

Overview

Linear algebra is fundamental to applications across nearly all disciplines within science and engineering. Despite its deceiving simplicity, the subject area extends well beyond what is presented in introductory courses. In fact, further development of industry applicable algorithms and software implementation are still necessary today. One such development is computing with parametric linear systems, where extensive case discussion is required, often leading to non-linear constraints. Few software packages properly address this type of computation and those that do often fail to provide complete case discussion. Since parametric linear systems arise in many applications (control theory, study of dynamical systems etc.) more research is needed to develop software capable of providing comprehensive case discussion.

An overview of the [ParametricMatrixTools](#) package implemented in the MAPLE computer algebra system is presented. When complete, this package will be capable of providing complete case discussion for many common operations on parametric matrices.

Background

Let \mathbb{K} be an [algebraically closed field](#) (e.g. complex numbers) or a [real closed field](#) (e.g. real numbers) and $\mathbb{K}[\alpha]$ the [ring of polynomials](#) in the variables $\alpha = \alpha_1, \dots, \alpha_m$ with coefficients in \mathbb{K} . We denote by $\mathbb{K}(\alpha)$ the [quotient field](#) of $\mathbb{K}[\alpha]$, that is, the field of multivariate rational functions in α with coefficients in \mathbb{K} .

A $m \times n$ [parametric matrix](#) takes the form:

$$A(\alpha) = \begin{bmatrix} f_{1,1}(\alpha) & f_{1,2}(\alpha) & \cdots & f_{1,n}(\alpha) \\ f_{2,1}(\alpha) & f_{2,2}(\alpha) & \cdots & f_{2,n}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1}(\alpha) & f_{m,2}(\alpha) & \cdots & f_{m,n}(\alpha) \end{bmatrix}$$

for $f_{i,j}(\alpha) \in \mathbb{K}(\alpha)$ with $1 \leq i \leq m$, $1 \leq j \leq n$. That is, $f_{i,j}(\alpha) = p_{i,j}(\alpha)/q_{i,j}(\alpha)$ where both $p_{i,j}(\alpha)$ and $q_{i,j}(\alpha)$ are polynomials in α .

A constructible (resp. semi-algebraic) set $S \subseteq \mathbb{K}^s$ is required such that the denominators $q_{i,j}(\alpha)$ of every entry of $A(\alpha)$ are nonzero everywhere on S .

Implementation

The theory of regular chains and its software implementation (the [RegularChains](#) library in MAPLE, see [RegularChains.org](#)) is extensively used in the implementation of this package. In particular the [ConstructibleSetTools](#) and [SemiAlgebraicSetTools](#) sub-packages are used for manipulating constructible sets and semi-algebraic sets respectively.

The implementation code together with a variety of test examples of various levels of difficulty can be downloaded from [StevenThornton.ca/Code/](#). In particular, some examples are taken from the literature on control system design and dynamical systems where parametric matrices arise.

Package Contents

The [ParametricMatrixTools](#) package will be available as a sub-package of the [RegularChains](#) library in MAPLE. Some of the methods we plan to include in this package are:

- Rank
- Linear system solving
- Row echelon forms
- Frobenius (Rational) canonical form
- Minimal polynomial
- Test for matrix similarity
- Jordan canonical form
- Weyr canonical form (see [2])

All implemented methods will have versions able to compute over both the real numbers subject to semi-algebraic constraints and over the complex numbers.

Example

Taking an example from [1] on [control theory](#), we look for the conditions on the parameters such that a block matrix derived from the system is **full rank** giving a **controllable system**.

Consider $E\ddot{x} = A_1\dot{x} + A_2x + Bu$ with

$$E = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \lambda & 3\lambda & \lambda \\ 3\lambda + \mu & \lambda + \mu & \lambda + 3\mu \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda, \mu \in \mathbb{C}$$

What is the **rank** of

$$C = \begin{bmatrix} -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\ -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\ A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & A_2 & -A_1 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & B \end{bmatrix} ?$$

$$\text{rank}(C) = \begin{cases} 18 & \text{if } \lambda \neq 0, \mu \neq 1/2 \\ 17 & \text{if } \lambda \neq 0, \mu = 1/2 \\ 16 & \text{if } \lambda = 0, \mu \neq -1 \\ 15 & \text{if } \lambda = 0, \mu = -1 \end{cases}$$

A second example taken from [3], we have a matrix depending on 6 complex parameters, z_{ij} for $i = 1, 2$, $j = 1, 2, 3$

$$X = \begin{bmatrix} -4z_{11} - 4z_{12} & -4z_{12} - 4z_{13} & 20z_{13} + 24z_{11} + 44z_{12} \\ -7z_{11} - 6z_{12} + z_{13} & -18z_{12} - 12z_{13} - 6z_{11} & 54z_{13} + 72z_{11} + 126z_{12} \\ -z_{21} + z_{23} & -12z_{22} - 6z_{21} - 6z_{23} & 24z_{23} + 60z_{22} + 36z_{21} \end{bmatrix}$$

Since $\det(X) \equiv 0$ we immediately know $\text{rank}(X) < 3$. Using the implemented rank algorithm we can verify that no cases exist where the rank is 3. The **row echelon** method is able to compute all row echelon forms of this matrix along with the corresponding conditions on the parameters from which solutions can be extracted. Sample cases include:

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{if} \quad \begin{cases} z_{22} + z_{23} = 0 \\ z_{12} + z_{13} \neq 0 \\ z_{21} - z_{23} \neq 0 \end{cases}$$

$$\begin{bmatrix} 1 & 6 & -36 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{if} \quad \begin{cases} z_{11} - z_{13} = 0 \\ z_{12} + z_{13} = 0 \\ z_{22} + z_{23} = 0 \\ z_{21} - z_{23} \neq 0 \end{cases}$$

References:

- [1] M Isabel García-Planas and J Clotet. Analyzing the set of uncontrollable second order generalized linear systems. *International Journal of Applied Mathematics and Informatics*, 1(2):76–83, 2007.
- [2] Kevin O'Meara, John Clark, and Charles Vinsonhaler. *Advanced topics in linear algebra: weaving matrix problems through the Weyr form*. Oxford University Press, 2011.
- [3] Bin Zhou and Guang-Ren Duan. An explicit solution to the matrix equation $AX - XF = BY$. *Linear Algebra and its applications*, 402:345–366, 2005.