MATHEMATICS 1208
QUADRATIC EQUATIONS

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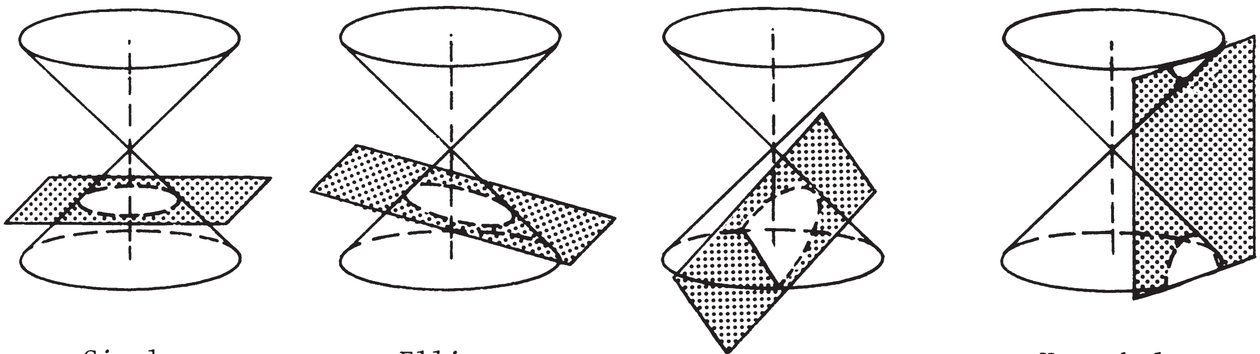
QUADRATIC EQUATIONS

The general form for all *quadratic equations* in x and y is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where the coefficients, A, B, C, D, E, and F *are real numbers and at least one coefficient,* A, B, or C, is a number other than zero.

The second-degree equations are derived from a plane intersecting a right circular cone.



Circle Ellipse Parabola Hyperbola

The circle, the ellipse, the parabola, and the hyperbola are called the *conic sections*. When the intersecting plane is perpendicular to the axis of the cone, the section formed is a *circle*. When the intersecting plane is rotated so that it is not perpendicular to the axis of the cone but intersects both edges of the cone, the section formed is an *ellipse*. When the intersecting plane is parallel to one edge of the cone, the section formed is a *parabola*. When the intersecting plane is parallel to the axis of the cone, the section formed is a *hyperbola*.

The graphs of the conic sections can assume any position on the coordinate axis. Transformation of the equation simplifies the equation and puts its graph in standard form.

OBJECTIVES

Read these objectives. The objectives tell you what you will be able to do when you have successfully completed this LIGHTUNIT.

When you have finished this LIGHTUNIT, you should be able:

1. To identify conic sections, find specific information relating to each of them, and graph related equations.
2. To rotate and translate an equation into a simpler equation and sketch its curve.

Survey the LIGHTUNIT Ask yourself some questions about this study. Write your questions here.

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OBJECTIVE

Read this objective. When you have completed this section, you should be able:

I. CONIC SECTIONS: CIRCLE AND ELLIPSE

1. To identify conic sections, find specific information relating to each of them, and graph related equations.

The equation for each of the conic sections will be derived from a geometric definition of *locus of points*.

DEFINITION

Locus of points is the set of all the points, and only those points, that satisfy a given condition.

You should become familiar with a formal definition of *conic* sections, sometimes simply called *conics*.

DEFINITION

Conic section is the locus of a point that moves so that the ratio of its distance from a fixed point to its distance from a fixed line is constant.

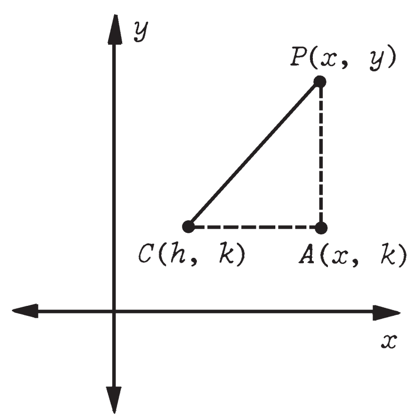
The ratio of a curve is called its *eccentricity* and is always denoted by e. When $e = 1$, the conic is a parabola. When $e < 1$, the conic is an *ellipse*. When $e > 1$, the conic is a hyperbola. As e approaches zero, the corresponding ellipses become more and more circular, approaching the circle as a limit. The fixed line is called the *directrix*.

THE CIRCLE

The circle is probably the most widely used geometric configuration in our world today. The circle appears equally as often in nature as it does in man's inventions. The study of the circle is responsible for the discovery of the number π by early mathematicians and also the study of irrational numbers.

DEFINITION

A *circle* is the locus of points that are at a constant distance from a fixed point.



The fixed point (h, k) is the *center* of the circle and the constant is the *radius*, r, of the circle. The point (h, k) and the constant r are the elements of the set of real numbers.

STANDARD FORM

On the coordinate axis, locate a point C with center (h, k) as the fixed point and point $P(x, y)$ as the radius, or the point that moves. Draw $\overline{PA} \perp \overline{CA}$. ΔCAP is a right triangle and $\overline{CP} = r$.

Using the distance formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

$$\overline{CA} = \sqrt{(x - h)^2 + (k - k)^2} = \sqrt{(x - h)^2} = |x - h|$$

and $\overline{AP} = \sqrt{(x - x)^2 + (y - k)^2} = \sqrt{(y - k)^2} = |y - k|$.

By the Pythagoreon Theorem, $(x - h)^2 + (y - k)^2 = r^2$, which is the standard form of the equation with (h, k) as the center and $|r|$ as the radius. If $(h, k) = (0, 0)$, which means that the center is at the origin, the equation becomes $x^2 + y^2 = r^2$.

STANDARD EQUATIONS OF THE CIRCLE
Center at origin: $x^2 + y^2 = r^2$
Center at (h, k): $(x - h)^2 + (y - k)^2 = r^2$

Model 1: Find the standard equation of a circle with center at (5, -1) and radius of 6.

Given: $(h, k) = (5, -1)$ and $r = 6$.

Using the distance formula square, $(x - h)^2 + (y - k)^2 = r^2$,

we obtain $(x - 5)^2 + (y + 1)^2 = 36$.

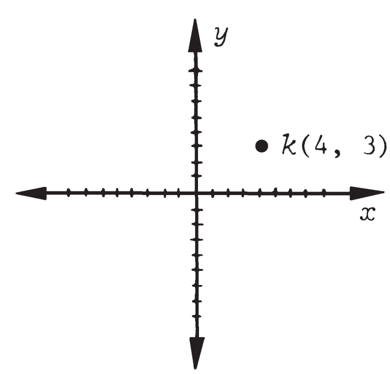
Model 2: Find the standard equation of a circle with center at the origin and passing through the point (4, 3).

Using the distance formula,

radius $= \sqrt{(4 - 0)^2 + (3 - 0)^2} = 5$.

Therefore, $x^2 + y^2 = r^2$ and the equation of the circle is

$$x^2 + y^2 = 25.$$



Note: If $r = 0$, $x^2 + y^2 = 0$. The only point belonging to this equation is (0, 0). If $r < 0$, the graph is also a point, being the empty set; hence, no circle.

[redacted] Write the standard equation of each of the following circles.

1.1 Center at (3, 5) and radius of 5.

1.2 Center at (-8, -6) and radius of 10.

1.3 Center at the origin and radius of 4.

1.4 $(h, k) = (6, 0)$ and $r = 6$.

1.5 $(h, k) = (1, -2)$ and passing through the origin.

1.6 Center on the positive x-axis and passing through the origin with radius of 2.

1.7 Center on the negative y-axis and passing through the origin with radius of 3.

[redacted] Describe the graph for each of the following circles given the standard equations.

1.8 $x^2 + y^2 = 36$

1.9 $(x - 2)^2 + (y + 3)^2 = 49$

1.10 $(x + 3)^2 + (y - 2)^2 = 0$

1.11 $x^2 + y^2 = -4$

1.12 $x^2 + y^2 = 0$

1.13 $(x - 5)^2 + (y + 4)^2 = 18$

1.14 $3x^2 + 3y^2 = -27$

1.15 $10(x - 3)^2 + 10(y + 4)^2 = 100$

We are given the equation of the circle $(x - h)^2 + (y - k)^2 = r^2$.

Expand to obtain

$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$ and

$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$.

Compare the result to the general form of the conic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The values of the coefficients in the equation for the circle can be identified from the general quadratic equation as $A = C = 1$, $B = 0$, $D = -2h$, $E = -2k$, and $h^2 + k^2 - r^2 = F$.

GENERAL FORM OF THE EQUATION OF A CIRCLE

$$x^2 + y^2 + Dx + Ey + F = 0$$

From the general form of the equation of a circle, we may easily determine the center, the radius, the range (values that y may take), and the domain (values that x may take).

Model 1: Given the circle $x^2 + y^2 + 6x - 4y - 12 = 0$ find the center, the radius, the domain, and the range, and sketch the curve.

Given: $x^2 + y^2 + 6x - 4y - 12 = 0$

Group like terms: $x^2 + 6x + y^2 - 4y = 12$

Complete the squares: $x^2 + 6x + 9 + y^2 - 4y + 4 = 12 + 9 + 4$

Simplify: $(x + 3)^2 + (y - 2)^2 = 25$

Therefore, the center $(h, k) = (-3, 2)$,

the radius = 5, the domain is $-8 \leq x \leq 2$,

and the range is $-3 \leq y \leq 7$.