

Behavioural equivalence in coalgebras of finitary set functors

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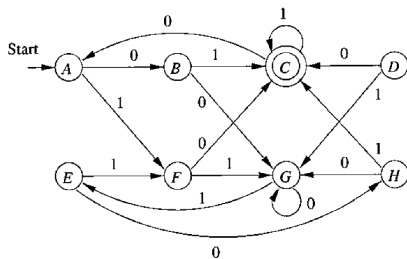
April 2, 2017

Overview

- 1 A coalgebraic view on DFAs
 - Language equivalent states
 - Good DFA relations
- 2 Coalgebras
 - Finitary functors
 - Substitutive relations
- 3 Logic
 - Soundness
 - Completeness

DFAs

A deterministic finite automata looks like this:

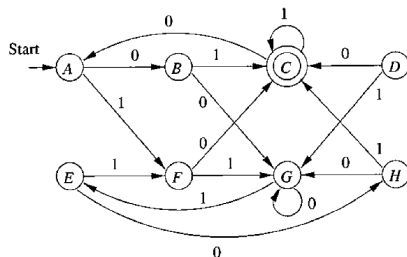


Formally, a DFA on the alphabet A is a quadruple consisting of:

- 1 Q
- 2 $\delta : Q \times A \rightarrow Q$
- 3 $f : Q \rightarrow 2$
- 4 $q_0 \in Q$

Behavioral equivalence

Key Question: When do two states accept the same language?

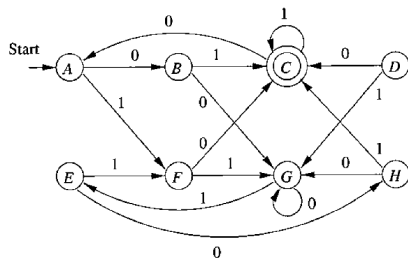


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- (A, C)?

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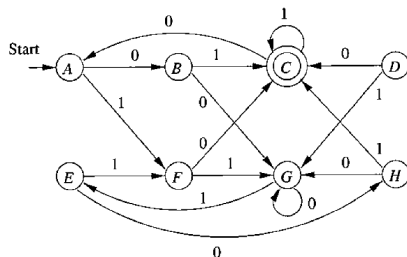


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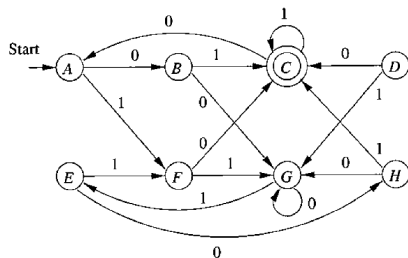


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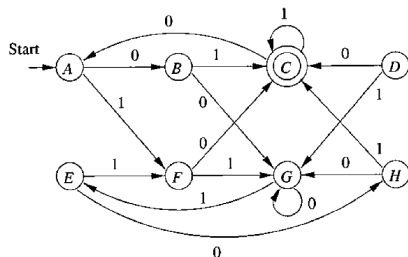


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- (B, H)? Yes.
- (A, E)?

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Coalgebraic view on DFAs

Curry $\delta : Q \times A \rightarrow Q$ to $\delta' : Q \rightarrow Q^A$ and pair with f . A DFA (without start) is now a set Q and a function of the form:

$$Q \xrightarrow{\langle f, \delta' \rangle} 2 \times Q^A$$

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Let $\mathcal{L} = \mathcal{P}(A^*)$, and define

- $\varepsilon? : \mathcal{L} \rightarrow 2$ by $\varepsilon?(L) = \text{"}\varepsilon \in L\text{"}$
- $\partial : \mathcal{L} \rightarrow \mathcal{L}^A$ by $\partial(L) = \lambda a. \{w : aw \in L\}$

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Even better: $\langle \varepsilon?, \partial \rangle$ is a bijection!

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This diagram says:

- $\varepsilon?(\ell(q)) = f(q)$
- $\forall a \in A, \partial(\ell(q))(a) = \{w : aw \in \ell(q)\} = \ell(\delta'(q)(a))$

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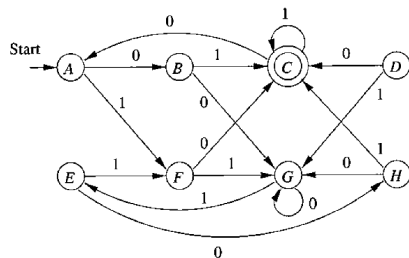
Idea: let's look at relations R on Q such that

$$R = \{(x, y) : f(x) = f(y) \text{ and } \forall a \in A, (\delta'(x)(a), \delta'(y)(a)) \in R\}$$

We call such relations “good DFA relations”. $\ker(\ell)$ is a good DFA relation.

Making good DFA relations: grow from below

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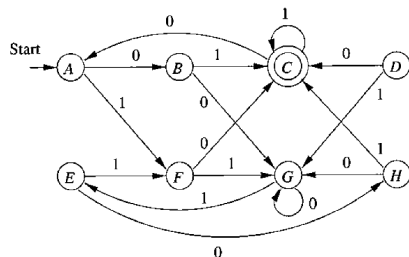


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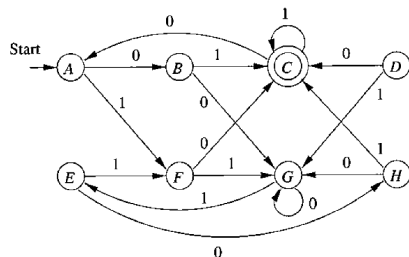


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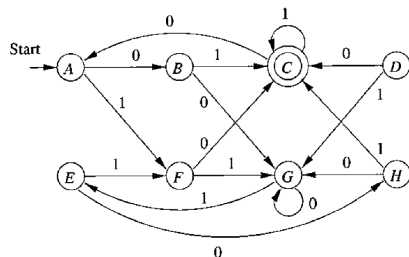


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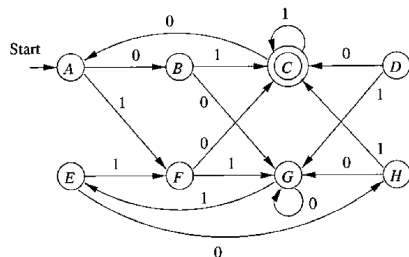


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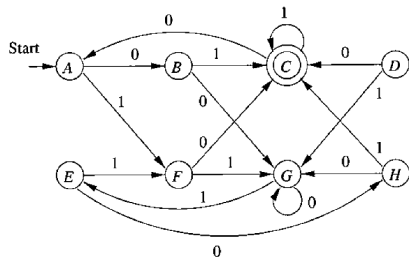
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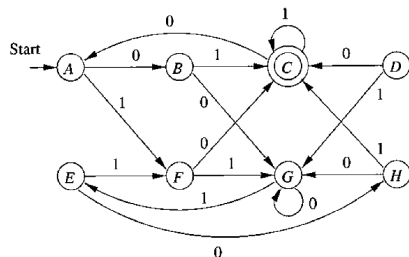
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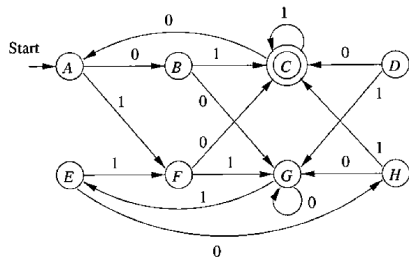
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Properties of good DFA relations

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Properties of good DFA relations

- 1 The diagonal Δ_Q is good.
- 2 Arbitrary unions of good relations are good.
- 3 The inverse of a good relation is good.
- 4 The composition of two good relations is good.
- 5 The equivalence closure of a good relation is good.

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The greatest good relation is created by “retreat from complete” and exactly coincides with $\ker(\ell)$. This is a coinductive proof principle:

Coinduction for DFAs

If R is any good DFA relation, $R \subseteq \ker(\ell)$.

Abstraction plan

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Given a Set endofunctor $H \dots$

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Examples of coalgebras

When H is...	H -coalgebras are...
$2 \times (-)^A$	DFAs
$B \times (-)^A$	Moore machines
$(B \times -)^A$	Mealy machines
$\mathcal{P}V \times \mathcal{P}_\omega-$	Kripke frames
$\mathbb{K} \times \mathbb{K}_\omega^-$	Weighted automata
$\mathcal{P}_\omega-$	Strongly extensional trees or non-wellfounded sets
$(\mathcal{P}_\omega-)^A$	Labeled transition systems

Executing the plan

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Finitary functors

Let Σ be a finitary, functional signature. A *signature* functor:

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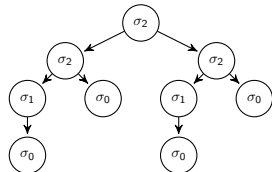
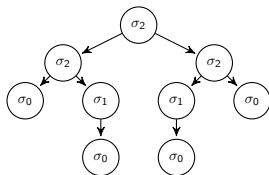
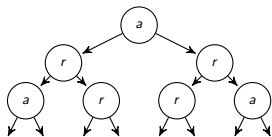
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Finitary functors always have final coalgebras.

$$\Sigma = \Sigma_2 = \{a, r\}$$

$$\Sigma_n = \{\sigma_n\}$$



Executing the plan

Given a Set endofunctor $H \dots$

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Theorem

Let (Q, d) be an H -coalgebra and R a relation on Q . TFAE:

- 1 R is a substitutive equivalence relation.
- 2 Q/R is an H -coalgebra and q_R is an H -coalgebra morphism.
- 3 R is the kernel of some H -coalgebra morphism out of (Q, d) .

Proof. (1 \Rightarrow 2) R is substitutive:

$$\begin{array}{ccc} R & \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} & Q \xrightarrow{d} HQ \\ & & \downarrow Hq_R \\ & & H(Q/R) \end{array}$$

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(2 \Rightarrow 3) R is the kernel of q_R .

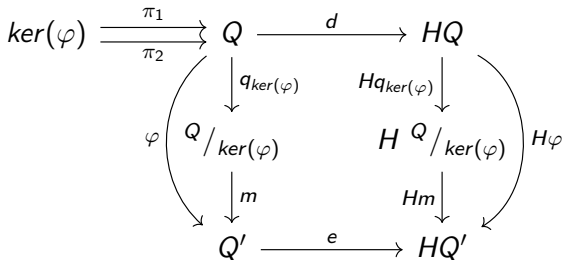
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Proof. (3 \Rightarrow 1)



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Properties of substitutive relations

Suppose (Q, d) is an H -coalgebra.

- 1 The diagonal Δ_Q is a substitutive equivalence.
- 2 Arbitrary unions of substitutive relations are substitutive.
- 3 The inverse of a substitutive relation is substitutive.
- 4 If $R, S \supseteq \Delta_Q$ are substitutive, so is $R \cdot S$.
- 5 If a relation is substitutive, its equivalence closure is substitutive.

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Theorem (Coinduction for H -coalgebras)

If (C, c) is the final H -coalgebra and $\ell : Q \rightarrow C$ is the universal map, then $\ker(\ell)$ is the greatest substitutive relation on Q .

$$Q \begin{array}{c} \xrightarrow{q_R} \\ \xrightarrow{\ell} \end{array} Q/R \xrightarrow{q} C$$

Behavioral equivalence in signature functor coalgebras

Suppose (Q, d) is an H_Σ -coalgebra. In the following, $x, y \in Q$, $\alpha, \beta \in H_\Sigma Q$, $R \subseteq Q$ and σ, τ are variables or terms.

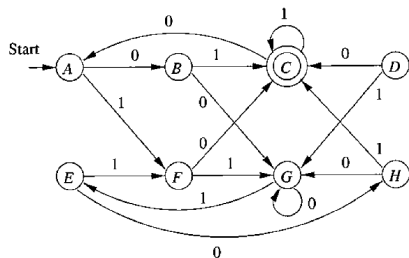
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$$\frac{R \vdash \tau = \sigma}{R \vdash \sigma = \tau} \text{ s} \quad \frac{R \vdash x_1 = y_1 \quad \dots \quad R \vdash x_{ar(f)} = y_{ar(f)}}{R \vdash f(x_1, \dots, x_{ar(f)}) = f(y_1, \dots, y_{ar(f)})} \text{ c}$$

$$\frac{R \cup S \vdash \varphi \quad \forall (x, y) \in S. R \cup S \vdash d(x) = d(y)}{R \vdash \varphi} \text{ i}$$

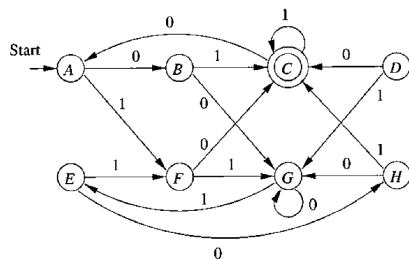
This logic is sound and complete with respect to behavioral equivalence.

Example proof



$$\left\{ \begin{array}{l} A = r(B, F) \\ B = r(G, C) \\ C = a(A, C) \\ E = r(H, F) \\ \vdots \\ H = r(G, C) \end{array} \right.$$

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Let $R = \{(A, E), (B, H)\}$

$$\frac{\overline{R \vdash A = E} \quad \overline{R \vdash B = H} \quad \overline{R \vdash F = F} \quad \overline{R \vdash r(B, F) = r(H, F)} \quad \overline{R \vdash r(G, C) = r(G, C)}}{\vdash A = E} \begin{array}{l} a \\ c \\ r \\ i \end{array}$$

Soundness

Ideal: $R \vdash x = y$ implies $(x, y) \in \ker(\ell)$

$R \vdash \alpha = \beta$ implies $Hq_{\ker(\ell)}(\alpha) = Hq_{\ker(\ell)}(\beta)$

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Given (Q, d) define the following monotone operator:

$$w_{SR}(S) = \{(x, y) \in S : Hq_S(d(x)) = Hq_S(d(y))\} \cup R$$

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We get its greatest fixpoint by “retreat from complete”:

$$\delta_R^0 = Q \times Q \quad \delta_R^{i+1} = ws_R(\delta_R^i) \quad \delta_R = \bigcap_{i \in \omega} \delta_R^i$$

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Reality: $R \vdash x = y$ implies $(x, y) \in e(\delta_R)$

$R \vdash \alpha = \beta$ implies $Hq_{\delta_R}(\alpha) = Hq_{\delta_R}(\beta)$

Soundness

Lemma

$(x, y) \in \delta_S$ if and only if $(x, y) \in S$ or $Hq_{\delta_S}(d(x)) = Hq_{\delta_S}(d(y))$.

Proposition

Let R, S be any relation on Q . If $Hq_{\delta_{RUS}}(d(x)) = Hq_{\delta_{RUS}}(d(y))$ for all $(x, y) \in S$, then $\delta_{RUS} = \delta_R$.

Proof.

$$\begin{aligned}\delta_{RUS} &= R \cup S \cup \{(x, y) \in \delta_{RUS} : Hq_{\delta_{RUS}}(d(x)) = Hq_{\delta_{RUS}}(d(y))\} \\ &= R \cup \{(x, y) \in \delta_{RUS} : Hq_{\delta_{RUS}}(d(x)) = Hq_{\delta_{RUS}}(d(y))\} \\ &= WS_R(\delta_{RUS})\end{aligned}$$



Completeness

- 1 If $(x, y) \in e(R)$, then $R \vdash x = y$.
- 2 If $Hq_R(\alpha) = Hq_R(\beta)$, then $R \vdash \alpha = \beta$.
- 3 If R is substitutive, then $\vdash x = y$ for all $(x, y) \in R$.

Behavioral equivalence in **finitary** functor coalgebras

Suppose (Q, f) is an F -coalgebra. There is a related H_Σ -coalgebra, $(Q, d = f \circ s)$.

$$\frac{}{R \vdash \sigma = \sigma} \text{r} \quad \frac{}{\{(x, y)\} \cup R \vdash x = y} \text{a} \quad \frac{R \vdash \sigma = \tau \quad R \vdash \tau = \rho}{R \vdash \sigma = \rho} \text{t}$$

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Thanks!