# MATHEMATICS 1002
# PROOF

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# PROOF

One of the main categories of items of our geometric system is the properties we call theorems. Theorems are general statements that can be proved. This LightUnit presents methods of proving theorems by using logical thinking and deductive reasoning.

## OBJECTIVES

**Read these objectives.** The objectives tell you what you will be able to do when you have successfully completed this LightUnit.

When you have finished this LightUnit, you should be able:

1. To identify the various compound sentences.
2. To use truth tables for compound sentences.
3. To explain the difference between *inductive* and *deductive* reasoning.
4. To use deductive reasoning in proofs.
5. To describe the six parts of a two-column proof.
6. To write an indirect proof.

**Survey the LightUnit.** Ask yourself some questions about this study. Write your questions here.

## SECTION OBJECTIVES

**Read these objectives.** When you have completed this section, you should be able:

# I. LOGIC

1. To identify the various compound sentences.
2. To use truth tables for compound sentences.

A "statement" in mathematics has a more narrow technical meaning than it does in everyday usage. We have learned from LightUnit 1001 that we must define our terms carefully in mathematics to avoid misunderstanding. Accordingly, we will make the following definition of a "statement."

> Statement: A sentence that is either true or false, but not both.

It does not matter whether or not we can decide if the statement is true or false.

Some of the following sentences are statements:

1. 3 times 5 equals 15.
2. $x + 8 = 20$.
3. A ray is a segment.
4. He created man in his own image.
5. The lights are on.
6. Points $A$ and $B$ determine a line.

For some sentences we can decide whether they are true or false. In others we know that they must be true or false but we can not determine which is the case. Some sentences do not contain enough information to make them true or false. Still others are not of the nature to be true or false.

Sentence 1 is a statement. From arithmetic facts we know the statement is true.

Sentence 3 is a statement. We can compare the definition of a ray and a segment and decide that sentence 3 is false.

Sentence 5 is a statement. We can see the light is on or touch the bulb to see if it is hot and then decide if sentence 5 is true or false.

Sentence 6 is a true statement. One of our postulates supports its truth.

We cannot decide if sentences 2 and 4 are true or not until we have more information: what number does $x$ represent, and who is *he*? So sentences 2 and 4 are not statements, according to the definition.

We could make sentence 2 a statement by changing $x$ to any number. Of course it would only be true if $x$ is replaced by 12. Number 4 could be made into a statement by putting any name in place of “He.” The name “God” would make it a true statement.

“The back side of the moon is made of green cheese.” This sentence is a statement because it is either true or false. Our inclination is to say it is false. But we can not really determine—no one has seen it but God and He won’t tell. (Avoid making “scientific” judgments on things science can not know.)

➔ **Study the definition of “statement.” Write *statement* before each sentence that is a statement.**

1.1 ____________________ God made the earth good.

1.2 ____________________ $5 \times 6 = 35$.

1.3 ____________________ Isaiah was a prophet in New Testament times.

1.4 ____________________ Water boils at 212° F.

1.5 ____________________ A line has two end points.

1.6 ____________________ Let not your heart be troubled.

1.7 ____________________ $x^2 + 3x + 12 = 0$

1.8 ____________________ They left the Garden of Eden in disgrace.

1.9 ____________________ Genesis is the first book of the Bible.

1.10 ____________________ $AB$ is the name of a line.

## CONJUNCTION

So far we have been considering simple statements. These simple statements can be combined to form more complicated ones. If we combine two statements with the word “and,” the new statement is called a *conjunction.* For example, the two statements: “God made man in His image. God built a woman from the man’s rib.” These can be combined with “and” to form one conjunction statement. God made man in His image and He built a woman from the man’s rib. (Note: The use of “He” in the second part of the conjunction is acceptable because it is made clear by its antecedent, otherwise “God” would be used in both clauses.) Following are examples of statements put together as conjunctions. Remember we are not concerned at this point about the truth of the statements.

a) Adam and Eve lived in the Garden and their first son was born there.

b) $3 + 1 = 4$ and $5 \times 5 = 25$.

c) $1 \times 2 = 2$ and $1 + 2 = 2$.

All of these statements can be put into the pattern of *p and* q where p represents the first statement and q represents the second. The new statement *p and q* is called the conjunction of p and q. In some cases both p and q are true while in others one is true and the other is false.

| Conjunction: | A compound statement formed by connecting two simple statements with the word *and*. |
|---|---|

Strictly speaking the two statements that make a conjunction do not have to be related to each other in any cause and effect way, but they are useless for all practical purposes unless they are.

"A triangle has three sides and water is wet" is a perfectly good conjunction but is of no practical benefit.

A more useful conjunction statement would be: "A right triangle has three sides and a right angle." Here both conditions must be met. A figure may have a right angle but four sides; it is not a right triangle. On the other hand it may have three sides but no right angle. Still it is not a right triangle.

The heart-smitten people in Jerusalem asked: "Men and brethren, what shall we do?" Peter putting two statements together with "and" answered using the conjunction: "Repent and be baptized . . ."

We can have one statement of a conjunction true and the other one false.

3 + 2 = 5 and 4 x 7 = 11

Since a conjunction is a statement, we want to be able to tell whether it is true or false. To do this, let us consider the analogy of a drawbridge. When the bridge is down, we can cross the stream. A conjunction is like two drawbridges over two streams. If both bridges are down, we can go over them and get home. A true statement is like a bridge that is down. In the above conjunction: The first statement 3 + 2 = 5 is true; the bridge is down and we can cross. But the second statement 4 x 7 = 11 is false; the bridge is up and we can not go over it to get home.

Think of the following conjunctions as bridges; which ones would take you home?

a) 2 x 2 = 4 and 2 + 2 = 4

b) 3 + 2 = 5 and 3 x 2 = 7

c) 5 + 1 = 7 and 5 x 1 = 5

d) 6 – 2 = 8 and 12 ÷ 6 = 72

The following table called a "truth table" sums up all the possible combinations of true and false for a conjunction.

| *Conjunction* | | |
|---|---|---|
| *p* | *q* | *p and q* |
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Note that *p and q* is true if and only if both p and q are true (T). (See the top row of the table.) It is false (F) in all other cases.

**Write *true* or *false* to tell whether the following conjunctions are true or false. Use the analogy of the drawbridges and the truth table if you need it.**

1.11 ____________ $6 + 3 = 9$ and $4 \cdot 4 = 20$.

1.12 ____________ Dogs have four legs and cats have ten lives.

1.13 ____________ January has 31 days and May has 31 days.

1.14 ____________ Sugar is sour and lemons are sweet.

1.15 ____________ A triangle has four sides and a rectangle has three sides.

1.16 ____________ A plane has at least three noncollinear points and a line has at least two points.

1.17 ____________ A square root of 16 is 4 and a square root of 9 is -3.

1.18 ____________ The intersection of two planes is a point and two lines intersect in a point.

1.19 ____________ The alphabet has 28 letters and a week has seven days.

1.20 ____________ $6 + 3 = 9$ and $4 \cdot 4 = 16$.

**Check Point — Check Answer Key**

## DISJUNCTION

Another way to combine two simple statements is by using the word *or*. Men will walk in light or they will walk in darkness. Statements like these are called *disjunctions*. We are more familiar with disjunctions made as follows: Noah died before the flood or Methuselah died before the flood. We put them together and say, "Either Noah or Methuselah died before the flood."

In the drawbridge analogy a disjunction is like two bridges over the same stream. If either bridge is down, we can go over.

$4 \times 3 = 12$ or $2 + 5 = 7$

These statements are of the form *p or q* where p represents the first statement and q represents the second. The new statement *p or q* is called the disjunction of *p and q*.

| | |
|---|---|
| Disjunction: | A compound statement formed by connecting two simple statements with the word *or*. |

The following mathematical statements are examples of disjunction.

a) $3 + 2 = 5$ *or* $7 \cdot 7 = 49$

b) $12 - 5 = 7$ *or* $5 - 5 = 10$

c) $5 \cdot 3 = 20$ *or* $7 > -2$

d) $16 \cdot 3 = 163$ *or* $-6 < -10$

The following truth table is for disjunction.

| *Disjunction* | | |
|---|---|---|
| $p$ | $q$ | $p$ *or* $q$ |
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

The English word "or" generally suggests a choice, alternatives.

You can see from the table that a disjunction is true if either or both simple statements are true. It is false only when both simple statements are false.

➔ **Write *true* or *false* to tell whether the following disjunctions are true or false. Picture drawbridges in your mind. Use the truth table if you need it.**

1.21 ____________________ January is the first month of the year or December is the last month.

1.22 ____________________ $5 \cdot 3 = 15$ or $7 + 5 = 20$

1.23 ____________________ A line has only one point or a plane has at least 3 points.

1.24 ____________________ George Washington was President of Mexico or Abe Lincoln was President of Spain.

1.25 ____________________ 5 is less than 7 or 7 is less than 12.

1.26 ____________________ A segment has a midpoint or $\overrightarrow{AB}$ is the name of a ray.

1.27 ____________________ A line and a point outside the line are in exactly one plane or two planes intersect in a plane.

1.28 ____________________ Some roses are red or some violets are blue.

1.29 ____________________ *Collinear* means points on the same plane or *coplanar* means points on the same line.

1.30 ____________________ In this diagram, $RS + ST = SR$ or $ST + RS = RT$.


**Check Point – Check Answer Key.**

When Jesus came to wash Peter's feet, Peter said, "Thou shalt never wash my feet." This statement was exactly the opposite of Jesus' intention. On the Day of Pentecost, when the Holy Ghost was poured out on the brethren, some of the onlooking Jews said, "These men are full of new wine," or "These are drunken." Peter answered, "These are not drunken."

Many times we have a simple statement and want to form a different statement that says exactly the opposite. That is, we wish to deny the first assertion. This new statement is called a "negation." If the first statement is $p$ the second statement is *not p* and is written ~p. The second statement is the negation of $p$ or the negative of $p$.

Negation: If $p$ is a statement, the new statement, *not p* or *p is false,* is called the negation of $p$.

Here are some examples of negations.

a) $p$: It is raining.
not $p$: It is not raining.

b) $p$: $6 + 3 = 12$
$\sim p$: $6 + 3 \neq 12$

c) $p$: The lights are on.
not $p$: The lights are not on; or
the lights are off.

d) $p$: A triangle has six sides.
$\sim p$: A triangle does not have six sides.

e) $p$: It is false that spinach is good for you.
$\sim p$: Spinach is good for you.

When we form the negation of a statement in words, we do not usually place *not* in front of the original statement. Logically, to do so would be correct; but the resulting statement might sound strange. Instead we place the *not* in the sentence where it sounds better. The drawbridge concept is very simple: If a bridge is not up it is down and we can travel. If it is not down, it is up; we can not travel.

The truth table for negation is quite simple:

| Negation | |
|---|---|
| $p$ | $\sim p$ |
| T | F |
| F | T |

If a statement is true, its negation is false.
If a statement is false, its negation is true.