

# A complete logic for behavioural equivalence in coalgebras of finitary set functors

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# Outline

Background

Finitary functors

Proof system

Bisimulation up to presentation

Modularity of the logic

Future directions

# Background

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Moschovakis, Moss, Wennstrom, Whitney:  $FLR_0$

- ▶ Language for expressing general recursive definitions
- ▶ Completeness results rare
- ▶ Success in the  $\mathcal{P}_\omega$  case

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Strategy:

- ▶ Use  $\Sigma$  as a language to express coalgebras
- ▶ Use  $\epsilon$  for additional axioms

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## Example presentations

Presentation: need  $\Sigma$  and  $\epsilon$  so that  $\epsilon : H_\Sigma \rightarrow F$

▶ Constant  $FX = A$ :

$\Sigma = \Sigma_0 = A$  and  $\epsilon_X : a() \mapsto a$

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- ▶ DFA's  $FX = 2 \times X^A$ :  
 $\Sigma = \Sigma_{|A|} = 2$  and  $\epsilon_X : b(x_1, \dots, x_{|A|}) \mapsto (b, \lambda i. x_i)$

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### Definition

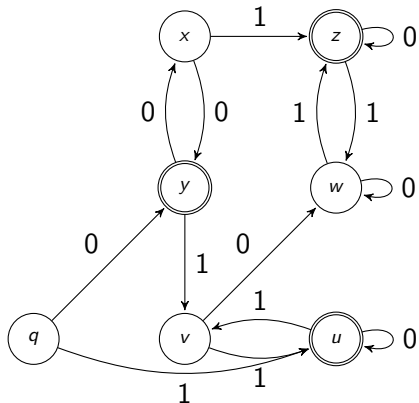
For  $\alpha, \beta \in H_\Sigma X$ , we write  $\alpha =_\epsilon \beta$  to mean  $\epsilon_X(\alpha) = \epsilon_X(\beta)$ .

# Specifications

$H_\Sigma$ - and  $F$ -coalgebras are related.

- ▶ Given  $X \xrightarrow{d} H_\Sigma X$ , get  $X \xrightarrow{\epsilon_X \circ d} FX$
- ▶ Given  $X \xrightarrow{f} FX$  and a section  $s$  of  $\epsilon_X$ , get  $X \xrightarrow{s \circ f} H_\Sigma X$

Example specification:  $FX = 2 \times X^2$



As a specification:

$$d(x) = \perp(y, z) \quad d(y) = \top(x, v)$$

$$d(v) = \perp(w, u) \quad d(u) = \top(u, v)$$

$$d(w) = \perp(w, z) \quad d(z) = \top(z, w)$$

$$d(q) = \perp(y, u)$$

Example specification:  $FX = \mathcal{P}_\omega X$



As a specification  $X \xrightarrow{d} H_\Sigma X$ :

$$d(x) = s_2(x, y)$$

$$d(y) = s_1(x)$$

As a coalgebra  $X \xrightarrow{f} FX$ :

$$f(x) = \{x, y\}$$

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As another specification  $X \xrightarrow{d'} H_\Sigma X$ :

$$d'(x) = s_2(y, x)$$

$$d'(y) = s_2(x, x)$$

## Proof system

Given a presentation  $(\Sigma, \epsilon)$  and a specification  $X \xrightarrow{d} H_\Sigma X$ .

Judgements of the form  $R \vdash \sigma = \tau$  where  $R \subseteq X \times X$  and  $(\sigma, \tau) \in X \times X + H_\Sigma X \times H_\Sigma X$ .

$$\frac{}{R \vdash \sigma = \sigma} \text{r} \qquad \frac{R \vdash \tau = \sigma}{R \vdash \sigma = \tau} \text{s} \qquad \frac{R \vdash \sigma = \tau \quad R \vdash \tau = \rho}{R \vdash \sigma = \rho} \text{t}$$

$$\frac{\{(x, y)\} \cup R \vdash x = y}{\{(x, y)\} \cup R \vdash x = y} \text{a} \qquad \frac{R \vdash x_1 = y_1 \quad \dots \quad R \vdash x_{ar(f)} = y_{ar(f)}}{R \vdash f(x_1, \dots, x_{ar(f)}) = f(y_1, \dots, y_{ar(f)})} \text{c}$$

$$\frac{\sigma =_\epsilon \tau}{R \vdash \sigma = \tau} \text{e} \qquad \frac{R \vdash \varphi \quad \forall (x, y) \in R. R \vdash d(x) = d(y)}{\vdash \varphi} \text{b}$$

Example proof:  $FX = \mathcal{P}_\omega X$



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$$d(x) = s_2(x, y)$$

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Let  $R = \{(x, y)\}$ .

$$\frac{\overline{R \vdash x = y} \quad a}{\overline{R \vdash s_2(x, y) = s_2(x, x)} \quad c} \frac{\overline{R \vdash x = x} \quad r \quad \overline{R \vdash x = y} \quad a}{\overline{R \vdash y = x} \quad s} \quad \frac{\overline{R \vdash s_2(x, x) = s_1(x)} \quad \epsilon}{\overline{R \vdash s_2(x, y) = s_1(x)} \quad t} \quad b$$

$$\frac{\overline{R \vdash x = y} \quad a}{\vdash x = y}$$



## Main result

Suppose  $F$  is a finitary functor which preserves weak pullbacks and  $(X, d)$  is a specification in a signature for that functor.

Then this proof system is **sound** and **complete**.

## Bisimulation up to presentation

$H_\Sigma$ - and  $F$ -coalgebras are related. So are their bisimulations.

### Definition

Given a specification  $(X, d)$  in the presentation  $(\Sigma, \epsilon)$ , the relation  $R \subseteq X \times X$  is an  $H_\Sigma$ -bisimulation on  $(X, d)$  if there is a  $\rho : R \rightarrow H_\Sigma R$  such that  $d \circ \pi_j = H_\Sigma \pi_j \circ \rho$ .

$$\begin{array}{ccc} R & \xrightarrow{\pi_j} & X \\ \rho \downarrow & & \downarrow d \\ H_\Sigma R & \xrightarrow{H_\Sigma \pi_j} & H_\Sigma X \end{array}$$

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### Theorem

*The following are equivalent:*

- ▶  *$R$  is an  $F$ -bisimulation on  $(X, \epsilon_X \circ d)$*
- ▶  *$R$  is a bisimulation up to presentation on  $(X, d)$*

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- ▶ For all  $(x, y) \in R$ , there is an  $f \in \Sigma_n$  and  $(x_1, y_1), \dots, (x_n, y_n) \in R$  such that:
  1.  $d(x) =_\epsilon f(x_1, \dots, x_n)$
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### Definition

Given  $R \subseteq X \times X$ , define:

- ▶  $c(R)$  by  $f(x_1, \dots, x_{ar(f)}) \ c(R) \ f(y_1, \dots, y_{ar(f)})$  iff  $x_i R y_i$ .
- ▶  $\sim_R$  by  $=_\epsilon \bullet \ c(R) \ \bullet \ =_\epsilon$ .

$R$  is a bisim. up to presentation iff  $x R y$  implies  $d(x) \sim_R d(y)$ .

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If any of the following are true:

1.  $xRy \rightarrow d(x) \sim_{R \cup \Delta_x} d(y)$
2.  $xRy \rightarrow d(x) \sim_{R \cup \sim} d(y)$
3.  $xRy \rightarrow d(x) \sim_{s(R)} d(y)$
4. \*  $xRy \rightarrow d(x) \sim_{e(R)} d(y)$
5. \*  $xRy \rightarrow d(x) \sim_{e(R \cup \sim)} d(y)$

then  $R \subseteq \sim$ .



## Sketch for soundness and completeness

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Soundness ( $\vdash x = y$  implies  $\vDash x = y$ ):

- ▶ If  $R \vdash \varphi$ , then  $\varphi \in e(R \cup \sim) + \sim_{e(R \cup \sim)}$  by induction.
- ▶ If  $\emptyset \vdash x = y$ , then  $(x, y) \in e(\sim) = \sim$ .
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Completeness ( $\vDash x = y$  implies  $\vdash x = y$ ):

- ▶ If  $\alpha \sim_R \beta$ , then  $R \vdash \alpha = \beta$  by explicit construction.
- ▶ If  $x$  and  $y$  are behaviourally equivalent, they're related by a bisimulation (up to presentation)  $R$ .
- ▶  $uRv \rightarrow d(u) \sim_R d(v) \rightarrow (R \vdash d(u) = d(v))$
- ▶ Conclude by  $b$  that  $\vdash x = y$ .

# Modularity of the logic

From presentations for  $F$  and  $G$ , you can naturally construct presentations for:

- ▶  $F \times G$
- ▶  $F + G$
- ▶  $F \circ G$

Modularity in both the language and axioms are a consequence.

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- ▶ Connections to enhanced bisimulation literature
- ▶ Relationships between  $F$  and  $G$  and relationships between  $\mathit{Coalg}_F$  and  $\mathit{Coalg}_G$ ?