

A Maple Package for Parametric Matrix Computations

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STEVEN THORNTON

Western University



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Find the **rank** of A .

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$$\text{rank}(A) = 2$$

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Find the **rank** of $A(\alpha)$ for $\alpha \in \mathbb{R}$.

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$$\text{rank}(A(\alpha)) = \begin{cases} 2 & \text{for } \alpha = 9 \\ 3 & \text{for } \alpha \neq 9 \end{cases}$$

Welcome to Grad School

Find the **rank** of $A(\alpha, \beta, \gamma)$ for $\alpha, \beta, \gamma \in \mathbb{C}$.

$$A(\alpha, \beta, \gamma) = \begin{bmatrix} \alpha\beta - \gamma^2 + 1 & \alpha & \beta - 2 \\ \gamma^2 - \alpha & 0 & \alpha^2 + 1 \\ \alpha - 5 & \beta - \gamma & \gamma \end{bmatrix}$$

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$$\text{rank}(A(\alpha, \beta, \gamma)) = 3 \quad \text{for} \quad \det(A(\alpha, \beta, \gamma)) \neq 0$$

Notation

- Parameters $\alpha = \alpha_1, \dots, \alpha_N$
- \mathbb{K} is an algebraically closed or real closed field
- $\mathbb{K}[\alpha]$ is the ring of polynomials in the parameters

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Parametric Matrix

Definition

A $m \times n$ **parametric matrix** takes the form:

$$A(\alpha) = \begin{bmatrix} f_{1,1}(\alpha) & f_{1,2}(\alpha) & \cdots & f_{1,n}(\alpha) \\ f_{2,1}(\alpha) & f_{2,2}(\alpha) & \cdots & f_{2,n}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1}(\alpha) & f_{m,2}(\alpha) & \cdots & f_{m,n}(\alpha) \end{bmatrix}$$

for

$$f_{i,j} \in \mathbb{K}[\alpha] \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Parametric Matrix Computations in CAS

Computer algebra systems are currently **unable** to provide case discussions for parametric matrices.

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Sage*

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What Operations/Properties are Discontinuous

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- Option to return **transformation matrices** for appropriate methods

Some Difficulties

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- **Best case:** Algorithms that are optimal in the constant case that lead to minimal splitting

Questions?