

Zigzag Form over Families of Parametric Matrices

June 18th, 2014

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joint work with

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Encuentro de Álgebra Computacional y Aplicaciones, 2014

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Motivation

Parametric Matrix Computations in CAS

Computer algebra systems are currently **unable** to handle case discussion for parametric matrices.

Example: Maple, Mathematica, Sage

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

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Maple

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$$\left[\begin{array}{c|c|c} -2 & & \\ \hline & -3 + \alpha & \\ \hline & & \alpha \end{array} \right]$$

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Sage

?

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Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\alpha=-2} \begin{bmatrix} -2 & 0 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

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Background

Notation

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- ▶ $\mathbb{K}[\alpha]$ is the ring of polynomials in the parameters
- ▶ $\mathbb{K}(\alpha)$ is the quotient field of $\mathbb{K}[\alpha]$

Constructible Set

If \mathbb{K} is **algebraically closed**, a *constructible set* S of $\mathbb{K}[\alpha]$ is the solution set to any polynomial system of the form:

$$\begin{aligned} f_1(\alpha) = \cdots = f_a(\alpha) &= 0 \\ g(\alpha) &\neq 0 \end{aligned}$$

with

$$f_1(\alpha), \dots, f_a(\alpha), g(\alpha) \in \mathbb{K}[\alpha]$$

Semi-Algebraic Set

If \mathbb{K} is **real closed**, a *semi-algebraic set* S of $\mathbb{K}[\alpha]$ is the solution set to any polynomial system of the form:

$$f_1(\alpha) = \cdots = f_a(\alpha) = 0$$

$$g(\alpha) \neq 0$$

$$p_1(\alpha) > 0, \dots, p_b(\alpha) > 0$$

$$q_1(\alpha) \geq 0, \dots, q_c(\alpha) \geq 0$$

with

$$f_1(\alpha), \dots, f_a(\alpha), g(\alpha), p_1(\alpha), \dots, p_b(\alpha), q_1(\alpha), \dots, q_c(\alpha) \in \mathbb{K}[\alpha]$$

Proposition

For two constructible (resp. semi-algebraic) sets $S_1, S_2 \in \mathbb{K}^m$ one can compute a **triangular decomposition** of their

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- ▶ intersection $S_1 \cap S_2$
- ▶ union $S_1 \cup S_2$
- ▶ set theoretic difference $S_1 \setminus S_2$

Remark

Let $S \subseteq \mathbb{K}^m$ be a constructible (resp. semi-algebraic) set, for every $f(\alpha) \in \mathbb{K}[\alpha]$ a **partition** $(S_{\text{eq}}, S_{\text{neq}})$ of S can be computed by

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$$S_{\text{eq}} = S \cap V(f(\alpha))$$

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$$S_{\text{neq}} = S \setminus V(f(\alpha)) = S \setminus S_{\text{eq}}.$$

- ▶ $f(\alpha)$ is **zero** everywhere on S_{eq}
- ▶ $f(\alpha)$ is **nonzero** everywhere on S_{neq}

Parametric Polynomial

A polynomial with parameters α takes the form

$$f(x; \alpha) = f_0(\alpha) + f_1(\alpha)x + \cdots + f_{r-1}(\alpha)x^{r-1} + x^r$$

with $f_0(\alpha), \dots, f_{r-1}(\alpha) \in \mathbb{K}(\alpha)$.

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Require constructible (resp. semi-algebraic) set S such that denominator of every coefficient is nonzero everywhere on S .

Companion Matrix

Frobenius companion matrix in x of parametric polynomial

$$f(x; \alpha) = f_0(\alpha) + f_1(\alpha)x + \cdots + f_{r-1}(\alpha)x^{r-1} + x^r$$

takes the form

$$C_{f(x;\alpha)} = \begin{bmatrix} 0 & \cdots & 0 & -f_0(\alpha) \\ 1 & \ddots & \vdots & \vdots \\ & \ddots & 0 & -f_{r-2}(\alpha) \\ & & 1 & -f_{r-1}(\alpha) \end{bmatrix}$$

Off-diagonal Block

Let $b \in \{0, 1\}$. Define

$$B_b = \begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

ZoZo

Form

Zigzag Matrix¹

$$\left[\begin{array}{cccccccc}
 C_{c_1(x;\alpha)} & & & & & & & \\
 & B_{b_1} & & & & & & \\
 & & C_{c_2(x;\alpha)}^T & & & & & \\
 & & & B_{b_2} & & & & \\
 & & & & C_{c_3(x;\alpha)} & & & \\
 & & & & & B_{b_3} & & \\
 & & & & & & C_{c_4(x;\alpha)}^T & \\
 & & & & & & & \ddots \\
 & & & & & & & & C_{c_{k-2}(x;\alpha)}^T & & & \\
 & & & & & & & & & B_{b_{k-2}} & & \\
 & & & & & & & & & & C_{c_{k-1}(x;\alpha)} & & B_{b_{k-1}} \\
 & & & & & & & & & & & C_{c_k(x;\alpha)}^T & \\
 \end{array} \right]$$

for k even

¹A. Storjohann. An $\mathcal{O}(n^3)$ algorithm for the Frobenius normal form. In *Proceedings of the 1998 international symposium on symbolic and algebraic computation*, pages 101-105. ACM, 1998.

Computation

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Theorem

Let \mathbb{K} , $\mathbb{K}[\alpha]$ and S be as previously defined. For **every** matrix $A(\alpha) \in \mathbb{K}^{n \times n}[\alpha]$, there exists a partition (S_1, \dots, S_N) of S such that for each S_i , there exists a matrix $Z_i(\alpha) \sim A(\alpha)$ in **Zigzag form** where the denominator of the entries of $Z_i(\alpha)$ are all nonzero everywhere on S_i .

Algorithm

The algorithm used extends the work of **Storjohann**² on the computation of the Zigzag form.

²**A. Storjohann.** An $\mathcal{O}(n^3)$ algorithm for the Frobenius normal form. In *Proceedings of the 1998 international symposium on symbolic and algebraic computation*, pages 101-105. ACM, 1998.

Algorithm

The algorithm used extends the work of **Storjohann**² on the computation of the Zigzag form.

Stages 1 and 3 are modified such that computation splits when searching for pivots.

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Examples

Algebraic Example over \mathbb{C}

$$A(\alpha) = \begin{bmatrix} -1 & -\alpha - 1 & 0 \\ -1/2 & \alpha - 2 & 1/2 \\ -2 & 3\alpha + 1 & -1 \end{bmatrix} \quad \text{with } \alpha \in \mathbb{C}$$

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$$Z_1(\alpha) = \begin{bmatrix} 0 & 0 & 4\alpha \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix} \quad \text{if } \alpha + 3 \neq 0$$

$$Z_2(\alpha) = \left[\begin{array}{cc|c} 0 & -4 & 1 \\ 1 & -4 & 0 \\ \hline & & -3 \end{array} \right] \quad \text{if } \alpha + 3 = 0$$

Algebraic Example over \mathbb{C}

```
> with (LinearAlgebra) : with (RegularChains) : with (ConstructibleSetTools) :
```

```
> A := Matrix( [[ [-1, -α - 1, 0], [-1/2, α - 2, 1/2], [-2, 3α + 1, -1]] ] )
```

$$A := \begin{bmatrix} -1 & -\alpha - 1 & 0 \\ -\frac{1}{2} & \alpha - 2 & \frac{1}{2} \\ -2 & 3\alpha + 1 & -1 \end{bmatrix}$$

(1)

```
> R := PolynomialRing( [α] ) :
```

```
> CS := GeneralConstruct( [ ], [ ], R ) :
```

```
> Display( ComplexZigzagForm( A, CS, R, R ) ;
```

$$\left[\left[\begin{bmatrix} 0 & 0 & 4\alpha \\ 1 & 0 & 4\alpha - 4 \\ 0 & 1 & -4 + \alpha \end{bmatrix}, \begin{bmatrix} 1 & \frac{2(5\alpha - 1)}{\alpha + 3} & -\frac{3\alpha + 1}{\alpha + 3} \\ 0 & \frac{2(3\alpha - 7)}{\alpha + 3} & -\frac{2(\alpha - 1)}{\alpha + 3} \\ 0 & -\frac{8}{\alpha + 3} & \frac{2}{\alpha + 3} \end{bmatrix}, \alpha + 3 \neq 0 \right], \right.$$

(2)

$$\left. \left[\begin{bmatrix} 0 & \alpha - 1 & 1 \\ 1 & \alpha - 1 & 0 \\ 0 & \frac{1}{2}\alpha + \frac{3}{2} & -3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -8 & 2 \end{bmatrix}, \alpha + 3 = 0 \right] \right]$$

Semi-Algebraic Example over \mathbb{R}

$$A(a, b) = \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix} \quad \text{with } a \geq 0, b > 0, a \neq b$$

Semi-Algebraic Example over \mathbb{R}

$$A(a, b) = \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix} \quad \text{with } a \geq 0, b > 0, a \neq b$$

$$Z_1(a, b) = \begin{bmatrix} 0 & 0 & 12a \\ 1 & 0 & 18 \\ 0 & 1 & b \end{bmatrix} \quad \text{if } a > 0, b > 0, a \neq b$$

$$Z_2(a, b) = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline & 0 & 1 \\ & 18 & b \end{array} \right] \quad \text{if } a = 0, b > 0$$

Semi-Algebraic Example over \mathbb{R}

```

> with(LinearAlgebra) : with(RegularChains) :
> A := Matrix([[0, 0, 2], [a, 0, 3], [0, 6, b]])

```

$$A := \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix} \quad (1)$$

```

> R := PolynomialRing([a, b]) :
> lrsas := RealTriangularize([], [a], [b], [a - b], R) :
> Display(RealZigzagForm(A, lrsas, R), R);

```

$$\left[\left[\begin{bmatrix} 0 & 0 & 12a \\ 1 & 0 & 18 \\ 0 & 1 & b \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{6a} \end{bmatrix}, [a > 0 \text{ and } b > 0 \text{ and } a - b \neq 0], \begin{bmatrix} 0 & 1 & 0 \\ a & 0 & 1 \\ 0 & 18 & b \end{bmatrix}, \right. \right. \\ \left. \left. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 12 & 2b \end{bmatrix}, \left[\begin{cases} a = 0 \\ b > 0 \end{cases} \right] \right] \quad (2)$$

Future

Work

Jordan Form Example

Jordan canonical form of:

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- ▶ Frobenius form
- ▶ minimal polynomial
- ▶ Jordan form
- ▶ Weyr form³

³K. O'Meara, J.Clark, and C. Vinsonhaler. *Advanced Topics in Linear Algebra: Weaving Matrix Problems Through the Weyr Form*. Oxford University Press, 2011

All current and future code and example worksheets will be available at StevenThornton.ca/Code

Information on the RegularChains package of MAPLE is available at RegularChains.org

Thank

You