

Up-to techniques in metric spaces

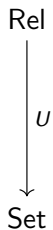
David Springer

National Institute of Informatics, Japan
ERATO MMSD Project

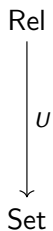
Shonan Meeting on Enhanced Coinduction
November 15, 2017

Relation Lifting

$$(X, R) \xrightarrow{f} (Y, S) \text{ means } R \subseteq f^*S$$



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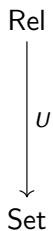


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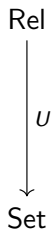


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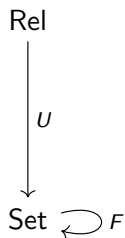
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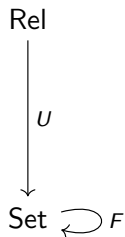
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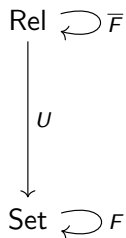
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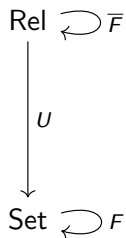
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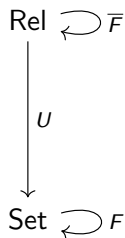
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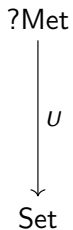
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$\alpha^* \circ \bar{F}$ is monotone on Rel_X .

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Metric Version

$$(X, r) \xrightarrow{f} (Y, s) \text{ means } f^*s \leq r$$



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?Met



U

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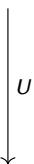
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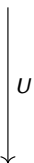
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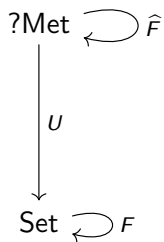
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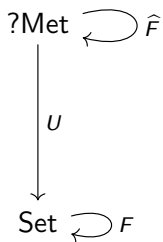
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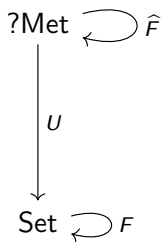
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A f is b -compatible when

$$b \circ f \leq f \circ b$$

Previous work

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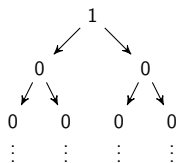
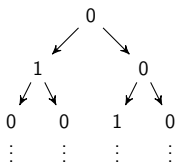
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And an account of enhanced coinduction in fibrations:

- Bonchi, Petrisan, Pous, Rot, 2017

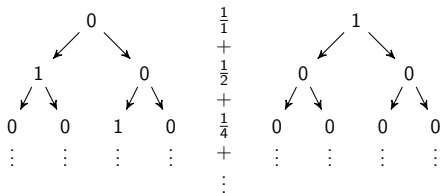
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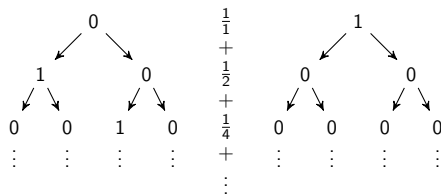
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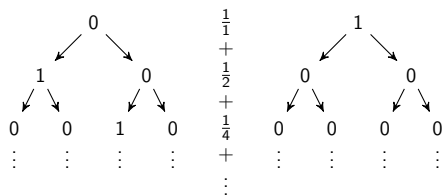
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If the above trees are the solutions for x and y in:

$$\begin{cases} \alpha(x) = (0, y, x) \\ \alpha(y) = (1, z, z) \\ \alpha(z) = (0, x, z) \end{cases}$$

how far apart are they?

Weighted graphs

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Behavioral distance on (X, α) is the least prefixed point of $\alpha^* \circ \widehat{F}$.

Compatible up-to techniques

We will make a technique called “up-to-metric closure” in analogy with “up-to-equivalence closure”:

Relation conditions	(Pseudo)Metric conditions
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$R(x, y) \leftrightarrow R(y, x)$	$r(x, y) = r(y, x)$
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(Compatible techniques are closed under meets for us.)

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For each X , define:

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A stronger but simpler condition is

$$\hat{F} eq_X \leq eq_{FX}$$

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$$met(r) = \bigwedge_{i \in \omega} \bigoplus^i (eq_X \wedge r \wedge conv(r))$$

Up-to-metric closure is compatible if:

- 1 $\widehat{F}eq_X \leq eq_{FX}$,
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Say a lifting is “good” if it satisfies (1-3)

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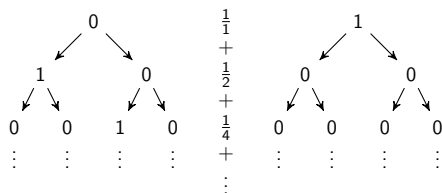
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Therefore, for $F = \mathbb{B} \times Id \times Id$, the following lift is good:

$$\widehat{F}(r)((b, x, y), (b', x', y')) = |b - b'| + \frac{1}{2}r(x, x') + \frac{1}{2}r(y, y')$$

Example: distances on trees, II

$$FX = \mathbb{B} \times X \times X$$



$$d((b, x, y), (b', x', y')) = |b - b'| + \frac{1}{2}d(x, x') + \frac{1}{2}d(y, y')$$

$$\begin{cases} \alpha(x) = (0, y, x) \\ \alpha(y) = (1, z, z) \\ \alpha(z) = (0, x, z) \end{cases}$$

Prove $d(y, z) \leq 5$.

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- Connections with existent literature.

Thanks!