

Jordan Canonical Form with Parameters from Frobenius Form with Parameters

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Motivation

Computer algebra systems generally give **generic answers** for computations on matrices with parameters.

Example

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

Maple

$$\begin{bmatrix} -2 & & \\ & \alpha & \\ & & \alpha - 3 \end{bmatrix}$$

Mathematica

$$\begin{bmatrix} -2 & & \\ & -3 + \alpha & \\ & & \alpha \end{bmatrix}$$

SymPy

$$\begin{bmatrix} -2 & & \\ & \alpha & \\ & & \alpha - 3 \end{bmatrix}$$

Sage

$$\begin{bmatrix} \alpha - 3 & & \\ & \alpha & \\ & & -2 \end{bmatrix}_2$$

Example

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\alpha = -2} \begin{bmatrix} -2 & 0 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Maple

$$\left[\begin{array}{c|cc} -5 & & \\ \hline & -2 & 1 \\ & & -2 \end{array} \right]$$

Mathematica

$$\left[\begin{array}{c|cc} -5 & & \\ \hline & -2 & 1 \\ & & -2 \end{array} \right]$$

SymPy

$$\left[\begin{array}{c|cc} -5 & & \\ \hline & -2 & 1 \\ & & -2 \end{array} \right]$$

Sage

$$\left[\begin{array}{c|cc} -5 & & \\ \hline & -2 & 1 \\ & & -2 \end{array} \right]$$

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Regular Chain Theory

- Let \mathbb{K} be a field,
- Let $X_1 < \dots < X_s$ be $s \geq 1$ ordered variables,
- $\mathbb{K}[X_1, \dots, X_s] = \mathbb{K}[X]$ is the ring of polynomials in $X = X_1, \dots, X_s$
- For a non-constant $p \in \mathbb{K}[X]$, **mvar**(p) denotes the greatest variable in p
- The leading coefficient of p w.r.t. **mvar**(p) is called the **initial** denoted **init**(p)

Triangular Set

A set T of non-constant polynomials in $\mathbb{K}[X]$ is called a **triangular set** if for all $p, q \in T$ with $p \neq q$, $\text{mvar}(p) \neq \text{mvar}(q)$.

Saturated Ideal

The **saturated ideal**, $\text{sat}(T)$ of T is the ideal $\langle T \rangle : h_T^\infty$, where h_T is the product of the initials of the polynomials in T

Regular Chain

T is a **regular chain** if $T = \emptyset$ or $T = T' \cup \{t\}$ for $t \in T$ with $\text{mvar}(t)$ maximum such that

- T' is a regular chain
- $\text{init}(t)$ is regular modulo $\text{sat}(T')$

Solution Set

For a set of polynomials $F \subset \mathbb{K}[X_1, \dots, X_s]$, $V(F) \subseteq \overline{\mathbb{K}}^s$ consists of all common roots of the polynomials in F

Quasi-component

The **quasi-component** of a triangular set is defined as $W(T) = V(T) \setminus V(h_T)$ where h_T is the product of the initials of T .

Regular System

Given a regular chain $T \subset \mathbb{K}[X]$ and a set of polynomials $H \subset \mathbb{K}[X]$, the pair $[T, H]$ is a **regular system** if each polynomial $h \in H$ is regular modulo $\text{sat}(T)$.

The **zero set** of $[T, H]$ is denoted $Z(T, H) = W(T) \setminus V(H)$.

Square-free Factorization over a Regular System

The function¹ **Squarefree_RC**(p, T, H) computes a set of triples $((b_{i,1}, \dots, b_{i,\ell_i}), T_i, H_i)$ with $1 \leq i \leq e$ such that $[T_1, H_1], \dots, [T_e, H_e]$ are regular systems forming a triangular decomposition of $Z(T, H)$. For all $1 \leq i \leq e$:

1. $b_{i,1}, \dots, b_{i,\ell_i}$ are polynomials with the same main variable $v = \mathbf{mvar}(p)$ such that $p \equiv \prod_{j=1}^{\ell_i} b_{i,j}^j \pmod{\mathbf{sat}(T_i)}$,
2. All discriminants $\mathbf{disc}(b_{i,j}, v)$ are regular modulo $\mathbf{sat}(T_i)$
3. All resultants $\mathbf{res}(b_{i,j}, b_{i,k}, v)$ are regular modulo $\mathbf{sat}(T_i)$
4. $\prod_{j=1}^{\ell_i} b_{i,j}^j$ is a square-free factorization of p modulo $\mathbf{sat}(T_i)$

¹Chen, C., Moreno Maza, M.: **Algorithms for computing triangular decomposition of polynomial systems**. *Journal of Symbolic Computation*, 47(6) (2012) 610–642

Linear Algebra Theory

Companion Matrix

Let $\psi(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$.

The Frobenius companion matrix of $\psi(x)$ is

$$C(\psi(x)) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

A matrix $F \in \mathbb{K}^{n \times n}$ is in **Frobenius form** if

$$F = \begin{bmatrix} C(\psi_1(x)) & & & \\ & C(\psi_2(x)) & & \\ & & \ddots & \\ & & & C(\psi_m(x)) \end{bmatrix}$$

and $\psi_{i+1} | \psi_i$ for $i = 1, \dots, m - 1$.

Properties of Frobenius Form

- The polynomials ψ_i are called the **invariant factors** of F .
- ψ_1 is the **minimal polynomial** of F
- $\prod_{i=1}^m \psi_i$ is the **characteristic polynomial** of F
- There exists a non-singular matrix Q such that $F = Q^{-1}AQ$ is in **Frobenius form**
- The **geometric multiplicity** of an eigenvalue λ of F is the number of invariant factors it is a solution of.

- Ozello, P. (1987). **Calcul exact des formes de Jordan et de Frobenius d'une matrice** (Doctoral dissertation, Université Joseph-Fourier-Grenoble I).
- Storjohann, A. **An $\mathcal{O}(n^3)$ algorithm for the Frobenius normal form.** In Proceedings of ISSAC 1998, ACM (1998) 101–105.

Jordan Block Matrix

A **Jordan block** is a matrix of the form

$$\mathbf{JBM}_m(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$$

- $\mathbf{JBM}_m(\lambda)$ has a single eigenvalue λ
- λ has **geometric multiplicity** 1

A matrix in **Jordan canonical form** is of the form

$$\begin{bmatrix} \mathbf{JBM}_{m_1}(\lambda_1) & & \\ & \ddots & \\ & & \mathbf{JBM}_{m_\ell}(\lambda_\ell) \end{bmatrix}$$

Frobenius Form to Jordan Form

$$F = \begin{bmatrix} C(\psi_1(x)) & & & \\ & C(\psi_2(x)) & & \\ & & \ddots & \\ & & & C(\psi_m(x)) \end{bmatrix}$$

The **Jordan canonical form** of F is

$$\begin{bmatrix} \text{JCF}(C(\psi_1(x))) & & & \\ & \text{JCF}(C(\psi_2(x))) & & \\ & & \ddots & \\ & & & \text{JCF}(C(\psi_m(x))) \end{bmatrix}$$

Jordan Form of a Companion Matrix

The JCF of a companion matrix $C(\psi(x))$ is

$$\text{JCF}(C(\psi(x))) = \begin{bmatrix} \text{JBM}_{m_1}(\lambda_1) & & & \\ & \text{JBM}_{m_2}(\lambda_2) & & \\ & & \ddots & \\ & & & \text{JBM}_{m_\ell}(\lambda_\ell) \end{bmatrix}$$

where

$$\psi(x) = \prod_{i=1}^{\ell} (x - \lambda_i)^{m_i}$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$.

JCF Over a Splitting Field

Example

Consider the companion matrix

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Maple computes the JCF as

$$\text{JCF}(A) = \begin{bmatrix} -1/3 \sqrt[3]{26 + 6\sqrt{33}} + 8/3 \frac{1}{\sqrt[3]{26 + 6\sqrt{33}}} + 1/3 & 0 & 0 \\ 0 & 1/6 \sqrt[3]{26 + 6\sqrt{33}} - 4/3 \frac{1}{\sqrt[3]{26 + 6\sqrt{33}}} + 1/3 - i/2\sqrt{3} \left(-1/3 \sqrt[3]{26 + 6\sqrt{33}} - 8/3 \frac{1}{\sqrt[3]{26 + 6\sqrt{33}}} \right) & 0 \\ 0 & 0 & 1/6 \sqrt[3]{26 + 6\sqrt{33}} - 4/3 \frac{1}{\sqrt[3]{26 + 6\sqrt{33}}} + 1/3 + i/2\sqrt{3} \left(-1/3 \sqrt[3]{26 + 6\sqrt{33}} - 8/3 \frac{1}{\sqrt[3]{26 + 6\sqrt{33}}} \right) \end{bmatrix}$$

Example

$$\text{JCF}(A) = \begin{bmatrix} 1 - y_2 - y_1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & y_2 \end{bmatrix}$$

where (y_1, y_2) is any point in the zero set $V(T)$ of

$$T = \begin{cases} y_1^2 + (y_2 - 1)y_1 + y_2^2 - y_2 + 3 \\ y_2^3 - y_2^2 + 3y_2 + 1 \end{cases}$$

Given a monic polynomial $\psi(x) \in \mathbb{K}[x]$, and its companion matrix $C(\psi(x)) \in \mathbb{K}^{n \times n}$, find a regular chain $T \subset \mathbb{K}[y_1, \dots, y_s]$ such that the JCF of $C(\psi(x))$ can be computed over the splitting field² generated by T . i.e. any point $p \in V(T)$ defines the splitting field of T : $\mathbb{L} = \mathbb{K}(p)$.

²Landau, S. (1985). Factoring polynomials over algebraic number fields. *SIAM Journal on Computing*, 14(1), 184-195.

Input: A square-free polynomial $p \in \mathbb{K}[x]$

Algorithm:

1. Initialize $i := 1$, $y_i := x$, $\mathbb{L} := \mathbb{K}$, $T := \{ \}$, $\mathcal{P} = \{ \}$ and $\mathcal{F} := \{p\}$.
2. While \mathcal{F} is not empty do
 - a. Pick a polynomial $f(y_i) \in \mathcal{F}$ over \mathbb{L} ,
 - b. Let α_i be a root of $f(y_i)$
 - c. Replace \mathbb{L} by $\mathbb{L}(\alpha_i)$
 - d. Replace T by $T \cup \{t_i(y_1, \dots, y_i)\}$ where $t_i(y_1, \dots, y_i)$ is obtained from $f(y_i)$ after replacing the algebraic numbers $\alpha_1, \dots, \alpha_{i-1}$ with the variables y_1, \dots, y_{i-1}
 - e. Replace \mathcal{P} by $\mathcal{P} \cup \{x - y_i\}$

- f. Factor $f(y_i)$ into irreducible factors over \mathbb{L} (Trager, Algebraic factoring and rational function integration)
- Add the degree 1 factors to \mathcal{P} after replacing $\alpha_1, \dots, \alpha_{i-1}$ with y_1, \dots, y_{i-1}
 - Add the degree greater than 1 factors to \mathcal{F} , replace y_i with y_{i+1}
- g. If \mathcal{F} is not empty, $i := i + 1$

After this algorithm has terminated, T is a regular chain in $\mathbb{K}[y_1, \dots, y_s]$ such that $\mathbb{K}[y_1, \dots, y_s] \setminus \langle T \rangle$ is isomorphic to the splitting field $\mathbb{K}(p)$.

JCF of a Matrix with Parameters

Frobenius Form

- Storjohann, A. **An $\mathcal{O}(n^3)$ algorithm for the Frobenius normal form.** In Proceedings of ISSAC 1998, ACM (1998) 101–105.
- Corless, R. M., Maza, M. M., & Thornton, S. E. (2015). **Zigzag Form over Families of Parametric Matrices.** ACM Communications in Computer Algebra, 48(3/4), 109-112.

Smith Form

The **invariant factors** of A are the non-constant polynomials on the diagonal of the **Smith form** of $xI - A$.

- Kaltofen, E., Krishnamoorthy, M., Saunders, B.D.: **Fast parallel algorithms for similarity of matrices.** In: Proceedings of the fifth ACM Symposium on Symbolic and Algebraic Computation, ACM (1986) 65–70

Objective

Given a monic univariate parametric polynomial $\psi(x; \alpha_1, \dots, \alpha_s) \in \mathbb{K}[\alpha_1, \dots, \alpha_s][x]$, and its companion matrix $C(\psi) \in \mathbb{K}^{n \times n}[\alpha_1, \dots, \alpha_s]$, find a partition (S_1, \dots, S_M) of $\overline{\mathbb{K}}^s$ and a set of matrices $\{J_1, \dots, J_M\}$, such that J_i is continuously the JCF of $C(\psi(x; \alpha^*))$ for all points $\alpha^* \in S_i$.

Let

$$p(x; \alpha) = \prod_{i=1}^{\ell} b_i(x; \alpha)^i$$

be a square-free factorization of $p(x; \alpha)$.

Specializes Well

b_1, \dots, b_ℓ specializes well at a point $\alpha^* \in \overline{\mathbb{K}}^S$ if

1. The degree of $b_i(x; \alpha^*)$ is the same as the degree of $b_i(x; \alpha)$ in x ,
2. $b_i(x; \alpha^*)$ is square-free,
3. The GCD of $b_i(x; \alpha^*)$ and $b_j(x; \alpha^*)$ has degree zero for all $i \neq j$.

b_1, \dots, b_ℓ specializes well at all points in the **complement** of

$$\text{Proviso}(b_1, \dots, b_\ell) = \left\{ \bigcup_{i=1}^{i=\ell} V(\Delta_i) \right\} \cup \left\{ \bigcup_{1 \leq i < j \leq \ell} V(R_{i,j}) \right\}$$

where

- Δ_i is the discriminant of b_i in x
- $R_{i,j}$ is the resultant of b_i and b_j w.r.t. x

- Let $\lambda_{i,1}, \dots, \lambda_{i,n_i}$ be the roots (in x) of b_i ($1 \leq i \leq \ell$) as functions of α ,
- The JCF of $C(\psi(x))$ in the complement of **Proviso**(b_1, \dots, b_ℓ) is

$$\bigoplus_{i=1}^{\ell} \bigoplus_{j=1}^{n_i} \mathbf{JBM}_i(\lambda_{i,j})$$

- Let $[T_1, H_1], \dots, [T_e, H_e]$ be a triangular decomposition of **Proviso** (b_1, \dots, b_ℓ) into regular systems,
- For each regular system $[T_i, H_i]$ ($1 \leq i \leq e$):
 - Let $((b_{j,1}, \dots, b_{j,\ell_j}), T_{i,j}, H_{i,j})$ be the output of **Squarefree_RC** $(\psi(x; \alpha), T_i, H_i)$,
 - For each j :
 - Let $\lambda_{1,k}, \dots, \lambda_{n,k}$ ($1 \leq k \leq \ell_j$) be the roots (in x) of $b_{j,1}, \dots, b_{j,\ell_j}$ as functions of α ,
 - The JCF of $C(\psi(x))$ for all points $\alpha^* \in Z(T_{i,j}, H_{i,j})$ is

$$\bigoplus_{k=1}^{\ell_j} \bigoplus_{i=1}^n \text{JBM}_k(\lambda_{i,k})$$

Example

Consider the Jacobian matrix of a dynamical system³ at equilibrium

$$A = \begin{bmatrix} 0 & 2\rho & 0 \\ a & 2\beta & 2v \\ b & -2v & 2\beta \end{bmatrix}$$

The JCF of A can take each of the 5 possible Jordan structures for a 3×3 matrix. We find 46 cases. For example, the JCF of A is

$$\begin{bmatrix} 2\beta & 0 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & \beta \end{bmatrix} \quad \text{with} \quad \begin{cases} 2\rho a + \beta^2 = 0 \\ v = 0 \\ a \neq 0, \rho \neq 0, \beta \neq 0 \end{cases}$$

³Van Gils, S., Krupa, M., Langford, W.F.: Hopf bifurcation with non-semisimple 1:1 resonance. *Nonlinearity* 3(3) (1990) 825

Future Work

Improvements to the JCF Computation

- Improved Frobenius form/Smith form implementation
- Real valued parameters
- Similarity transformation matrix
- Complexity estimate

A Maple Package for Parametric Matrix Computations

The `ParametricMatrixTools` package contains methods for computing:

- Rank
- Smith form
- Frobenius form
- Matrix similarity
- Minimal polynomial
- Jordan form
- Weyr form⁴

⁴K. O'Meara, J. Clark, and C. I. Vinsonhaler. *Advanced Topics in Linear Algebra: Weaving Matrix Problems through the Weyr Form*. Oxford, USA, 2011.

The `ParametricMatrixTools` package is available at
github.com/steventhornton/ParametricMatrixTools

Information on the `RegularChains` package of Maple is available at
RegularChains.org