

A Maple Package for Parametric Matrix Computations

Robert M. Corless, Marc Moreno Maza
and **Steven E. Thornton**

Department of Applied Mathematics, Western University
Ontario Research Centre for Computer Algebra

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Outline

1. Motivation
2. Regular Chains
3. Background
4. Rank
5. Zigzag Form
6. Future Work

Parametric Matrix Computations in CAS

Computer algebra systems are currently **unable** to handle case discussion for parametric matrices.

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

Jordan Canonical Form: Maple, Mathematica, Sage

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$$\begin{bmatrix} -2 & & \\ & -3 + \alpha & \\ & & \alpha \end{bmatrix}$$

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Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\alpha = -2} \begin{bmatrix} -2 & 0 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

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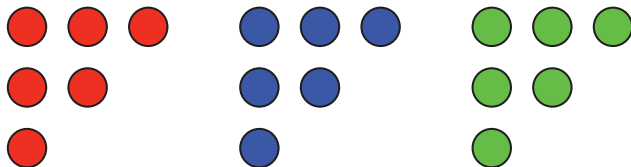
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REGULAR CHAINS



Multivariate Polynomials

- ▶ Let \mathbb{K} be an algebraically closed of real closed field
- ▶ Let $x_1 < \dots < x_n$ be $n \geq 1$ ordered variables
- ▶ $\mathbb{K}[x_1, \dots, x_n]$ is the ring of polynomials in x_1, \dots, x_n
- ▶ For non-constant $p \in \mathbb{K}[x_1, \dots, x_n]$, $\text{mvar}(p)$ denotes the greatest variable in p
- ▶ Leading coefficient of p w.r.t $\text{mvar}(p)$ is called the **initial** denoted $\text{init}(p)$

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Regular Chain

Definition (Triangular Set)

A set T of non-constant polynomials in $\mathbb{K}[x_1, \dots, x_n]$ is called a **triangular set** if for all $p, q \in T$ with $p \neq q$, $\text{mvar}(p) \neq \text{mvar}(q)$.

Definition (Saturated Ideal)

The **saturated ideal** $\text{sat}(T)$ of T is the ideal $\langle T \rangle : h_T^\infty$, where h_T is the product of the initials of the polynomials in T

Definition (Regular Chain)

T is a **regular chain** if $T = \emptyset$ or $T = T' \cup \{t\}$ for $t \in T$ with $\text{mvar}(t)$ maximum such that

- ▶ T' is a regular chain
- ▶ $\text{init}(t)$ is regular modulo $\text{sat}(T')$

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Definition (Constructible Set)

A **constructible set** is the disjunction of systems of polynomial equations and inequations. Where the systems of equations are conjunctions of constraints.

Semi-algebraic Set

Definition (Semi-algebraic System)

A **semi-algebraic system** of $\mathbb{K}[x_1, \dots, x_n]$ is any polynomial system S of the form

$$\left\{ \begin{array}{l} f_1 = \dots = f_a = 0 \\ g \neq 0 \\ p_1 > 0, \dots, p_b > 0 \\ q_1 \geq 0, \dots, q_c \geq 0 \end{array} \right\}$$

for $f_1, \dots, f_a, g, p_1, \dots, p_b, q_1, \dots, q_c \in \mathbb{K}[x_1, \dots, x_n]$.

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Notation

- ▶ Parameters $\alpha = \alpha_1, \dots, \alpha_s$
- ▶ \mathbb{K} is an algebraically closed or real closed field
- ▶ $\mathbb{K}[\alpha]$ is the ring of polynomials in the parameters
- ▶ $\mathbb{K}(\alpha)$ is the quotient field of $\mathbb{K}[\alpha]$
- ▶ S is a constructible (resp. semi-algebraic) set of \mathbb{K}^s

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Proposition^{1,2}

For two constructible (resp. semi-algebraic) sets $S_1, S_2 \subseteq \mathbb{K}^s$ one can compute a **triangular decomposition** of their

- ▶ intersection $S_1 \cap S_2$
- ▶ union $S_1 \cup S_2$
- ▶ set theoretic difference $S_1 \setminus S_2$

¹**C. Chen, J. H. Davenport, J. P. May, M. Moreno Maza, B. Xia, and R. Xiao.** Triangular decomposition of semi-algebraic systems. *J. Symb. Comput.*, 49:326, 2013.

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Remark

Let $S \subseteq \mathbb{K}^s$ be a constructible (resp. semi-algebraic) set; for every $f(\alpha) \in \mathbb{K}[\alpha]$ a **partition** $(S_{\text{eq}}, S_{\text{neq}})$ of S can be computed by

$$\begin{aligned} S_{\text{eq}} &= S \cap V(f(\alpha)) \\ S_{\text{neq}} &= S \setminus V(f(\alpha)) = S \setminus S_{\text{eq}} . \end{aligned}$$

- ▶ $f(\alpha)$ is **zero** everywhere on S_{eq}
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Parametric Matrix

Definition

A $m \times n$ **parametric matrix** takes the form:

$$A(\alpha) = \begin{bmatrix} f_{1,1}(\alpha) & f_{1,2}(\alpha) & \cdots & f_{1,n}(\alpha) \\ f_{2,1}(\alpha) & f_{2,2}(\alpha) & \cdots & f_{2,n}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1}(\alpha) & f_{m,2}(\alpha) & \cdots & f_{m,n}(\alpha) \end{bmatrix}$$

for

$$f_{i,j} \in \mathbb{K}(\alpha) \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Require constructible (resp. semi-algebraic) set $S \subseteq \mathbb{K}^s$ such that denominator of every entry of $A(\alpha)$ is nonzero everywhere on S .

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Parametric Rank Algorithm (Complex Case)

- Input:**
- ▶ Parametric matrix $A(\alpha)$
 - ▶ Constraints on the parameter values given in a set S by a polynomial system of the form

$$f_1(\alpha) = \cdots = f_a(\alpha) = 0, g(\alpha) \neq 0$$

- Output:**
- ▶ A list with elements of the form $[r_i, S_i]$ where S_i gives conditions on α such that the rank of $A(\alpha)$ is r_i
 - ▶ S_i 's form a partition of S

$$\bigcup_i S_i = S, \quad S_i \cap S_j = \emptyset \quad \forall i \neq j$$

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Parametric Rank Algorithm (Complex Case)

Step 1: Define polynomial system S' as **union** of equations of $A(\alpha)X = 0$ and S

Step 2: Let

$$\mathcal{T} := \text{Triangularize}(S', \mathbb{K}[\alpha_1 < \cdots < \alpha_s < x_1 < \cdots < x_n])$$

Step 3: For $0 \leq r \leq n$, let C_r be the **constructible set** of \mathbb{K}^s given by all regular systems $[T_j \cap \mathbb{K}[\alpha_1 < \cdots < \alpha_s], h_j]$ such that $[T_j, h_j] \in \mathcal{T}$ and the number of polynomials of T_j of positive degree in (at least) one of the variables $x_1 < \cdots < x_n$ is exactly $n - r$.

Step 4: For $r := n$ down to 1 do

$$C_r := \text{Difference}(C_r, C_{r-1} \cup \cdots \cup C_0)$$

Step 5: Project out the x_i variables

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Example³ over \mathbb{C}

Example

Consider $E\ddot{x} = A_1\dot{x} + A_2x + Bu$

$$E = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \lambda & 3\lambda & \lambda \\ 3\lambda + \mu & \lambda + \mu & \lambda + 3\mu \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda, \mu \in \mathbb{C}$$

³**M. I. García-Planas and J. Clotet.** Analyzing the set of uncontrollable second order generalized linear systems. *International Journal of Applied Mathematics and Informatics*, 2007.

Example³ over \mathbb{C}

Example (cont.)

What is the **rank** of:

$$C = \begin{bmatrix} -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\ -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\ A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & A_2 & -A_1 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & B \end{bmatrix}$$

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Example³ over \mathbb{C}

Example (cont.)

$$\text{Rank}(C) = \begin{cases} 18 & \text{if } \lambda \neq 0, \mu \neq 1/2 \\ 17 & \text{if } \lambda \neq 0, \mu = 1/2 \\ 16 & \text{if } \lambda = 0, \mu \neq -1 \\ 15 & \text{if } \lambda = 0, \mu = -1 \end{cases}$$

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Zigzag Form of Parametric Matrices

Zigzag Form

Parametric Polynomial

Define a polynomial in x with parameters α as

$$f(x; \alpha) = f_0(\alpha) + f_1(\alpha)x + \cdots + f_{r-1}(\alpha)x^{r-1} + x^r$$

with $f_0(\alpha), \dots, f_{r-1}(\alpha) \in \mathbb{K}(\alpha)$.

Require constructible (resp. semi-algebraic) set $S \subseteq \mathbb{K}^k$ such that denominator of every coefficient is nonzero everywhere on S .

Parametric Polynomial

Define a polynomial in x with parameters α as

$$f(x; \alpha) = f_0(\alpha) + f_1(\alpha)x + \cdots + f_{r-1}(\alpha)x^{r-1} + x^r$$

with $f_0(\alpha), \dots, f_{r-1}(\alpha) \in \mathbb{K}(\alpha)$.

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Companion Matrix

Frobenius companion matrix in x of a parametric polynomial

$$f(x; \alpha) = f_0(\alpha) + f_1(\alpha)x + \cdots + f_{r-1}(\alpha)x^{r-1} + x^r$$

takes the form

$$C_{f(x;\alpha)} = \begin{bmatrix} 0 & \cdots & 0 & -f_0(\alpha) \\ 1 & \ddots & \vdots & \vdots \\ & \ddots & 0 & -f_{r-2}(\alpha) \\ & & 1 & -f_{r-1}(\alpha) \end{bmatrix}$$

Off-diagonal Block

Let $b \in \{0, 1\}$. Define

$$B_b = \begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Zigzag Matrix⁴

$$\left[\begin{array}{cccccccc} C_{c_1(x;\alpha)} & B_{b_1} & & & & & & \\ & C_{c_2(x;\alpha)}^T & & & & & & \\ & B_{b_2} & C_{c_3(x;\alpha)} & B_{b_3} & & & & \\ & & C_{c_4(x;\alpha)}^T & & & & & \\ & & & \ddots & & & & \\ & & & & & C_{c_{d-2}(x;\alpha)}^T & & \\ & & & & & B_{b_{d-2}} & C_{c_{d-1}(x;\alpha)} & B_{b_{d-1}} \\ & & & & & & & C_{c_d(x;\alpha)}^T \end{array} \right]$$

for d even

⁴A. Storjohann. An $\mathcal{O}(n^3)$ algorithm for the Frobenius normal form. In *Proceedings of ISSAC 1998*, pages 101-105. ACM, 1998.

Theorem

For **every** matrix $A(\alpha) \in \mathbb{K}^{n \times n}[\alpha]$, there exists a partition (S_1, \dots, S_N) of the input constructible (resp. semi-algebraic) set S such that for each S_i , there exists a matrix $Z_i(\alpha) \sim A(\alpha)$ in **Zigzag form** where the denominator of the entries of $Z_i(\alpha)$ are all nonzero everywhere on S_i .

- ▶ The algorithm extends the work of **Storjohann**⁴ on the computation of the Zigzag form.
- ▶ The algorithm only uses **similarity transformations** to obtain the Zigzag form.
- ▶ **Stages 1 and 3** of Storjohann's algorithm⁴ are modified such that computation splits when searching for pivots.

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Semi-Algebraic Example over \mathbb{R}

Example

$$A(a, b) = \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix} \quad \text{with } a \geq 0, b > 0, a \neq b$$

Semi-Algebraic Example over \mathbb{R}

Example

$$A(a, b) = \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix} \quad \text{with } a \geq 0, b > 0, a \neq b$$

$$Z_1(a, b) = \begin{bmatrix} 0 & 0 & 12a \\ 1 & 0 & 18 \\ 0 & 1 & b \end{bmatrix} \quad \text{if } a > 0, b > 0, a \neq b$$

$$Z_2(a, b) = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline & 0 & 1 \\ & 18 & b \end{array} \right] \quad \text{if } a = 0, b > 0$$

Algebraic Example over \mathbb{C}

Example

$$A(\alpha) = \begin{bmatrix} -1 & -\alpha - 1 & 0 \\ -1/2 & \alpha - 2 & 1/2 \\ -2 & 3\alpha + 1 & -1 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

Algebraic Example over \mathbb{C}

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$$Z_1(\alpha) = \begin{bmatrix} 0 & 0 & 4\alpha \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix} \quad \text{if } \alpha + 3 \neq 0$$

$$Z_2(\alpha) = \left[\begin{array}{cc|c} 0 & -4 & 1 \\ 1 & -4 & 0 \\ \hline & & -3 \end{array} \right] \quad \text{if } \alpha + 3 = 0$$

Outline

1. Motivation
2. Regular Chains
3. Background
4. Rank
5. Zigzag Form
- 6. Future Work**

Future Work

Research is currently being conducted to create a **MAPLE** package **ParametricMatrixTools** with methods to compute

- ▶ Rank ✓
- ▶ Frobenius (Rational) form
- ▶ Minimal polynomial
- ▶ Test for similarity
- ▶ Jordan form
- ▶ Weyr form⁵

⁵K. O'Meara, J.Clark, and C. Vinsonhaler. *Advanced Topics in Linear Algebra: Weaving Matrix Problems Through the Weyr Form*. Oxford University Press, 2011

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All current and future code and example worksheets will be available at StevenThornton.ca/Code

Information on the RegularChains package of MAPLE is available at RegularChains.org

Thank You!