

Linearization of Automatic Arrays and Weave Specifications

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The original intention was to see how to extend the work

[6] C. Grabmayer, J. Endrullis, D. Hendricks, J.W. Klop, and L.S. Moss.
Automatic sequences and zip-specifications.

from zip-specifications of automatic sequences to automatic arrays.

Outline:

- 1 Background
- 2 The connection between sequences and arrays
- 3 Consequences

Let Δ be a finite set.

A **sequence in Δ** is a map $\mathbb{N} \rightarrow \Delta$ which we will usually represent like

$$\sigma = \sigma_0 \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \dots$$

An **array in Δ** is a map $\mathbb{N} \times \mathbb{N} \rightarrow \Delta$ which we will usually represent like

$$a = \begin{array}{|cccc} \vdots & \vdots & \vdots & \vdots \\ a_{20} & a_{21} & a_{22} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{00} & a_{01} & a_{02} & \dots \end{array}$$

We use the functor $F_k : \mathit{Set} \rightarrow \mathit{Set}$ which acts on objects by $F_k X = \Delta \times X^k$ and on morphisms by $F_k f = \text{id}_\Delta \times f$.

Sequences as coalgebras

Let $S = \{\sigma \in \Delta^\omega\}$ be the set of infinite sequences of symbols from Δ . We can make it into an F_k -coalgebra as follows.

The **sequence projection** $\pi_{\frac{i}{k}} : S \rightarrow S$ is defined by

$$\pi_{\frac{i}{k}}(\sigma) = \pi_{\frac{i}{k}}(\sigma_0\sigma_1\sigma_2\sigma_3\dots) = \sigma_i\sigma_{i+k}\sigma_{i+2k}\sigma_{i+3k}\dots$$

(More standardly, $\pi_{\frac{i}{k}}$ is written as $\pi_{i,k}$.)

The **head function** $\text{hd} : S \rightarrow \Delta$ is defined by

$$\text{hd}(\sigma) = \sigma_0$$

The F_k -coalgebra structure on S is given by

$$\langle \text{hd}, \pi_{\frac{0}{k}}, \pi_{\frac{1}{k}}, \dots, \pi_{\frac{k-1}{k}} \rangle : S \rightarrow \Delta \times S^k.$$

Theorem (Kupke & Rutten and [6])

This coalgebra is final for the subcategory of F_k -coalgebras $(C, \langle o, c_0, c_1, \dots, c_{k-1} \rangle)$ which share the property that $o \circ c_0 = o$. (Called **zero-consistent** coalgebras)

Roughly, the reason for finality is that the sequence projection maps give nearly unique addresses for each of the entries:

$$\boxed{\sigma_0} \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 \sigma_9 \sigma_{10} \sigma_{11} \sigma_{12} \sigma_{13} \sigma_{14} \sigma_{15} \sigma_{16} \dots$$

$$\pi_{\frac{0}{2}} \quad \pi_{\frac{1}{2}} \quad \boxed{\text{hd}}$$

To get the entry σ_5 we must apply $\pi_{\frac{1}{2}}$, then $\pi_{\frac{0}{2}}$, then finally $\pi_{\frac{1}{2}}$ and hd.

$$\sigma_0 \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_8 \sigma_9 \sigma_{10} \sigma_{11} \sigma_{12} \sigma_{13} \sigma_{14} \sigma_{15} \sigma_{16} \dots$$

DFAOs as F -coalgebras

We can make the previous DFAO into an F_2 -coalgebra using its set of states, Q , along with the transition and final output maps, δ and f respectively:

$$Q \xrightarrow{\langle f, \delta(\cdot, 0), \delta(\cdot, 1) \rangle} \Delta \times Q \times Q$$

By finality we immediately get a map, seq , into sequences in Δ

$$\begin{array}{ccc}
 Q & \xrightarrow{\langle f, \delta(\cdot, 0), \delta(\cdot, 1) \rangle} & \Delta \times Q \times Q \\
 \text{seq} \downarrow & & \downarrow \langle \text{id}_\Delta, \text{seq}, \text{seq} \rangle \\
 S & \xrightarrow{\langle \text{hd}, \pi_{\frac{0}{2}}, \pi_{\frac{1}{2}} \rangle} & \Delta \times S \times S
 \end{array}$$

The seq map takes a state $q \in Q$ to the sequence the DFAO generates when using that state as the start state.

An array coalgebra

0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0

$$\boxed{c} : A \rightarrow \Delta$$

$$\pi_{\frac{0}{2}, \frac{0}{2}} : A \rightarrow A$$

$$\pi_{\frac{0}{2}, \frac{1}{2}} : A \rightarrow A$$

$$\pi_{\frac{1}{2}, \frac{0}{2}} : A \rightarrow A$$

$$\pi_{\frac{1}{2}, \frac{1}{2}} : A \rightarrow A$$

Proposition

The array coalgebra $(A, \langle c, \pi_{\dot{k}, \dot{l}} \rangle)$ is final for the zero-consistent coalgebras of $F_{kl}X = \Delta \times X^{kl}$.

Recall the sequence coalgebra $(S, \langle \text{hd}, \pi_{\dot{kl}} \rangle)$ is *also* final for this functor. This means

Corollary

There is an isomorphism of coalgebras, $\varphi_{k,l}$, as shown:

$$\begin{array}{ccc}
 S & \xrightarrow{\langle \text{hd}, \pi_{\dot{kl}} \rangle} & \Delta \times S^{kl} \\
 \varphi \updownarrow & & \updownarrow F\varphi \\
 A & \xrightarrow{\langle c, \pi_{\dot{k}, \dot{l}} \rangle} & \Delta \times A^{kl}
 \end{array}$$

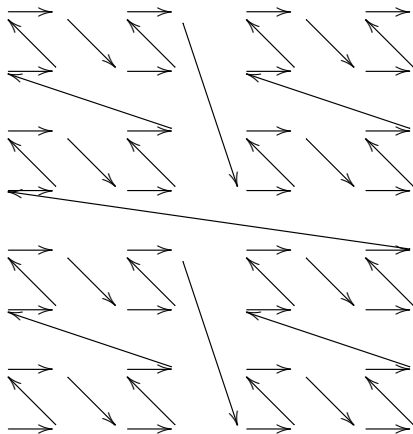
What does $\varphi_{k,l}$ look like?

We choose a correspondence between sequence and array projections. For simplicity, we'll take $k = l = 2$ and choose

S		A
hd	\leftrightarrow	c
$\pi_{\frac{0}{4}}$	\leftrightarrow	$\pi_{\frac{0}{2}, \frac{0}{2}}$
$\pi_{\frac{1}{4}}$	\leftrightarrow	$\pi_{\frac{0}{2}, \frac{1}{2}}$
$\pi_{\frac{2}{4}}$	\leftrightarrow	$\pi_{\frac{1}{2}, \frac{0}{2}}$
$\pi_{\frac{3}{4}}$	\leftrightarrow	$\pi_{\frac{1}{2}, \frac{1}{2}}$

So, $\sigma_5 = \text{hd}(\pi_{\frac{1}{4}}(\pi_{\frac{1}{4}}(\sigma)))$ corresponds to $c(\pi_{\frac{0}{2}, \frac{1}{2}}(\pi_{\frac{0}{2}, \frac{1}{2}}(\varphi_{2,2}(\sigma))))$.

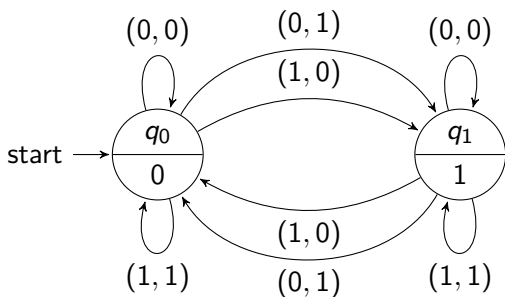
What does $\varphi_{2,2}$ look like?



We refer to $\varphi_{k,l} : S \rightarrow A$ as “ord” and $\varphi_{k,l}^{-1} : A \rightarrow S$ as “lin”.

k, l -automatic arrays

A k, l -DFAO is like a DFAO except we make the additional assumption that its input alphabet consists of pairs of digits from base k and base l . An example:



We determine the entry in the array at position (m, n) by pairing the digits of $[m]_k$ and $[n]_l$ and running the DFAO on those pairs.

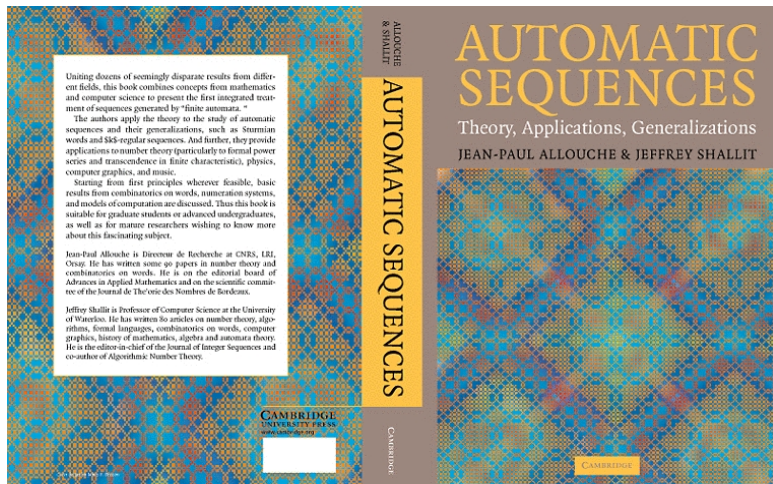
Fractals: Sierpinski gasket

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1
0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1
0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1
0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Fractals: Sierpinski carpet

1	1	1	1	1	1	1	1	1
1	0	1	1	0	1	1	0	1
1	1	1	1	1	1	1	1	1
1	1	1	0	0	0	1	1	1
1	0	1	0	0	0	1	0	1
1	1	1	0	0	0	1	1	1
1	1	1	1	1	1	1	1	1
1	0	1	1	0	1	1	0	1
1	1	1	1	1	1	1	1	1

Pretty book covers



lin preserves automaticity

Theorem (Cobham and Allouche & Shallit)

A sequence σ is kl -automatic \Leftrightarrow The kl -kernel of σ is finite



An array a is k, l -automatic \Leftrightarrow The k, l -kernel of a is finite

$$\begin{array}{ccc}
 S & \xrightarrow{\langle \text{hd}, \pi_{kl} \cdot \rangle} & \Delta \times S^{kl} \\
 \updownarrow \varphi & & \updownarrow F\varphi \\
 A & \xrightarrow{\langle c, \pi_{k,l} \cdot \rangle} & \Delta \times A^{kl}
 \end{array}$$

Theorem

An array a is k, l -automatic iff $\text{lin}(a)$ is kl -automatic. Similarly, a sequence σ is kl -automatic iff $\text{ord}(\sigma)$ is k, l -automatic.

Zip specifications

Automatic sequences have compact descriptions in terms of zipping other sequences. For example, the Thue-Morse sequence from earlier

LRRLLRLLRLLRLLRLLRLLRLLRLLR...

can be described with the “zip equations” $\begin{cases} x = \mathbf{zip}(x, y), & \text{hd}(x) = L \\ y = \mathbf{zip}(y, x), & \text{hd}(y) = R \end{cases}$

$$\begin{cases} x^* = LRRLLRLLR \\ y^* = RLLRLLRLL \end{cases}$$

Theorem (Grabmayer, Endrullis, Hendriks, Klop & Moss)

A sequence is k -automatic iff it has a \mathbf{zip}_k specification. Moreover, it is decidable whether two zip specifications generate the same sequence.

Weave specifications

1	0	0	1	0	1	1	0
0	1	1	0	1	0	0	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	0	0	1	0	1	1	0
0	1	1	0	1	0	0	1

We can describe this array with a system of equations and corner points, as below:

$$x = \mathbf{wv} \begin{pmatrix} y & x \\ x & y \end{pmatrix} \text{ and } c(x) = 0$$

$$y = \mathbf{wv} \begin{pmatrix} x & y \\ y & x \end{pmatrix} \text{ and } c(y) = 1$$

$$\mathbf{wv} \begin{pmatrix} C & D \\ A & B \end{pmatrix} = \begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{10} & d_{10} & c_{11} & d_{11} & \dots \\ a_{10} & b_{10} & a_{11} & b_{11} & \dots \\ c_{00} & d_{00} & c_{01} & d_{01} & \dots \\ a_{00} & b_{00} & a_{01} & b_{01} & \dots \end{array}$$

$$\begin{aligned} \text{lin} \left(\mathbf{wv} \begin{pmatrix} C & D \\ A & B \end{pmatrix} \right) &= a_{00} b_{00} c_{00} d_{00} a_{01} b_{01} c_{01} d_{01} a_{10} b_{10} c_{10} \dots \\ &= \mathbf{zip}(\text{lin}(A), \text{lin}(B), \text{lin}(C), \text{lin}(D)) \\ \mathbf{wv} \begin{pmatrix} C & D \\ A & B \end{pmatrix} &= \text{ord}(\mathbf{zip}(\text{lin}(A), \text{lin}(B), \text{lin}(C), \text{lin}(D))) \end{aligned}$$

We can design a reasonable system of weave specifications where every weave term is the z-ordering of a zip term. Conversely, the z-ordering of a zip specification yields a weave specification.

The array a is k, l -automatic

\Updownarrow^1

a has a weave specification

\Updownarrow^2

The sequence $\text{lin}(a)$ is kl -automatic \Leftrightarrow^3 $\text{lin}(a)$ has a zip specification

1 lin preserves automaticity

2 weave specifications are z-orderings of zip specifications

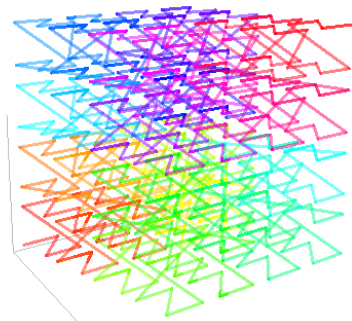
3 Grabmayer, Endrullis, Hendriks, Klop and Moss

Theorem

An array is k, l -automatic iff it has a $\mathbf{wv}_{k,l}$ specification. Moreover, it is decidable whether two weave specifications have the same solution.

Further research

- 1 This suggests a method for investigating automatic arrays.
- 2 Higher dimensions?



- 3 (Partial result) Progress toward zip specifications of mixed arity?