

# Chapter 8 : 2 dimensional bifurcations

- all bif from chap 3 (1D) have counterparts in 2D.

They are very similar

- saddle node, transcritical, super + subcritical pitchfork all involve a

stable f.p loses stability

in 2D stable <sup>node</sup> means

$$\lambda_1, \lambda_2 \text{ are } \text{Re} + \leq 0$$

saddle node : stable node collides w/ saddle node  
annihilate

transcritical : stable node exchanges w/ saddle node

supercritical : generates 2 new stable points

subcritical : absorbs 2 saddle points

- Behavior is fundamentally the same in 2D.

- What about when the eigenvalues are complex?

How can a stable spiral lose stability?

→ Hopf bifurcations

↳ will discuss later.

saddle node :  $\dot{x} = \mu - x^2$   
 $\dot{y} = -y$

transcritical  $\dot{x} = \mu x - x^2$   $\dot{y} = -y$

supercrit pit.  $\dot{x} = \mu x - x^3$   $\dot{y} = -y$

subcrit pit.  $\dot{x} = \mu x + x^3$   $\dot{y} = -y$

Book goes thru S.n. + supercrit pit.

So we'll analyze trans. in class

# Chapter 8

~~Chapter 8~~

## Saddle Node

$$\begin{aligned}\dot{x} &= \mu - x^2 \\ \dot{y} &= -y\end{aligned}$$

$$A = \begin{bmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{bmatrix}$$

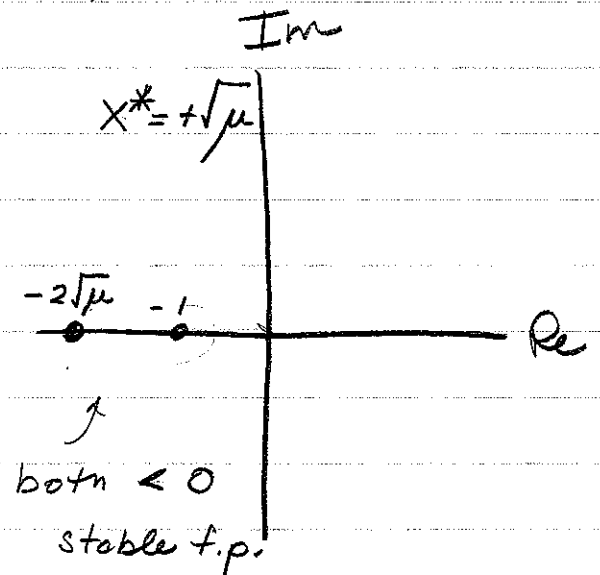
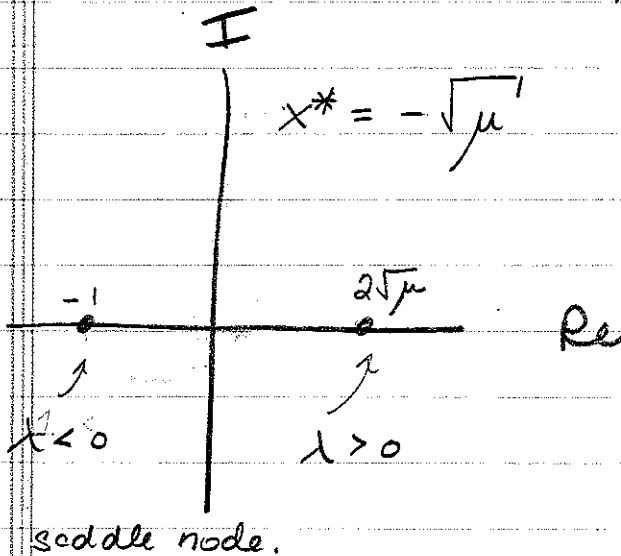
$$= \begin{bmatrix} -2x & 0 \\ 0 & -1 \end{bmatrix} (x^*, y^*)$$

$$\begin{aligned}(x^*, y^*)_1 &= (+\sqrt{\mu}, 0) \\ (x^*, y^*)_2 &= (-\sqrt{\mu}, 0)\end{aligned}$$

$$\lambda_1 = -2x^*$$

$$\lambda_2 = -1$$

Plot  $\lambda$

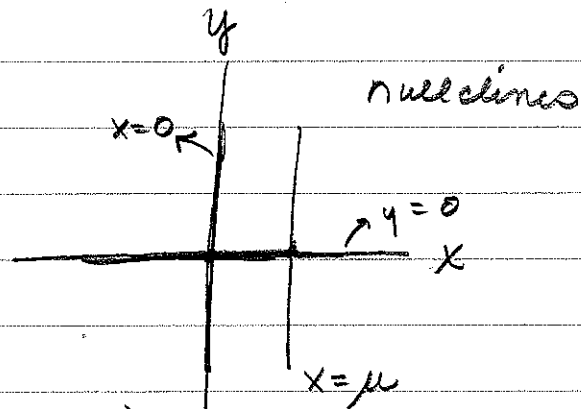


Saddle node is a ...  
eigenvalue "crosses thru  
zero"  
Zero eigenvalue bif.

Transcritical

$$\dot{x} = \mu x - x^2$$

$$\dot{y} = -y$$



$$\dot{x} = 0 = \mu x - x^2 = x(\mu - x)$$

$$x^* = 0 \text{ or } x^* = \mu$$

$$\dot{y} = 0$$

$$y^* = 0$$

$(0, 0)$ ;  $(\mu, 0)$  2 possible fixed points

→ Jacobian:  $A = \begin{bmatrix} \mu - 2x & 0 \\ 0 & -1 \end{bmatrix}_{(x^*, y^*)}$

$$\tau = \mu - 2x + -1 = \mu - 2x - 1 = \lambda_1 + \lambda_2$$

$$\Delta = -\mu + 2x = \lambda_1 \lambda_2$$

$$\lambda = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\Delta} \right)$$

$$\tau^2 - 4\Delta = (\mu - 2x + 1)^2 - 4(-\mu + 2x)$$

→ at  $(0, 0)$

$$A = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}$$

$$\tau = \mu - 1$$

$$\Delta = -\mu$$

$$\tau^2 - 4\Delta = (\mu - 1)^2 + 4\mu = \mu^2 - 2\mu + 1 + 4\mu$$

$$= (\mu + 1)^2 > 0$$

λ always pos.

$$\lambda_1 = \mu \quad \lambda_2 = -1$$

$\mu > 0 \quad (0,0) \Rightarrow \Rightarrow$  saddle node.

$$\Delta < 0$$

$$z^2 - 4\Delta > 0$$

$\mu < 0 \quad (0,0) \Rightarrow \Rightarrow$  stable node

$$z < 0$$

$$\Delta > 0$$

$$z^2 - 4\Delta > 0$$

$\mu = 0$  : bifurcation!

$\lambda_1, \lambda_2$  are never complex

$\rightarrow (\mu, 0)$

$$A = \begin{bmatrix} \mu - 2\mu & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -\mu & 0 \\ 0 & -1 \end{bmatrix}$$

$$\lambda_1 = -\mu \quad \lambda_2 = -1$$

$$z = -\mu - 1 = -1(\mu + 1)$$

$$\Delta = \mu$$

$$z^2 - 4\Delta = (-1)^2(\mu + 1)^2 - 4(\mu)$$

$$= (1)(\mu^2 + 2\mu + 1 - 4\mu) = (\mu^2 - 2\mu + 1)$$

$$= (\mu - 1)^2 > 0$$

So never complex!

$\mu > 0 \quad (\mu, 0)$

$$z < 0$$

$$\Delta > 0$$

$\Rightarrow$  stable

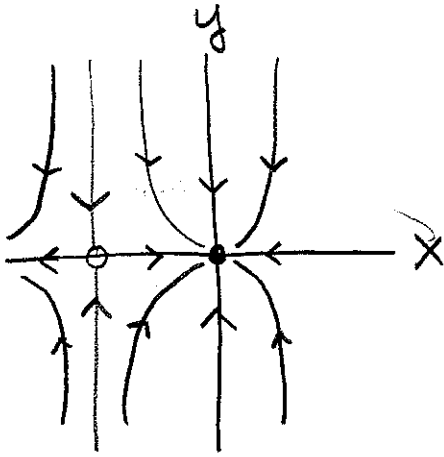
$\mu < 0 \quad (\mu, 0)$

$$\Delta < 0$$

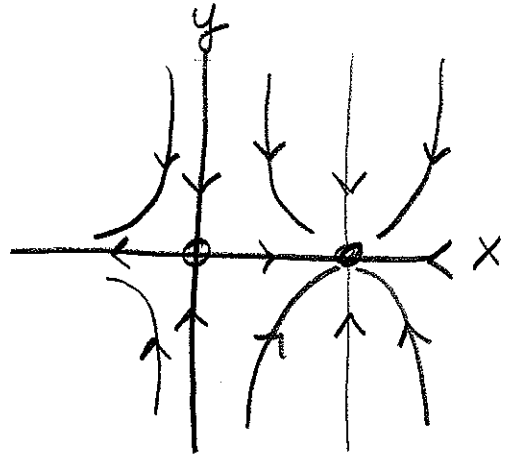
$\Rightarrow$  saddle.

# Phase Portraits for Transcritical

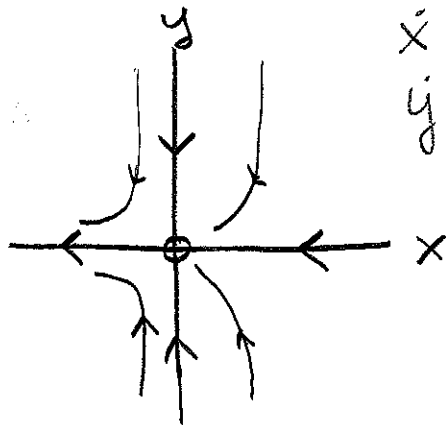
$\mu < 0$



$\mu > 0$



$\mu = 0$



$$\dot{x} = -x^2$$

$$\dot{y} = -y$$

$$x^*, y^* = (0, 0)$$

$$x > 0 \quad \dot{x} < 0$$

$$x < 0 \quad \dot{x} < 0$$