

# A MAPLE Package for Parametric Matrix Computations

Robert M. Corless, Marc Moreno Maza  
and **Steven E. Thornton**

Department of Applied Mathematics, Western University  
Ontario Research Centre for Computer Algebra

SONAD 2015  
London, Canada



# Outline

1. Motivation
2. Regular Chains
3. Background
4. Rank
5. Zigzag Form
6. Frobenius Form
7. Future Work

# Parametric Matrix Computations in CAS

Computer algebra systems are currently **unable** to handle case discussion for parametric matrices.

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

# Jordan Canonical Form: Maple, Mathematica, Sage

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

Maple

$$\begin{bmatrix} -2 & & \\ & \alpha & \\ & & \alpha - 3 \end{bmatrix}$$

# Jordan Canonical Form: Maple, Mathematica, Sage

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

Maple

$$\begin{bmatrix} -2 & & \\ & \alpha & \\ & & \alpha - 3 \end{bmatrix}$$

Mathematica

$$\begin{bmatrix} -2 & & \\ & -3 + \alpha & \\ & & \alpha \end{bmatrix}$$

# Jordan Canonical Form: Maple, Mathematica, Sage

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

Maple

$$\begin{bmatrix} -2 & & \\ & \alpha & \\ & & \alpha - 3 \end{bmatrix}$$

Mathematica

$$\begin{bmatrix} -2 & & \\ & -3 + \alpha & \\ & & \alpha \end{bmatrix}$$

Sage

?

# Jordan Canonical Form: Maple, Mathematica, Sage

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\alpha = -2} \begin{bmatrix} -2 & 0 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$



# Jordan Canonical Form: Maple, Mathematica, Sage

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\alpha = -2} \begin{bmatrix} -2 & 0 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Maple

$$\left[ \begin{array}{c|cc} -5 & & \\ \hline & -2 & 1 \\ & & -2 \end{array} \right]$$

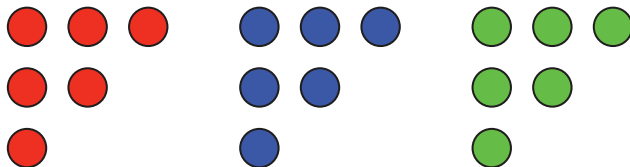
Mathematica

$$\left[ \begin{array}{c|cc} -5 & & \\ \hline & -2 & 1 \\ & & -2 \end{array} \right]$$

Sage

$$\left[ \begin{array}{c|cc} -5 & & \\ \hline & -2 & 1 \\ & & -2 \end{array} \right]$$

# REGULAR CHAINS



# Notation

- ▶ Let  $\mathbb{K}$  be an algebraically closed or real closed field
- ▶ Let  $x_1 < \dots < x_n$  be  $n \geq 1$  ordered variables
- ▶  $\mathbb{K}[x_1, \dots, x_n]$  is the ring of polynomials in  $x_1, \dots, x_n$
- ▶ For non-constant  $p \in \mathbb{K}[x_1, \dots, x_n]$ ,  $\text{mvar}(p)$  denotes the greatest variable in  $p$
- ▶ Leading coefficient of  $p$  w.r.t  $\text{mvar}(p)$  is called the **initial** denoted  $\text{init}(p)$

# Notation

- ▶ Let  $\mathbb{K}$  be an algebraically closed or real closed field
- ▶ Let  $x_1 < \dots < x_n$  be  $n \geq 1$  ordered variables
- ▶  $\mathbb{K}[x_1, \dots, x_n]$  is the ring of polynomials in  $x_1, \dots, x_n$
- ▶ For non-constant  $p \in \mathbb{K}[x_1, \dots, x_n]$ ,  $\text{mvar}(p)$  denotes the greatest variable in  $p$
- ▶ Leading coefficient of  $p$  w.r.t  $\text{mvar}(p)$  is called the **initial** denoted  $\text{init}(p)$

# Notation

- ▶ Let  $\mathbb{K}$  be an algebraically closed or real closed field
- ▶ Let  $x_1 < \dots < x_n$  be  $n \geq 1$  ordered variables
- ▶  $\mathbb{K}[x_1, \dots, x_n]$  is the ring of polynomials in  $x_1, \dots, x_n$
- ▶ For non-constant  $p \in \mathbb{K}[x_1, \dots, x_n]$ ,  $\text{mvar}(p)$  denotes the greatest variable in  $p$
- ▶ Leading coefficient of  $p$  w.r.t  $\text{mvar}(p)$  is called the **initial** denoted  $\text{init}(p)$

# Notation

- ▶ Let  $\mathbb{K}$  be an algebraically closed or real closed field
- ▶ Let  $x_1 < \dots < x_n$  be  $n \geq 1$  ordered variables
- ▶  $\mathbb{K}[x_1, \dots, x_n]$  is the ring of polynomials in  $x_1, \dots, x_n$
- ▶ For non-constant  $p \in \mathbb{K}[x_1, \dots, x_n]$ ,  $\text{mvar}(p)$  denotes the greatest variable in  $p$
- ▶ Leading coefficient of  $p$  w.r.t  $\text{mvar}(p)$  is called the **initial** denoted  $\text{init}(p)$

# Notation

- ▶ Let  $\mathbb{K}$  be an algebraically closed or real closed field
- ▶ Let  $x_1 < \dots < x_n$  be  $n \geq 1$  ordered variables
- ▶  $\mathbb{K}[x_1, \dots, x_n]$  is the ring of polynomials in  $x_1, \dots, x_n$
- ▶ For non-constant  $p \in \mathbb{K}[x_1, \dots, x_n]$ ,  $\text{mvar}(p)$  denotes the greatest variable in  $p$
- ▶ Leading coefficient of  $p$  w.r.t  $\text{mvar}(p)$  is called the **initial** denoted  $\text{init}(p)$

# Regular Chain

## Definition (Triangular Set)

A set  $T$  of non-constant polynomials in  $\mathbb{K}[x_1, \dots, x_n]$  is called a **triangular set** if for all  $p, q \in T$  with  $p \neq q$ ,  $\text{mvar}(p) \neq \text{mvar}(q)$ .

## Definition (Saturated Ideal)

The **saturated ideal**  $\text{sat}(T)$  of  $T$  is the ideal  $\langle T \rangle : h_T^\infty$ , where  $h_T$  is the product of the initials of the polynomials in  $T$

## Definition (Regular Chain)

$T$  is a **regular chain** if  $T = \emptyset$  or  $T = T' \cup \{t\}$  for  $t \in T$  with  $\text{mvar}(t)$  maximum such that

- ▶  $T'$  is a regular chain
- ▶  $\text{init}(t)$  is regular modulo  $\text{sat}(T')$



# Regular Chain

## Definition (Triangular Set)

A set  $T$  of non-constant polynomials in  $\mathbb{K}[x_1, \dots, x_n]$  is called a **triangular set** if for all  $p, q \in T$  with  $p \neq q$ ,  $\text{mvar}(p) \neq \text{mvar}(q)$ .

## Definition (Saturated Ideal)

The **saturated ideal**  $\text{sat}(T)$  of  $T$  is the ideal  $\langle T \rangle : h_T^\infty$ , where  $h_T$  is the product of the initials of the polynomials in  $T$

## Definition (Regular Chain)

$T$  is a **regular chain** if  $T = \emptyset$  or  $T = T' \cup \{t\}$  for  $t \in T$  with  $\text{mvar}(t)$  maximum such that

- ▶  $T'$  is a regular chain
- ▶  $\text{init}(t)$  is regular modulo  $\text{sat}(T')$

# Regular Chain

## Definition (Triangular Set)

A set  $T$  of non-constant polynomials in  $\mathbb{K}[x_1, \dots, x_n]$  is called a **triangular set** if for all  $p, q \in T$  with  $p \neq q$ ,  $\text{mvar}(p) \neq \text{mvar}(q)$ .

## Definition (Saturated Ideal)

The **saturated ideal**  $\text{sat}(T)$  of  $T$  is the ideal  $\langle T \rangle : h_T^\infty$ , where  $h_T$  is the product of the initials of the polynomials in  $T$

## Definition (Regular Chain)

$T$  is a **regular chain** if  $T = \emptyset$  or  $T = T' \cup \{t\}$  for  $t \in T$  with  $\text{mvar}(t)$  maximum such that

- ▶  $T'$  is a regular chain
- ▶  $\text{init}(t)$  is regular modulo  $\text{sat}(T')$

## Definition (Constructible Set)

A **constructible set** is the disjunction of systems of polynomial equations and inequations. Where the systems of equations are conjunctions of constraints.

# Semi-algebraic Set

## Definition (Semi-algebraic System)

A **semi-algebraic system** of  $\mathbb{K}[x_1, \dots, x_n]$  is any polynomial system  $S$  of the form

$$\left\{ \begin{array}{l} f_1 = \dots = f_a = 0 \\ g \neq 0 \\ p_1 > 0, \dots, p_b > 0 \\ q_1 \geq 0, \dots, q_c \geq 0 \end{array} \right\}$$

for  $f_1, \dots, f_a, g, p_1, \dots, p_b, q_1, \dots, q_c \in \mathbb{K}[x_1, \dots, x_n]$ .

## Definition (Semi-algebraic Set)

A **semi-algebraic set**  $S$  of  $\mathbb{K}[x_1, \dots, x_n]$  is the solution set of a semi-algebraic system.

# Semi-algebraic Set

## Definition (Semi-algebraic System)

A **semi-algebraic system** of  $\mathbb{K}[x_1, \dots, x_n]$  is any polynomial system  $S$  of the form

$$\left\{ \begin{array}{l} f_1 = \dots = f_a = 0 \\ g \neq 0 \\ p_1 > 0, \dots, p_b > 0 \\ q_1 \geq 0, \dots, q_c \geq 0 \end{array} \right\}$$

for  $f_1, \dots, f_a, g, p_1, \dots, p_b, q_1, \dots, q_c \in \mathbb{K}[x_1, \dots, x_n]$ .

## Definition (Semi-algebraic Set)

A **semi-algebraic set**  $S$  of  $\mathbb{K}[x_1, \dots, x_n]$  is the solution set of a semi-algebraic system.

# Example

## Example

RealTriangularize applied to *sofa* and *cylinder*

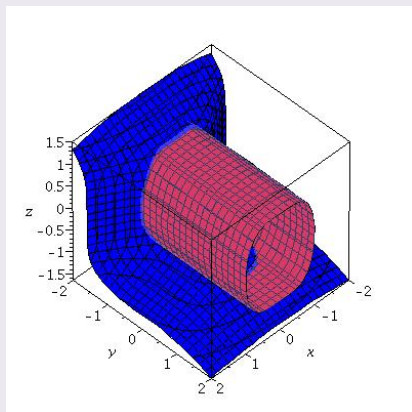
$$x^2 + y^3 + z^5 = x^4 + z^2 - 1 = 0$$

# Example

## Example

RealTriangularize applied to *sofa* and *cylinder*

$$x^2 + y^3 + z^5 = x^4 + z^2 - 1 = 0$$



# Example

```
> R := PolynomialRing([z, y, x]);
F := [x^2 + y^3 + z^5, x^4 + z^2 - 1];
RealTriangularize(F, R, output = record);
```

$R := \text{polynomial\_ring}$

$F := [z^5 + y^3 + x^2, x^4 + z^2 - 1]$

$$\left\{ \begin{array}{l} (x^8 - 2x^4 + 1)z + y^3 + x^2 = 0 \\ y^6 + 2x^2y^3 + x^{20} - 5x^{16} + 10x^{12} - 10x^8 + 6x^4 - 1 = 0 \\ x < 1 \\ x + 1 > 0 \\ x^{12} - 4x^8 + 5x^4 - 1 \neq 0 \end{array} \right. , \left\{ \begin{array}{l} (x^8 - 2x^4 + 1)z - x^2 = 0 \\ y^3 + 2x^2 = 0 \\ x^{12} - 4x^8 + 5x^4 - 1 = 0 \end{array} \right. ,$$

$$\left\{ \begin{array}{l} (x^8 - 2x^4 + 1)z + x^2 = 0 \\ y = 0 \\ x^{12} - 4x^8 + 5x^4 - 1 = 0 \end{array} \right. , \left\{ \begin{array}{l} z = 0 \\ y + 1 = 0 \\ x + 1 = 0 \end{array} \right. , \left\{ \begin{array}{l} z = 0 \\ y + 1 = 0 \\ x - 1 = 0 \end{array} \right.$$



# Outline

1. Motivation
2. Regular Chains
- 3. Background**
4. Rank
5. Zigzag Form
6. Frobenius Form
7. Future Work

# Notation

- ▶ Parameters  $\alpha = \alpha_1, \dots, \alpha_s$
- ▶  $\mathbb{K}$  is an algebraically closed or real closed field
- ▶  $\mathbb{K}[\alpha]$  is the ring of polynomials in the parameters
- ▶  $\mathbb{K}(\alpha)$  is the quotient field of  $\mathbb{K}[\alpha]$
- ▶  $S$  is a constructible (resp. semi-algebraic) set of  $\mathbb{K}^s$

# Notation

- ▶ Parameters  $\alpha = \alpha_1, \dots, \alpha_s$
- ▶  $\mathbb{K}$  is an algebraically closed or real closed field
- ▶  $\mathbb{K}[\alpha]$  is the ring of polynomials in the parameters
- ▶  $\mathbb{K}(\alpha)$  is the quotient field of  $\mathbb{K}[\alpha]$
- ▶  $S$  is a constructible (resp. semi-algebraic) set of  $\mathbb{K}^s$

# Notation

- ▶ Parameters  $\alpha = \alpha_1, \dots, \alpha_s$
- ▶  $\mathbb{K}$  is an algebraically closed or real closed field
- ▶  $\mathbb{K}[\alpha]$  is the ring of polynomials in the parameters
- ▶  $\mathbb{K}(\alpha)$  is the quotient field of  $\mathbb{K}[\alpha]$
- ▶  $S$  is a constructible (resp. semi-algebraic) set of  $\mathbb{K}^s$

# Notation

- ▶ Parameters  $\alpha = \alpha_1, \dots, \alpha_s$
- ▶  $\mathbb{K}$  is an algebraically closed or real closed field
- ▶  $\mathbb{K}[\alpha]$  is the ring of polynomials in the parameters
- ▶  $\mathbb{K}(\alpha)$  is the quotient field of  $\mathbb{K}[\alpha]$
- ▶  $S$  is a constructible (resp. semi-algebraic) set of  $\mathbb{K}^s$

# Notation

- ▶ Parameters  $\alpha = \alpha_1, \dots, \alpha_s$
- ▶  $\mathbb{K}$  is an algebraically closed or real closed field
- ▶  $\mathbb{K}[\alpha]$  is the ring of polynomials in the parameters
- ▶  $\mathbb{K}(\alpha)$  is the quotient field of  $\mathbb{K}[\alpha]$
- ▶  $S$  is a constructible (resp. semi-algebraic) set of  $\mathbb{K}^s$

# Proposition<sup>1,2</sup>

For two constructible (resp. semi-algebraic) sets  $S_1, S_2 \subseteq \mathbb{K}^s$  one can compute a **triangular decomposition** of their

- ▶ intersection  $S_1 \cap S_2$
- ▶ union  $S_1 \cup S_2$
- ▶ set theoretic difference  $S_1 \setminus S_2$

<sup>1</sup>**C. Chen, J. H. Davenport, J. P. May, M. Moreno Maza, B. Xia, and R. Xiao.** Triangular decomposition of semi-algebraic systems. *J. Symb. Comput.*, 49:326, 2013.

<sup>2</sup>**C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza, W. Pan.** Comprehensive triangular decomposition. In Proc. of *CASC07. Volume 4770 of Lecture Notes in Computer Science.* (2007) 73101.

# Proposition<sup>1,2</sup>

For two constructible (resp. semi-algebraic) sets  $S_1, S_2 \subseteq \mathbb{K}^s$  one can compute a **triangular decomposition** of their

- ▶ intersection  $S_1 \cap S_2$
- ▶ union  $S_1 \cup S_2$
- ▶ set theoretic difference  $S_1 \setminus S_2$

<sup>1</sup>**C. Chen, J. H. Davenport, J. P. May, M. Moreno Maza, B. Xia, and R. Xiao.** Triangular decomposition of semi-algebraic systems. *J. Symb. Comput.*, 49:326, 2013.

<sup>2</sup>**C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza, W. Pan.** Comprehensive triangular decomposition. In Proc. of *CASC07. Volume 4770 of Lecture Notes in Computer Science.* (2007) 73101.



# Proposition<sup>1,2</sup>

For two constructible (resp. semi-algebraic) sets  $S_1, S_2 \subseteq \mathbb{K}^s$  one can compute a **triangular decomposition** of their

- ▶ intersection  $S_1 \cap S_2$
- ▶ union  $S_1 \cup S_2$
- ▶ set theoretic difference  $S_1 \setminus S_2$

<sup>1</sup>**C. Chen, J. H. Davenport, J. P. May, M. Moreno Maza, B. Xia, and R. Xiao.** Triangular decomposition of semi-algebraic systems. *J. Symb. Comput.*, 49:326, 2013.

<sup>2</sup>**C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza, W. Pan.** Comprehensive triangular decomposition. In Proc. of *CASC07. Volume 4770 of Lecture Notes in Computer Science*. (2007) 73101.

# Proposition<sup>1,2</sup>

For two constructible (resp. semi-algebraic) sets  $S_1, S_2 \subseteq \mathbb{K}^s$  one can compute a **triangular decomposition** of their

- ▶ intersection  $S_1 \cap S_2$
- ▶ union  $S_1 \cup S_2$
- ▶ set theoretic difference  $S_1 \setminus S_2$

<sup>1</sup>C. Chen, J. H. Davenport, J. P. May, M. Moreno Maza, B. Xia, and R. Xiao. Triangular decomposition of semi-algebraic systems. *J. Symb. Comput.*, 49:326, 2013.

<sup>2</sup>C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza, W. Pan. Comprehensive triangular decomposition. In Proc. of CASC07. Volume 4770 of Lecture Notes in Computer Science. (2007) 73101.

# Remark

Let  $S \subseteq \mathbb{K}^s$  be a constructible (resp. semi-algebraic) set; for every  $f(\alpha) \in \mathbb{K}[\alpha]$  a **partition**  $(S_{\text{eq}}, S_{\text{neq}})$  of  $S$  can be computed by

$$\begin{aligned}S_{\text{eq}} &= S \cap V(f(\alpha)) \\S_{\text{neq}} &= S \setminus V(f(\alpha)) = S \setminus S_{\text{eq}}.\end{aligned}$$

- ▶  $f(\alpha)$  is **zero** everywhere on  $S_{\text{eq}}$
- ▶  $f(\alpha)$  is **nonzero** everywhere on  $S_{\text{neq}}$

# Remark

Let  $S \subseteq \mathbb{K}^s$  be a constructible (resp. semi-algebraic) set; for every  $f(\alpha) \in \mathbb{K}[\alpha]$  a **partition**  $(S_{\text{eq}}, S_{\text{neq}})$  of  $S$  can be computed by

$$\begin{aligned}S_{\text{eq}} &= S \cap V(f(\alpha)) \\S_{\text{neq}} &= S \setminus V(f(\alpha)) = S \setminus S_{\text{eq}}.\end{aligned}$$

- ▶  $f(\alpha)$  is **zero** everywhere on  $S_{\text{eq}}$
- ▶  $f(\alpha)$  is **nonzero** everywhere on  $S_{\text{neq}}$

# Remark

Let  $S \subseteq \mathbb{K}^s$  be a constructible (resp. semi-algebraic) set; for every  $f(\alpha) \in \mathbb{K}[\alpha]$  a **partition**  $(S_{\text{eq}}, S_{\text{neq}})$  of  $S$  can be computed by

$$\begin{aligned}S_{\text{eq}} &= S \cap V(f(\alpha)) \\S_{\text{neq}} &= S \setminus V(f(\alpha)) = S \setminus S_{\text{eq}} .\end{aligned}$$

- ▶  $f(\alpha)$  is **zero** everywhere on  $S_{\text{eq}}$
- ▶  $f(\alpha)$  is **nonzero** everywhere on  $S_{\text{neq}}$

# Remark

Let  $S \subseteq \mathbb{K}^s$  be a constructible (resp. semi-algebraic) set; for every  $f(\alpha) \in \mathbb{K}[\alpha]$  a **partition**  $(S_{\text{eq}}, S_{\text{neq}})$  of  $S$  can be computed by

$$\begin{aligned} S_{\text{eq}} &= S \cap V(f(\alpha)) \\ S_{\text{neq}} &= S \setminus V(f(\alpha)) = S \setminus S_{\text{eq}} . \end{aligned}$$

- ▶  $f(\alpha)$  is **zero** everywhere on  $S_{\text{eq}}$
- ▶  $f(\alpha)$  is **nonzero** everywhere on  $S_{\text{neq}}$

# Parametric Matrix

## Definition

A  $m \times n$  **parametric matrix** takes the form:

$$A(\alpha) = \begin{bmatrix} f_{1,1}(\alpha) & f_{1,2}(\alpha) & \cdots & f_{1,n}(\alpha) \\ f_{2,1}(\alpha) & f_{2,2}(\alpha) & \cdots & f_{2,n}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1}(\alpha) & f_{m,2}(\alpha) & \cdots & f_{m,n}(\alpha) \end{bmatrix}$$

for

$$f_{i,j} \in \mathbb{K}(\alpha) \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Require constructible (resp. semi-algebraic) set  $S \subseteq \mathbb{K}^s$  such that denominator of every entry of  $A(\alpha)$  is nonzero everywhere on  $S$ .

# Parametric Matrix

## Definition

A  $m \times n$  **parametric matrix** takes the form:

$$A(\alpha) = \begin{bmatrix} f_{1,1}(\alpha) & f_{1,2}(\alpha) & \cdots & f_{1,n}(\alpha) \\ f_{2,1}(\alpha) & f_{2,2}(\alpha) & \cdots & f_{2,n}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1}(\alpha) & f_{m,2}(\alpha) & \cdots & f_{m,n}(\alpha) \end{bmatrix}$$

for

$$f_{i,j} \in \mathbb{K}(\alpha) \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Require constructible (resp. semi-algebraic) set  $S \subseteq \mathbb{K}^s$  such that denominator of every entry of  $A(\alpha)$  is nonzero everywhere on  $S$ .



# Outline

1. Motivation
2. Regular Chains
3. Background
- 4. Rank**
5. Zigzag Form
6. Frobenius Form
7. Future Work

# Parametric Rank Algorithm (Complex Case)

- Input:**
- ▶ Parametric matrix  $A(\alpha)$
  - ▶ Constraints on the parameter values given in a set  $S$  by a polynomial system of the form

$$f_1(\alpha) = \cdots = f_a(\alpha) = 0, g(\alpha) \neq 0$$

- Output:**
- ▶ A list with elements of the form  $[r_i, S_i]$  where  $S_i$  gives conditions on  $\alpha$  such that the rank of  $A(\alpha)$  is  $r_i$
  - ▶  $S_i$ 's form a partition of  $S$

$$\bigcup_i S_i = S, \quad S_i \cap S_j = \emptyset \quad \forall i \neq j$$

# Parametric Rank Algorithm (Complex Case)

- Input:**
- ▶ Parametric matrix  $A(\alpha)$
  - ▶ Constraints on the parameter values given in a set  $S$  by a polynomial system of the form

$$f_1(\alpha) = \dots = f_a(\alpha) = 0, g(\alpha) \neq 0$$

- Output:**
- ▶ A list with elements of the form  $[r_i, S_i]$  where  $S_i$  gives conditions on  $\alpha$  such that the rank of  $A(\alpha)$  is  $r_i$
  - ▶  $S_i$ 's form a partition of  $S$

$$\bigcup_i S_i = S, \quad S_i \cap S_j = \emptyset \quad \forall i \neq j$$

# Parametric Rank Algorithm (Complex Case)

**Step 1:** Define polynomial system  $S'$  as **union** of equations of  $A(\alpha)X = 0$  and  $S$

**Step 2:** Let

$$\mathcal{T} := \text{Triangularize}(S', \mathbb{K}[\alpha_1 < \cdots < \alpha_s < x_1 < \cdots < x_n])$$

**Step 3:** For  $0 \leq r \leq n$ , let  $C_r$  be the **constructible set** of  $\mathbb{K}^s$  given by all regular systems  $[T_j \cap \mathbb{K}[\alpha_1 < \cdots < \alpha_s], h_j]$  such that  $[T_j, h_j] \in \mathcal{T}$  and the number of polynomials of  $T_j$  of positive degree in (at least) one of the variables  $x_1 < \cdots < x_n$  is exactly  $n - r$ .

**Step 4:** For  $r := n$  down to 1 do

$$C_r := \text{Difference}(C_r, C_{r-1} \cup \cdots \cup C_0)$$

**Step 5:** Project out the  $x_i$  variables

# Parametric Rank Algorithm (Complex Case)

**Step 1:** Define polynomial system  $S'$  as **union** of equations of  $A(\alpha)X = 0$  and  $S$

**Step 2:** Let

$$\mathcal{T} := \text{Triangularize}(S', \mathbb{K}[\alpha_1 < \cdots < \alpha_s < x_1 < \cdots < x_n])$$

**Step 3:** For  $0 \leq r \leq n$ , let  $C_r$  be the **constructible set** of  $\mathbb{K}^s$  given by all regular systems  $[T_j \cap \mathbb{K}[\alpha_1 < \cdots < \alpha_s], h_j]$  such that  $[T_j, h_j] \in \mathcal{T}$  and the number of polynomials of  $T_j$  of positive degree in (at least) one of the variables  $x_1 < \cdots < x_n$  is exactly  $n - r$ .

**Step 4:** For  $r := n$  down to 1 do

$$C_r := \text{Difference}(C_r, C_{r-1} \cup \cdots \cup C_0)$$

**Step 5:** Project out the  $x_i$  variables

# Parametric Rank Algorithm (Complex Case)

**Step 1:** Define polynomial system  $S'$  as **union** of equations of  $A(\alpha)X = 0$  and  $S$

**Step 2:** Let

$$\mathcal{T} := \text{Triangularize}(S', \mathbb{K}[\alpha_1 < \cdots < \alpha_s < x_1 < \cdots < x_n])$$

**Step 3:** For  $0 \leq r \leq n$ , let  $C_r$  be the **constructible set** of  $\mathbb{K}^s$  given by all regular systems  $[T_j \cap \mathbb{K}[\alpha_1 < \cdots < \alpha_s], h_j]$  such that  $[T_j, h_j] \in \mathcal{T}$  and the number of polynomials of  $T_j$  of positive degree in (at least) one of the variables  $x_1 < \cdots < x_n$  is exactly  $n - r$ .

**Step 4:** For  $r := n$  down to 1 do

$$C_r := \text{Difference}(C_r, C_{r-1} \cup \cdots \cup C_0)$$

**Step 5:** Project out the  $x_i$  variables

# Parametric Rank Algorithm (Complex Case)

**Step 1:** Define polynomial system  $S'$  as **union** of equations of  $A(\alpha)X = 0$  and  $S$

**Step 2:** Let

$$\mathcal{T} := \text{Triangularize}(S', \mathbb{K}[\alpha_1 < \cdots < \alpha_s < x_1 < \cdots < x_n])$$

**Step 3:** For  $0 \leq r \leq n$ , let  $C_r$  be the **constructible set** of  $\mathbb{K}^s$  given by all regular systems  $[T_j \cap \mathbb{K}[\alpha_1 < \cdots < \alpha_s], h_j]$  such that  $[T_j, h_j] \in \mathcal{T}$  and the number of polynomials of  $T_j$  of positive degree in (at least) one of the variables  $x_1 < \cdots < x_n$  is exactly  $n - r$ .

**Step 4:** For  $r := n$  down to 1 do

$$C_r := \text{Difference}(C_r, C_{r-1} \cup \cdots \cup C_0)$$

**Step 5:** Project out the  $x_i$  variables

# Parametric Rank Algorithm (Complex Case)

**Step 1:** Define polynomial system  $S'$  as **union** of equations of  $A(\alpha)X = 0$  and  $S$

**Step 2:** Let

$$\mathcal{T} := \text{Triangularize}(S', \mathbb{K}[\alpha_1 < \cdots < \alpha_s < x_1 < \cdots < x_n])$$

**Step 3:** For  $0 \leq r \leq n$ , let  $C_r$  be the **constructible set** of  $\mathbb{K}^s$  given by all regular systems  $[T_j \cap \mathbb{K}[\alpha_1 < \cdots < \alpha_s], h_j]$  such that  $[T_j, h_j] \in \mathcal{T}$  and the number of polynomials of  $T_j$  of positive degree in (at least) one of the variables  $x_1 < \cdots < x_n$  is exactly  $n - r$ .

**Step 4:** For  $r := n$  down to 1 do

$$C_r := \text{Difference}(C_r, C_{r-1} \cup \cdots \cup C_0)$$

**Step 5:** Project out the  $x_i$  variables



# Example<sup>3</sup> over $\mathbb{C}$

## Example

Consider  $E\ddot{x} = A_1\dot{x} + A_2x + Bu$

$$E = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \lambda & 3\lambda & \lambda \\ 3\lambda + \mu & \lambda + \mu & \lambda + 3\mu \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda, \mu \in \mathbb{C}$$

<sup>3</sup>M. I. García-Planas and J. Clotet. Analyzing the set of uncontrollable second order generalized linear systems. *International Journal of Applied Mathematics and Informatics*, 2007.

# Example<sup>3</sup> over $\mathbb{C}$

## Example (cont.)

What is the **rank** of:

$$C = \begin{bmatrix} -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\ -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\ A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & A_2 & -A_1 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & B \end{bmatrix}$$

<sup>3</sup>M. I. García-Planas and J. Clotet. Analyzing the set of uncontrollable second order generalized linear systems. *International Journal of Applied Mathematics and Informatics*, 2007.

# Example<sup>3</sup> over $\mathbb{C}$

## Example (cont.)

$$\text{Rank}(C) = \begin{cases} 18 & \text{if } \lambda \neq 0, \mu \neq 1/2 \\ 17 & \text{if } \lambda \neq 0, \mu = 1/2 \\ 16 & \text{if } \lambda = 0, \mu \neq -1 \\ 15 & \text{if } \lambda = 0, \mu = -1 \end{cases}$$

<sup>3</sup>**M. I. García-Planas and J. Clotet.** Analyzing the set of uncontrollable second order generalized linear systems. *International Journal of Applied Mathematics and Informatics*, 2007.

# Zigzag Form of Parametric Matrices

Zigzag Form

# Parametric Polynomial

Define a polynomial in  $x$  with parameters  $\alpha$  as

$$f(x; \alpha) = f_0(\alpha) + f_1(\alpha)x + \cdots + f_{r-1}(\alpha)x^{r-1} + x^r$$

with  $f_0(\alpha), \dots, f_{r-1}(\alpha) \in \mathbb{K}(\alpha)$ .

Require constructible (resp. semi-algebraic) set  $S \subseteq \mathbb{K}^k$  such that denominator of every coefficient is nonzero everywhere on  $S$ .

# Parametric Polynomial

Define a polynomial in  $x$  with parameters  $\alpha$  as

$$f(x; \alpha) = f_0(\alpha) + f_1(\alpha)x + \cdots + f_{r-1}(\alpha)x^{r-1} + x^r$$

with  $f_0(\alpha), \dots, f_{r-1}(\alpha) \in \mathbb{K}(\alpha)$ .

Require constructible (resp. semi-algebraic) set  $S \subseteq \mathbb{K}^k$  such that denominator of every coefficient is nonzero everywhere on  $S$ .

# Companion Matrix

*Frobenius companion matrix* in  $x$  of a parametric polynomial

$$f(x; \alpha) = f_0(\alpha) + f_1(\alpha)x + \cdots + f_{r-1}(\alpha)x^{r-1} + x^r$$

takes the form

$$C_{f(x;\alpha)} = \begin{bmatrix} 0 & \cdots & 0 & -f_0(\alpha) \\ 1 & \ddots & \vdots & \vdots \\ & \ddots & 0 & -f_{r-2}(\alpha) \\ & & 1 & -f_{r-1}(\alpha) \end{bmatrix}$$

# Off-diagonal Block

Let  $b \in \{0, 1\}$ . Define

$$B_b = \begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$



# Zigzag Matrix<sup>4</sup>

$$\left[ \begin{array}{cccccccc} C_{c_1(x;\alpha)} & B_{b_1} & & & & & & \\ & C_{c_2(x;\alpha)}^T & & & & & & \\ & B_{b_2} & C_{c_3(x;\alpha)} & B_{b_3} & & & & \\ & & C_{c_4(x;\alpha)}^T & & & & & \\ & & & \ddots & & & & \\ & & & & & C_{c_{d-2}(x;\alpha)}^T & & \\ & & & & & B_{b_{d-2}} & C_{c_{d-1}(x;\alpha)} & B_{b_{d-1}} \\ & & & & & & & C_{c_d(x;\alpha)}^T \end{array} \right]$$

for  $d$  even

<sup>4</sup>A. Storjohann. An  $\mathcal{O}(n^3)$  algorithm for the Frobenius normal form. In *Proceedings of ISSAC 1998*, pages 101-105. ACM, 1998.

# Theorem

## Theorem

For **every** matrix  $A(\alpha) \in \mathbb{K}^{n \times n}[\alpha]$ , there exists a partition  $(S_1, \dots, S_N)$  of the input constructible (resp. semi-algebraic) set  $S$  such that for each  $S_i$ , there exists a matrix  $Z_i(\alpha) \sim A(\alpha)$  in **Zigzag form** where the denominator of the entries of  $Z_i(\alpha)$  are all nonzero everywhere on  $S_i$ .

- ▶ The algorithm extends the work of **Storjohann**<sup>4</sup> on the computation of the Zigzag form.
- ▶ The algorithm only uses **similarity transformations** to obtain the Zigzag form.
- ▶ **Stages 1 and 3** of Storjohann's algorithm<sup>4</sup> are modified such that computation splits when searching for pivots.

<sup>4</sup>**A. Storjohann.** An  $\mathcal{O}(n^3)$  algorithm for the Frobenius normal form. In *Proceedings of ISSAC 1998*, pages 101-105. ACM, 1998.

- ▶ The algorithm extends the work of **Storjohann**<sup>4</sup> on the computation of the Zigzag form.
- ▶ The algorithm only uses **similarity transformations** to obtain the Zigzag form.
- ▶ **Stages 1 and 3** of Storjohann's algorithm<sup>4</sup> are modified such that computation splits when searching for pivots.

<sup>4</sup>**A. Storjohann.** An  $\mathcal{O}(n^3)$  algorithm for the Frobenius normal form. In *Proceedings of ISSAC 1998*, pages 101-105. ACM, 1998.

- ▶ The algorithm extends the work of **Storjohann**<sup>4</sup> on the computation of the Zigzag form.
- ▶ The algorithm only uses **similarity transformations** to obtain the Zigzag form.
- ▶ **Stages 1 and 3** of Storjohann's algorithm<sup>4</sup> are modified such that computation splits when searching for pivots.

<sup>4</sup>**A. Storjohann.** An  $\mathcal{O}(n^3)$  algorithm for the Frobenius normal form. In *Proceedings of ISSAC 1998*, pages 101-105. ACM, 1998.

# Semi-Algebraic Example over $\mathbb{R}$

## Example

$$A(a, b) = \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix} \quad \text{with } a \geq 0, b > 0, a \neq b$$

---

# Semi-Algebraic Example over $\mathbb{R}$

## Example

$$A(a, b) = \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix} \quad \text{with } a \geq 0, b > 0, a \neq b$$

$$Z_1(a, b) = \begin{bmatrix} 0 & 0 & 12a \\ 1 & 0 & 18 \\ 0 & 1 & b \end{bmatrix} \quad \text{if } a > 0, b > 0, a \neq b$$

$$Z_2(a, b) = \left[ \begin{array}{c|cc} 0 & 1 & 0 \\ \hline & 0 & 1 \\ & 18 & b \end{array} \right] \quad \text{if } a = 0, b > 0$$

# Outline

1. Motivation
2. Regular Chains
3. Background
4. Rank
5. Zigzag Form
- 6. Frobenius Form**
7. Future Work



# What is a Frobenius Matrix

A matrix in **Frobenius Normal Form** has the following structure:

$$F = \begin{bmatrix} C_{f_1} & & & \\ & C_{f_2} & & \\ & & \ddots & \\ & & & C_{f_n} \end{bmatrix}$$

where  $f_n \mid f_{n-1} \mid \cdots \mid f_1$ .

# What is a Frobenius Matrix

A matrix in **Frobenius Normal Form** has the following structure:

$$F = \begin{bmatrix} C_{f_1} & & & \\ & C_{f_2} & & \\ & & \ddots & \\ & & & C_{f_n} \end{bmatrix}$$

where  $f_n \mid f_{n-1} \mid \cdots \mid f_1$ .

$f_1$  is the **minimal polynomial** of  $F$ .

# Theorem

## Theorem

For **every** matrix  $A(\alpha) \in \mathbb{K}^{n \times n}[\alpha]$ , there exists a partition  $(S_1, \dots, S_N)$  of the input constructible (resp. semi-algebraic) set  $S$  such that for each  $S_i$ , there exists a matrix  $Z_i(\alpha) F_i(\alpha) \sim A(\alpha)$  in **Zigzag Frobenius form** where the denominator of the entries of  $Z_i(\alpha) F_i(\alpha)$  are all nonzero everywhere on  $S_i$ .

# Gcd Algorithm (Complex)

- Input:**
- ▶ Two polynomials  $p(x; \alpha)$ ,  $q(x; \alpha)$  with parameters  $\alpha$
  - ▶ Constraints on the parameter values given in a set  $S$  by a polynomial system of the form

$$f_1(\alpha) = \cdots = f_a(\alpha) = 0, g(\alpha) \neq 0$$

- Output:**
- ▶ A list with elements of the form  $[g_i, S_i]$  where  $S_i$  gives conditions on  $\alpha$  such that the gcd of  $p$  and  $q$  is  $g_i$
  - ▶  $S_i$ 's form a partition of  $S$

$$\bigcup_i S_i = S, \quad S_i \cap S_j = \emptyset \quad \forall i \neq j$$

# Gcd Algorithm (Complex)

- Input:**
- ▶ Two polynomials  $p(x; \alpha)$ ,  $q(x; \alpha)$  with parameters  $\alpha$
  - ▶ Constraints on the parameter values given in a set  $S$  by a polynomial system of the form

$$f_1(\alpha) = \cdots = f_a(\alpha) = 0, g(\alpha) \neq 0$$

- Output:**
- ▶ A list with elements of the form  $[g_i, S_i]$  where  $S_i$  gives conditions on  $\alpha$  such that the gcd of  $p$  and  $q$  is  $g_i$
  - ▶  $S_i$ 's form a partition of  $S$

$$\bigcup_i S_i = S, \quad S_i \cap S_j = \emptyset \quad \forall i \neq j$$

# Gcd Algorithm (Complex)

**Input:**

- ▶  $p, q$
- ▶  $CS$
- ▶  $R$

**Algorithm:**  $src := \text{SubresultantChain}(p, q, x)$   
 $r := \text{resultant}(p, q, x)$   
 $DS := CS \cap V(r)$   
for each  $rs \in DS$

- $es := z(rs) \cap V(\text{init}(p)) \cap V(\text{init}(q))$
- Recursive call with  $\text{tail}(p), \text{tail}(q), es$
- $es := rs \setminus es$
- Proceed bottom up in  $src$  and get first subresultant who's initial does not vanish on  $es$ , it is the gcd

# Gcd Example

## Example

Consider polynomials  $p$  and  $q$  in  $x$

$$p(x; \alpha) = (x - 1)(x + \alpha)$$

$$q(x; \alpha) = \alpha x^2 + 2$$

$\alpha$  is a complex valued parameter.

# Gcd Example

## Example

Consider polynomials  $p$  and  $q$  in  $x$

$$p(x; \alpha) = (x - 1)(x + \alpha)$$

$$q(x; \alpha) = \alpha x^2 + 2$$

$\alpha$  is a complex valued parameter.

$$\gcd(p, q) = \begin{cases} (-\alpha^2 + \alpha)x + \alpha^2 + 2 & \text{if } \alpha^3 + 2 = 0 \\ x - 1 & \text{if } \alpha + 2 = 0 \\ 1 & \text{otherwise} \end{cases}$$



# Smith Form

A matrix in **Smith normal form** has the following structure:

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & f_1 & & \\ & & & & \ddots & \\ & & & & & f_n \end{bmatrix}$$

where  $f_1 \mid f_2 \mid \cdots \mid f_n$ .

# Smith and Frobenius Form

## Theorem

For every square matrix  $A$

$\Rightarrow$  **Smith form** of  $xI - A$  is  $\text{diag}(1, \dots, 1, f_1, \dots, f_n)$

# Smith and Frobenius Form

## Theorem

For every square matrix  $A$

⇒ **Smith form** of  $xI - A$  is  $\text{diag}(1, \dots, 1, f_1, \dots, f_n)$

⇒ **Frobenius form** of  $A$  is given by  $\text{diag}(C_{f_n}, \dots, C_{f_1})$

# Computing the Smith Form

- ▶ Algorithm relies on gcd computations

# Computing the Smith Form

- ▶ Algorithm relies on gcd computations
- ▶ Computing  $xI - A$  gives polynomials for Frobenius form

# Computing the Smith Form

- ▶ Algorithm relies on gcd computations
- ▶ Computing  $xI - A$  gives polynomials for Frobenius form
- ▶ Faster to compute  $xI - Z$  for  $Z$  in zigzag form

# Example

## Example

$$A(\alpha) = \begin{bmatrix} -1 & -\alpha - 1 & 0 \\ -1/2 & \alpha - 2 & 1/2 \\ -2 & 3\alpha + 1 & -1 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

---

# Example

## Example

$$A(\alpha) = \begin{bmatrix} -1 & -\alpha - 1 & 0 \\ -1/2 & \alpha - 2 & 1/2 \\ -2 & 3\alpha + 1 & -1 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

$$Z_1(\alpha) = \begin{bmatrix} 0 & 0 & 4\alpha \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix} \quad \text{if } \alpha + 3 \neq 0$$

$$Z_2(\alpha) = \left[ \begin{array}{cc|c} 0 & -4 & 1 \\ 1 & -4 & 0 \\ \hline & & -3 \end{array} \right] \quad \text{if } \alpha + 3 = 0$$



# Example

## Example

$$Z_1(\alpha) = \begin{bmatrix} 0 & 0 & 4\alpha \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix} \quad \text{if } \alpha + 3 \neq 0$$

# Example

## Example

$$Z_1(\alpha) = \begin{bmatrix} 0 & 0 & 4\alpha \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix} \quad \mathbf{if} \quad \alpha + 3 \neq 0$$

**Smith Form** of  $xI - Z_1(\alpha)$

$$S_1(\alpha) = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & x^3 - (\alpha - 4)x^2 - (4\alpha - 4)x - 4\alpha & \end{bmatrix}$$

# Example

## Example

$$Z_1(\alpha) = \begin{bmatrix} 0 & 0 & 4\alpha \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix} \quad \text{if } \alpha + 3 \neq 0$$

**Smith Form** of  $xI - Z_1(\alpha)$

$$S_1(\alpha) = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & x^3 - (\alpha - 4)x^2 - (4\alpha - 4)x - 4\alpha & \end{bmatrix}$$

**Frobenius Form**

$$F_1(\alpha) = \begin{bmatrix} 0 & 0 & 4\alpha \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix}$$

# Example

## Example

$$Z_2(\alpha) = \left[ \begin{array}{cc|c} 0 & -4 & 1 \\ 1 & -4 & 0 \\ \hline & & -3 \end{array} \right] \quad \text{if } \alpha + 3 = 0$$

# Example

## Example

$$Z_2(\alpha) = \left[ \begin{array}{cc|c} 0 & -4 & 1 \\ 1 & -4 & 0 \\ \hline & & -3 \end{array} \right] \quad \text{if } \alpha + 3 = 0$$

**Smith Form** of  $xI - Z_2(\alpha)$

$$S_2(\alpha) = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3 - (\alpha - 4)x^2 - (4\alpha - 4)x - 3\alpha + 3 \end{array} \right]$$

# Example

## Example

$$Z_2(\alpha) = \left[ \begin{array}{cc|c} 0 & -4 & 1 \\ 1 & -4 & 0 \\ \hline & & -3 \end{array} \right] \quad \text{if } \alpha + 3 = 0$$

**Smith Form** of  $xI - Z_2(\alpha)$

$$S_2(\alpha) = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3 - (\alpha - 4)x^2 - (4\alpha - 4)x - 3\alpha + 3 \end{array} \right]$$

**Frobenius Form**

$$F_2(\alpha) = \left[ \begin{array}{ccc} 0 & 0 & 3\alpha - 3 \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{array} \right]$$

# Example

## Example

$$A(\alpha) = \begin{bmatrix} -1 & -\alpha - 1 & 0 \\ -1/2 & \alpha - 2 & 1/2 \\ -2 & 3\alpha + 1 & -1 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

$$F_1(\alpha) = \begin{bmatrix} 0 & 0 & 4\alpha \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix} \quad \text{if } \alpha + 3 \neq 0$$

$$F_2(\alpha) = \begin{bmatrix} 0 & 0 & 3\alpha - 3 \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix} \quad \text{if } \alpha + 3 = 0$$

# Outline

1. Motivation
2. Regular Chains
3. Background
4. Rank
5. Zigzag Form
6. Frobenius Form
- 7. Future Work**



# Future Work

Research is currently being conducted to create a **MAPLE** package **ParametricMatrixTools** with methods to compute

- ▶ Rank ✓
- ▶ Frobenius (Rational) ✓
- ▶ Minimal polynomial ✓
- ▶ Test for similarity ✓
- ▶ Jordan form
- ▶ Weyr form<sup>5</sup>

<sup>5</sup>K. O'Meara, J.Clark, and C. Vinsonhaler. *Advanced Topics in Linear Algebra: Weaving Matrix Problems Through the Weyr Form*. Oxford University Press, 2011

# Future Work

Research is currently being conducted to create a **MAPLE** package **ParametricMatrixTools** with methods to compute

- ▶ Rank ✓
- ▶ Frobenius (Rational) ✓
- ▶ Minimal polynomial ✓
- ▶ Test for similarity ✓
- ▶ Jordan form
- ▶ Weyr form<sup>5</sup>

<sup>5</sup>K. O'Meara, J.Clark, and C. Vinsonhaler. *Advanced Topics in Linear Algebra: Weaving Matrix Problems Through the Weyr Form*. Oxford University Press, 2011

# Future Work

Research is currently being conducted to create a **MAPLE** package **ParametricMatrixTools** with methods to compute

- ▶ Rank ✓
- ▶ Frobenius (Rational) ✓
- ▶ Minimal polynomial ✓
- ▶ Test for similarity ✓
- ▶ Jordan form
- ▶ Weyr form<sup>5</sup>

<sup>5</sup>K. O'Meara, J.Clark, and C. Vinsonhaler. *Advanced Topics in Linear Algebra: Weaving Matrix Problems Through the Weyr Form*. Oxford University Press, 2011

# Future Work

Research is currently being conducted to create a **MAPLE** package **ParametricMatrixTools** with methods to compute

- ▶ Rank ✓
- ▶ Frobenius (Rational) ✓
- ▶ Minimal polynomial ✓
- ▶ Test for similarity ✓
- ▶ Jordan form
- ▶ Weyr form<sup>5</sup>

<sup>5</sup>K. O'Meara, J.Clark, and C. Vinsonhaler. *Advanced Topics in Linear Algebra: Weaving Matrix Problems Through the Weyr Form*. Oxford University Press, 2011

# Future Work

Research is currently being conducted to create a **MAPLE** package **ParametricMatrixTools** with methods to compute

- ▶ Rank ✓
- ▶ Frobenius (Rational) ✓
- ▶ Minimal polynomial ✓
- ▶ Test for similarity ✓
- ▶ Jordan form
- ▶ Weyr form<sup>5</sup>

<sup>5</sup>K. O'Meara, J.Clark, and C. Vinsonhaler. *Advanced Topics in Linear Algebra: Weaving Matrix Problems Through the Weyr Form*. Oxford University Press, 2011

# Future Work

Research is currently being conducted to create a **MAPLE** package **ParametricMatrixTools** with methods to compute

- ▶ Rank ✓
- ▶ Frobenius (Rational) ✓
- ▶ Minimal polynomial ✓
- ▶ Test for similarity ✓
- ▶ Jordan form
- ▶ Weyr form<sup>5</sup>

<sup>5</sup>K. O'Meara, J.Clark, and C. Vinsonhaler. *Advanced Topics in Linear Algebra: Weaving Matrix Problems Through the Weyr Form*. Oxford University Press, 2011

# Future Work

Research is currently being conducted to create a **MAPLE** package **ParametricMatrixTools** with methods to compute

- ▶ Rank ✓
- ▶ Frobenius (Rational) ✓
- ▶ Minimal polynomial ✓
- ▶ Test for similarity ✓
- ▶ Jordan form
- ▶ Weyr form<sup>5</sup>

<sup>5</sup>**K. O'Meara, J.Clark, and C. Vinsonhaler.** *Advanced Topics in Linear Algebra: Weaving Matrix Problems Through the Weyr Form.* Oxford University Press, 2011

- ▶ Speed improvements
  - ▶ Minimize unnecessary splitting
  - ▶ Parallel implementation
- ▶ Cost analysis
- ▶ Matrix factorizations



- ▶ Speed improvements
  - ▶ Minimize unnecessary splitting
  - ▶ Parallel implementation
- ▶ Cost analysis
- ▶ Matrix factorizations

- ▶ Speed improvements
  - ▶ Minimize unnecessary splitting
  - ▶ Parallel implementation
- ▶ Cost analysis
- ▶ Matrix factorizations

- ▶ Speed improvements
  - ▶ Minimize unnecessary splitting
  - ▶ Parallel implementation
- ▶ Cost analysis
- ▶ Matrix factorizations

- ▶ Speed improvements
  - ▶ Minimize unnecessary splitting
  - ▶ Parallel implementation
- ▶ Cost analysis
- ▶ Matrix factorizations

All current and future code and example worksheets will be available at [StevenThornton.ca/Code](http://StevenThornton.ca/Code)

Information on the RegularChains package of MAPLE is available at [RegularChains.org](http://RegularChains.org)

Thank You!