The Weyr Canonical Form

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Introduction

The Jordan canonical form is a well known and highly studied matrix canonical form. This poster discusses its younger, less popular cousin, the Weyr canonical form. Originally discovered in 1885 by Czech mathematician Eduard Weyr [3], the Weyr form has had difficulty gaining the attention of mathematicians. Some applications that are unique to the Weyr form include: "commutativity problems, approximate simultaneous diagonalization, and algebraic geometry" [1].

This poster focuses on the computation of the Weyr canonical form though first computing the Jordan canonical form. An algorithm can be found in Chapter 2.5 of [1] for the direct computation of the Weyr form. When we implemented this algorithm it was found to be too expensive to be of practical use. We discuss a simple alternative below.

Weyr Matrix

Basic Weyr Matrix

An $n \times n$ matrix W is a **basic Weyr matrix** with eigenvalue λ if, for integers n_i, $1 \le i \le r$, where $n_1 + n_2 + \cdots + n_r = n$ and $n_1 \ge n_2 \ge \cdots \ge n_r \ge 1$, the following properties hold.

- It can be viewed as a **block upper bidiagonal matrix** with r diagonal blocks ($r-1$ super-diagonal block).
- Each of the **diagonal blocks** are of the form λI where I is an $n_i \times n_i$ identity matrix.
- The **super-diagonal blocks** $W_{i,i+1}$ are of the form $[I \ \mathbf{0}]^T$ where I is an $n_{i+1} \times n_{i+1}$ identity matrix and $\mathbf{0}$ is a zero matrix of size $(n_i - n_{i+1}) \times n_{i+1}$.
- All other blocks are zero.

Properties of a Basic Weyr Matrix

- The eigenvalue λ has **algebraic multiplicity** n and **geometric multiplicity** n_1.
- The **Weyr structure** (also known as the Weyr characteristic [2]) of a basic Weyr matrix is the sequence $(n_1, n_2, \ldots, n_r)$.

Examples

Weyr structure $(3,2,2)$

$$\left[\begin{array}{ccc|cc|cc} \lambda & 0 & 0 & 1 & 0 & & \\ 0 & \lambda & 0 & 0 & 1 & & \\ 0 & 0 & \lambda & 0 & 0 & & \\ \hline & & & \lambda & 0 & 1 & 0 \\ & & & 0 & \lambda & 0 & 1 \\ \hline & & & & & \lambda & 0 \\ & & & & & 0 & \lambda \end{array}\right]$$

Weyr structure $(3,3,1)$

$$\left[\begin{array}{ccc|ccc|c} \lambda & 0 & 0 & 1 & 0 & 0 & \\ 0 & \lambda & 0 & 0 & 1 & 0 & \\ 0 & 0 & \lambda & 0 & 0 & 1 & \\ \hline & & & \lambda & 0 & 0 & 1 \\ & & & 0 & \lambda & 0 & 0 \\ & & & 0 & 0 & \lambda & 0 \\ \hline & & & & & & \lambda \end{array}\right]$$

Weyr Matrix

A matrix W with eigenvalues $\lambda_1, \ldots, \lambda_k$, $\lambda_i \ne \lambda_j$ for $i \ne j$ is in **Weyr form** (and is a **Weyr matrix**) if it is the direct sum of basic Weyr matrices, where W_i has eigenvalue λ_i. It has the form:

$$W = \begin{bmatrix} W_1 & & & \\ & W_2 & & \\ & & \ddots & \\ & & & W_k \end{bmatrix} \tag{1}$$

Weyr Canonical Form

Theorem 1 *Every square matrix A over an algebraically closed field F is similar to a matrix in Weyr form (see [1] for a proof).*

Corollary 1 *For every square matrix A over an algebraically closed field F, there exists an invertible matrix Q such that $W = QAQ^{-1}$ gives a Weyr matrix, known as the **Weyr canonical form** of A.*

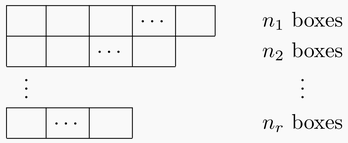
Corollary 2 *The Weyr canonical form of a matrix is unique up to permutations of the basic Weyr matrices. That is, there is no required ordering for the basic Weyr matrices W_i in equation (1).*

Duality between the Jordan and Weyr Forms

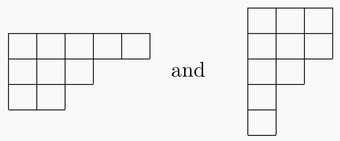
Let a **basic Jordan matrix** be the direct sum of Jordan block matrices where each block has the same eigenvalue and the blocks are arranged in order of decreasing size along the diagonal.

The **Jordan structure** of a basic Jordan matrix is the sequence of the sizes of the Jordan blocks along the diagonal.

Each partition $(n_1, n_2, \ldots, n_r)$ of n determines a *Young tableau* (or *Ferrer's diagram*):



"Transposing" the tableau gives a Young tableau that corresponds to the **dual** partition $(m_1, m_2, \ldots, m_s)$ of $(n_1, n_2, \ldots, n_r)$. In terms of the original tableau, m_1 is the number of boxes in the first column ($= r$), m_2 is the number in the second column, and so on. For example,



are the tableaux corresponding to the dual partitions $(5,3,2)$ and $(3,3,2,1,1)$.

Theorem 2 *The Weyr and Jordan structures of an $n \times n$ matrix with a single eigenvalue are dual partitions of n.*

The General Case

Let J be a matrix in Jordan form. There exists a matrix J^* similar to J, also in Jordan form, where J^* is the direct sum of basic Jordan matrices J_i^*. Each J_i^* has one eigenvalue λ_i where $\lambda_i \ne \lambda_j$ for $i \ne j$. Let P_i be the Jordan structure of J^* and let Q_i be the dual of P_i. We can define W_i to be the basic Weyr matrix with eigenvalue λ_i and Weyr structure Q_i. The matrix $W = \operatorname{diag}(W_1, W_2, \ldots)$ is the Weyr form of J.

Example

Both J and J^* are in Jordan form with J similar to J^*. J^* is the direct sum of two basic Jordan matrices. Computing the Weyr form of J is equivalent to computing the Weyr form of J^*. This can be done by computing the basic Weyr matrices that are similar to each basic Jordan matix in J^*. The Weyr form of J and J^* is W.

$$J = \left[\begin{array}{c|cc|cc|c|c} 5 & & & & & & \\ \hline & 2 & 1 & & & & \\ & & 2 & & & & \\ \hline & & & 5 & 1 & & \\ & & & & 5 & & \\ \hline & & & & & 2 & \\ \hline & & & & & & 5 \end{array}\right] \quad J^* = \left[\begin{array}{cc|cc|cc|c} 5 & 1 & & & & & \\ & 5 & & & & & \\ \hline & & 5 & & & & \\ & & & 5 & & & \\ \hline & & & & 2 & 1 & \\ & & & & & 2 & \\ \hline & & & & & & 2 \end{array}\right] \quad W = \left[\begin{array}{ccc|c} 5 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ \hline & & & 5 \end{array}\right] \left[\begin{array}{ccc|cc} & & & 2 & 0 & 1 \\ & & & 0 & 2 & 0 \\ \hline & & & & & 2 \end{array}\right]$$

Example

Consider the 8×8 matrix J in Jordan canonical form with Jordan structure $(3,2,2,1)$

$$J = \left[\begin{array}{ccc|cc|cc|c} 2 & 1 & 0 & & & & & \\ & 2 & 1 & & & & & \\ & & 2 & & & & & \\ \hline & & & 2 & 1 & & & \\ & & & & 2 & & & \\ \hline & & & & & 2 & 1 & \\ & & & & & & 2 & \\ \hline & & & & & & & 2 \end{array}\right].$$

We will find a permutation matrix P such that the Weyr form W of J is given by $W = PJP^{-1}$.

We can view J as the matrix of a linear transformation T of $\mathbb{C}^8$ relative to some (ordered) basis $\mathcal{B} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. The action of T on the basis vectors can be represented as:

$$\begin{array}{l} 0 \leftarrow v_1 \leftarrow v_2 \leftarrow v_3 \\ 0 \leftarrow v_4 \leftarrow v_5 \\ 0 \leftarrow v_6 \leftarrow v_7 \\ 0 \leftarrow v_8 \end{array}$$

If we reorder the basis vectors in $\mathcal{B}$ by running through them in column order, and taking the vectors v_i to be the standard basis vectors, we get the basis

$$\mathcal{B}' = \{v_1, v_4, v_6, v_8, v_2, v_5, v_7, v_3\} = \left\{ \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix} \right\}.$$

Rewriting the basis vectors in a permutation matrix P gives

$$P = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0 \end{bmatrix}. \quad \text{Thus } W = PJP^{-1} = \left[\begin{array}{ccc|ccc|c} 2&0&0&0&1&0&0 \\ 0&2&0&0&0&1&0 \\ 0&0&2&0&0&0&1 \\ 0&0&0&2&0&0&0 \\ \hline & & & & 2&0&0&1 \\ & & & & 0&2&0&0 \\ & & & & 0&0&2&0 \\ \hline & & & & & & & 2 \end{array}\right].$$

Implementation

An implementation in the Maple computer algebra system for computing the Weyr form of a matrix can be found at: `github.com/StevenThornton/WeyrForm`

References

[1] Kevin O'Meara et al. *Advanced Topics in Linear Algebra: Weaving Matrix Problems through the Weyr Form.* Oxford University Press, USA, 2011.

[2] Helene Shapiro. The Weyr characteristic. *The American mathematical monthly*, 106(10):919–929, 1999.

[3] Eduard Weyr. Répartition des matrices en espèces et formation de toutes les espèces. *CR Acad. Sci. Paris*, 100:966–969, 1885.