concept has had major implications for a wide range of fields. More to come on this as we get to higher dimension systems.

Nonlinear versus Linear Systems

Why are you taking this class?

“The elegant body of mathematical theory pertaining to linear systems (Fourier analysis, orthogonal functions, and so on), and its successful application to many fundamentally linear problems in the physical sciences, tends to dominate even moderately advanced University courses in mathematics and theoretical physics. The mathematical intuition so developed ill equips the student to confront the bizarre behaviour exhibited by the simplest of discrete nonlinear systems, such as equation (3). Yet such nonlinear systems are surely the rule, not the exception, outside the physical sciences.

I would therefore urge that people be introduced to, say, equation (3) early in their mathematical education. This equation can be studied phenomenologically by iterating it on a calculator, or even by hand. Its study does not involve as much conceptual sophistication as does elementary calculus. Such study would greatly enrich the student's intuition about nonlinear systems.

Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realized that simple nonlinear systems do not necessarily possess simple dynamical properties.”

From Robert M. May, Published in Nature, Vol. 261, p.459, June 10 1976.

This class is about building your mathematical intuition to learn about the system NOT just executing the math steps.



Refresher of BASIC DEFINITIONS



$\frac{dx}{dt}$ & \dot{x} will be used interchangeably.

Linear Terms: one that is *first degree* in its *dependent* variables

x is 1st degree and therefore a linear term
xt is 1st degree in x and therefore a linear term
 x^2 is 2nd degree in x and there not a linear term

Nonlinear Terms: higher powers, products and transcendentals of the *dependent* variable

sin x nonlinear term in the dependent variable
sin t not a nonlinear term in the dependent variable

linear equation: consists of a sum of linear terms

$y = x + 2$
 $y(t) + x(t) = N$
 $dy/dt = x + \sin t$

nonlinear equation: all other equations

$y + x^2 = 2$
 $x(t) * y(t) = N$
 $dy / dt = xy + \sin x$

Trajectory: the path of the dependent variable in time, through the phase space.

Phase or State Space: Is the n-dimensional space occupied by the possible values of the dependent variables. Looking at the phase space removes time from the point of view and gives the trajectory. This view of the phase space is called a phase portrait. The phase portrait for a 1 dimensional system is a line. The phase portrait for a 2 dimensional system is a plane.

Defining the dimensionality of the system:

A general system of first order differential equations:

$$\frac{dx_1}{dt} = f(x_1, x_2, \dots, x_n)$$

$$\frac{dx_2}{dt} = f(x_1, x_2, \dots, x_n)$$

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$$\frac{dx_n}{dt} = f(x_1, x_2, \dots, x_n)$$

has the dimension n.

This system requires n initial conditions.

This is the most useful and general form that we will be dealing with. We will refer to this as a state space model form.

The values of $\{x_1, x_2, \dots, x_n\}$ at any given t defines the state of the system at that point in time.

Ultimately we want to know how $\{x_1, x_2, \dots, x_n\}$ travel through time relative to each other for all possible initial conditions.

Converting To State Space

It is often convenient to convert a higher **order** equation to set of n 1st order differential equations.

Example:

$a_1 \frac{d^2x}{dt^2} + a_2 \frac{dx}{dt} + a_3x = 0$ is a 2nd order differential equation that can be converted into a 2 dimensional system of 1st order differential equations.

Rewritten in shorter notation: $a_1\ddot{x} + a_2\dot{x} + a_3x = 0$

To start the conversion first define our state variables, x_1 and x_2 , as $x_1 = x$ $x_2 = \dot{x}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{x} = \frac{-a_3x - a_2\dot{x}}{a_1} = -\frac{a_3}{a_1}x_1 - \frac{a_2}{a_1}x_2$$

so the system can be written as a 2 dimensional system of 1st order equations with 2 state (dependent) variables x_1 and x_2 .

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{a_3}{a_1}x_1 - \frac{a_2}{a_1}x_2$$

converted a second order differential eq to a set (2), 1st order differential equation. This is a Linear System – a 2nd order or 2nd dimension system, $x_1 \dots x_n$, $n = \text{dimension}$

Handling Autonomous Equations:

Autonomous: An autonomous ODE is not *explicitly* a function of the independent variable time, t .

$$\frac{dx}{dt} = 2x^2 + x + 1 \quad \text{is time autonomous. } x(t) = f(x) \quad \text{not } f(x,t) \text{ or } f(t).$$

In other words your initial condition can start anywhere in time. It doesn't matter.

Nonautonomous: A nonautonomous ODE is explicitly a function of independent variable time, t .

$$\frac{dx}{dt} = t(2x^2 + x) \quad \text{is nonautonomous}$$

How does one handle nonautonomous equations in state space?

You can still convert a nonautonomous differential equation to state space by defining a new, additional state variable that includes time, e.g. $x_3 = t$

adds a dimension to the system

2nd order ODE, autonomous \rightarrow 2nd dimension system

2nd order ODE, nonautonomous \rightarrow 3rd dimension system