

Overview

Given an $m \times n$ parametric matrix $A(\alpha)$ and a polynomial system S of constraints on the parameters α , an algorithm is presented for computing the rank of $A(\alpha)$ subject to the conditions given by S . This algorithm relies on the theory of regular chains. An implementation is discussed for the MAPLE computer algebra system implemented on top of the **RegularChains** library. We give an example of the type of computations this implementation is capable of. An overview of other operations on parametric matrices to be included in a future MAPLE package is mentioned.

Introduction

Determining the rank of a matrix is a simple computation traditionally presented in introductory linear algebra courses. The computation becomes tedious when the matrix contains parameters as the rank can change discontinuously requiring comprehensive case discussion. Currently, no computer algebra system is capable of providing automatic case discussion for computations on parametric matrices. We present an algorithm for computing the rank of matrices whose entries are multivariate polynomials whose indeterminates are regarded as parameters and are subject to a system of polynomial equations and inequalities. This algorithm relies on the theory of regular chains and is implemented on top of the **RegularChains** library in MAPLE.

The algorithm presented avoids the usage of general tools for solving parametric polynomial systems, such as *comprehensive Gröbner bases*, *comprehensive triangular decomposition*, or *dynamic evaluation*. We rely on the non-comprehensive triangular decomposition algorithms presented in [2] and [1] for the complex and real cases, respectively in order to control the growth of the number of cases. The notion of a *regular chain*, introduced independently in [4] and [5], is closely related to that of a triangular decomposition of a polynomial system.

Parametric Matrix

Let \mathbb{K} be an algebraically closed or real closed field. $\mathbb{K}[\alpha]$ is the ring of multivariate polynomials in the parameters $\alpha = \alpha_1, \dots, \alpha_k$.

An $m \times n$ **parametric matrix** takes the form:

$$A(\alpha) = \begin{bmatrix} f_{1,1}(\alpha) & f_{1,2}(\alpha) & \cdots & f_{1,n}(\alpha) \\ f_{2,1}(\alpha) & f_{2,2}(\alpha) & \cdots & f_{2,n}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1}(\alpha) & f_{m,2}(\alpha) & \cdots & f_{m,n}(\alpha) \end{bmatrix}$$

for $f_{i,j} \in \mathbb{K}[\alpha]$ $1 \leq i \leq m$, $1 \leq j \leq n$.

Regular Chains

A set \mathcal{T} of non-constant polynomials in $\mathbb{K}[x_1 < \cdots < x_n]$ is a **triangular set** if for all $p, q \in \mathcal{T}$, the leading variables of p and q are different.

The **saturated ideal** $\text{sat}(\mathcal{T})$ of \mathcal{T} is the ideal $\langle \mathcal{T} \rangle : h_{\mathcal{T}}^\infty$, where $h_{\mathcal{T}}$ is the product of the initials of the polynomials in \mathcal{T} .

\mathcal{T} is a **regular chain** if $\mathcal{T} = \emptyset$ or $\mathcal{T} = \mathcal{T}' \cup \{t\}$ for $t \in \mathcal{T}$ with $\text{mvar}(t)$ maximum such that

- \mathcal{T}' is a regular chain
- $\text{init}(t)$ is regular modulo $\text{sat}(\mathcal{T}')$

A **constructible set** is any finite union of zero sets of systems (i.e. conjunctions) of polynomial constraints (equations and inequations).

Algorithm

Input: An $m \times n$ parametric $A(\alpha)$ for $\alpha = a_1, \dots, a_k$, an algebraically closed field \mathbb{K} and a set $S \in \mathbb{K}^k$ of polynomial constraints on the parameters.

Output: Disjoint cells C_0, \dots, C_m such that for all $0 \leq r \leq m$ and $\alpha \in C_r$, the rank of the specialized matrix $A(\alpha)$ is r .

Step 1: Let $S' \in \mathbb{K}[\alpha_1 < \cdots < \alpha_k < x_1 < \cdots < x_n]$ be the system obtained by adding S to the equations of $A(\alpha)X = 0$ where X is a column vector of unknowns x_1, \dots, x_n

Step 2: Let \mathcal{T} be a triangular decomposition representing the zero set of S'

Step 3: For $0 \leq r \leq m$, let C_r be the constructible set of \mathbb{K}^{k+n} given by all regular systems $[T_j, h_j] \in \mathcal{T}$ where the number of polynomials of T_j of positive degree in (at least) one of x_1, \dots, x_n is exactly r

Step 4: For $r := m$ down to 1 do

$$C_r := C_r \setminus \{C_{r-1} \cup \cdots \cup C_0\}$$

Step 5: Return $C_0 \cap \mathbb{K}[\alpha_1, \dots, \alpha_k], \dots, C_m \cap \mathbb{K}[\alpha_1, \dots, \alpha_k]$.

Implementation

The above algorithm has been implemented in MAPLE on top of the **RegularChains** library. This implementation is capable of computations for both the case where \mathbb{K} is algebraically closed, as is given by the algorithm above, and when \mathbb{K} is a real closed field. The algorithm for the real case is analogous to the given algorithm.

Future

Many operations and properties of parametric matrices are discontinuous. We are developing a MAPLE package to address such computations including:

- Rank
- Singularity
- Hermite form
- Smith form
- Frobenius form
- Matrix similarity
- Minimal polynomial
- Jordan form
- Eigenvalue multiplicity
- Eigenvectors

Example

From control theory [3], consider the differential system $E\ddot{x} = A_1\dot{x} + A_2x + Bu$.

$$E = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \lambda & 3\lambda & \lambda \\ 3\lambda + \mu & \lambda + \mu & \lambda + 3\mu \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda, \mu \in \mathbb{C}$$

We are interested in finding the conditions on μ and λ such that the following 18×18 matrix is full rank and thus the system is controllable.

$$C = \begin{bmatrix} -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\ -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\ A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & A_2 & -A_1 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & B \end{bmatrix}$$

$$\text{Rank}(C) = \begin{cases} 18 & \text{if } \lambda \neq 0, \mu \neq 1/2 \\ 17 & \text{if } \lambda \neq 0, \mu = 1/2 \\ 16 & \text{if } \lambda = 0, \mu \neq -1 \\ 15 & \text{if } \lambda = 0, \mu = -1 \end{cases}$$

The conditions on μ were overlooked in [3].

References

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