Rewriting dataflow diagrams

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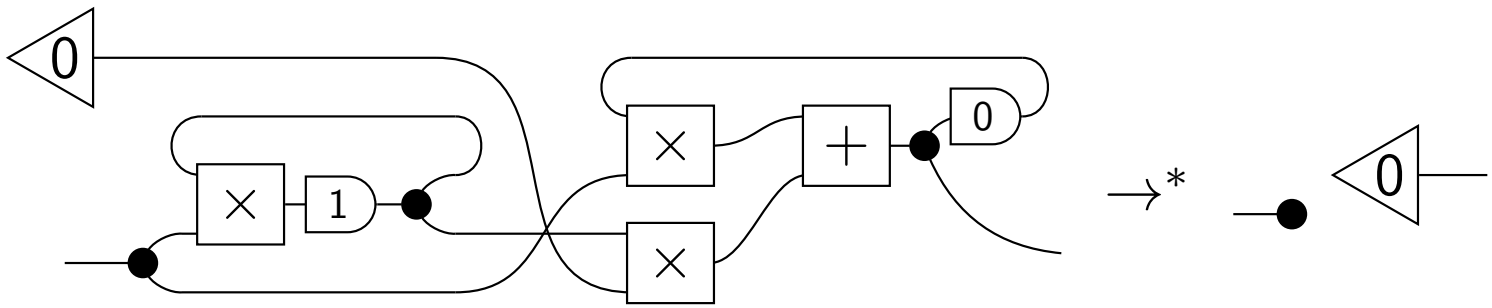
Disclaimer

I don't know ~~anything~~ much about term rewriting.

Where we're heading

I would like to use rewriting tools to simplify diagrams representing certain kinds of functions on sequences:



Causal functions on sequences

A^ω is the set of A-*valued infinite sequences*. The entries of $\sigma \in A^\omega$ are $\sigma_k \in A$ for $k \in \mathbb{N}$, so $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_k, \ldots)$.

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Definition

A *causal* function $f : A^\omega \to B^\omega$ satisfies

$$\forall \sigma, \tau \in A^\omega, \forall k \in \mathbb{N}, \sigma_{0:k} = \tau_{0:k} \to f(\sigma)_{0:k} = f(\tau)_{0:k}$$

Intuitively, the first k outputs of f only depend on the first k inputs.

Examples and non-examples

We specialize from arbitrary sets A to $\mathbb{R}$.

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3. $\mathtt{runProd} : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined by $[\mathtt{runProd}(\sigma)]_k = \prod_{i=0}^{k} \sigma_i$ is causal. The kth output depends on all the first k inputs.

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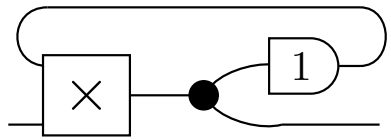
4. $\mathtt{tl} : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined by $[\mathtt{tl}(\sigma)]_k = \sigma_{k+1}$ is **not causal**, since output k is input $k + 1$.

Dataflow diagrams

Dataflow diagrams are a common visual language for defining causal functions. For example, this is one of the example causal functions:



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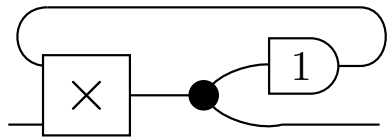
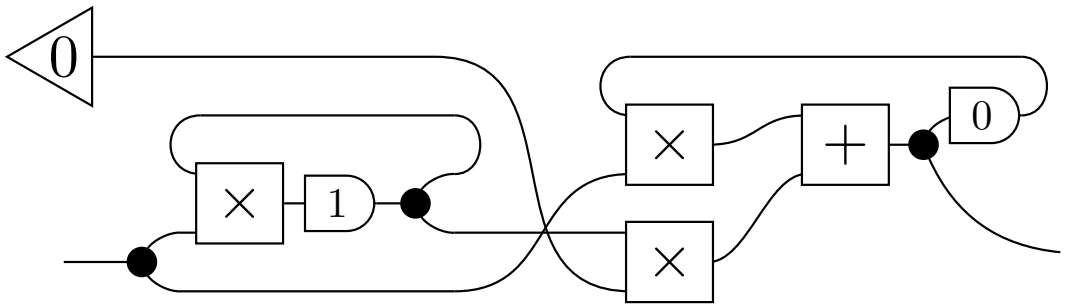
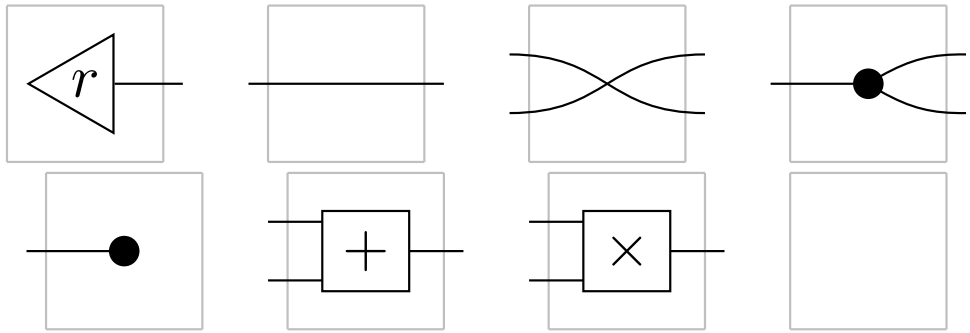


The first diagram we saw is another example, related to the derivative of the diagram above:

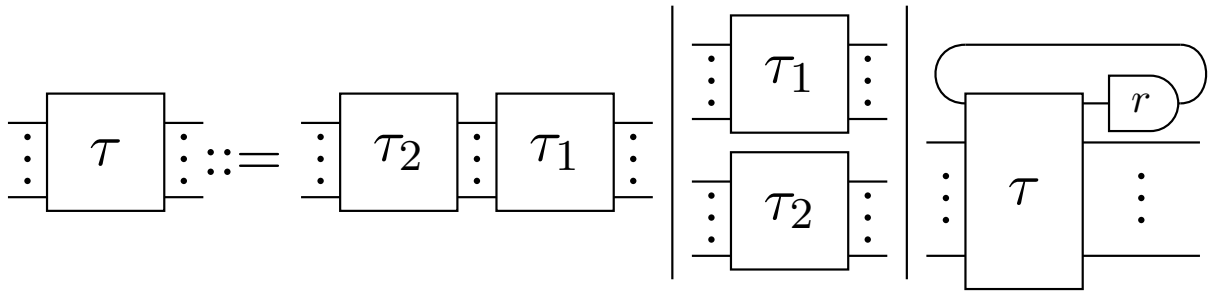


Dataflow diagrams

Dataflow diagrams are constructed from these basic components:



And closed under these operations:



In the above rules, $r \in \mathbb{R}$ is arbitrary.

Linearized language

We can use a 1-D syntax set up to parallel the 2-D syntax of diagrams. Types are $[n, m]$ (think of $[n, m] \leftrightsquigarrow \mathbb{R}^n \to \mathbb{R}^m$), and symbols are:

$$\texttt{const}_r : [0, 1] \mid \texttt{id}_1 : [1, 1] \mid \texttt{swap} : [2, 2] \mid \texttt{dup} : [1, 2] \mid$$
$$\texttt{del} : [1, 0] \mid + : [2, 1] \mid \times : [2, 1] \mid \texttt{id}_0 : [0, 0]$$

Operations take the following types (superscripts usually omitted):

$$\circ^{\ell,m,n} : [m, n] \times [\ell, m] \to [\ell, n]$$
$$\otimes^{n,p,\ell,m} : [n, p] \times [\ell, m] \to [n + \ell, p + m]$$
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We may also use variables in terms.

(Vague) problem statement

Given:

- a set of equations between terms in this language
- a size function $\|\cdot\|$ sending closed terms to non-negative real numbers

Produce:

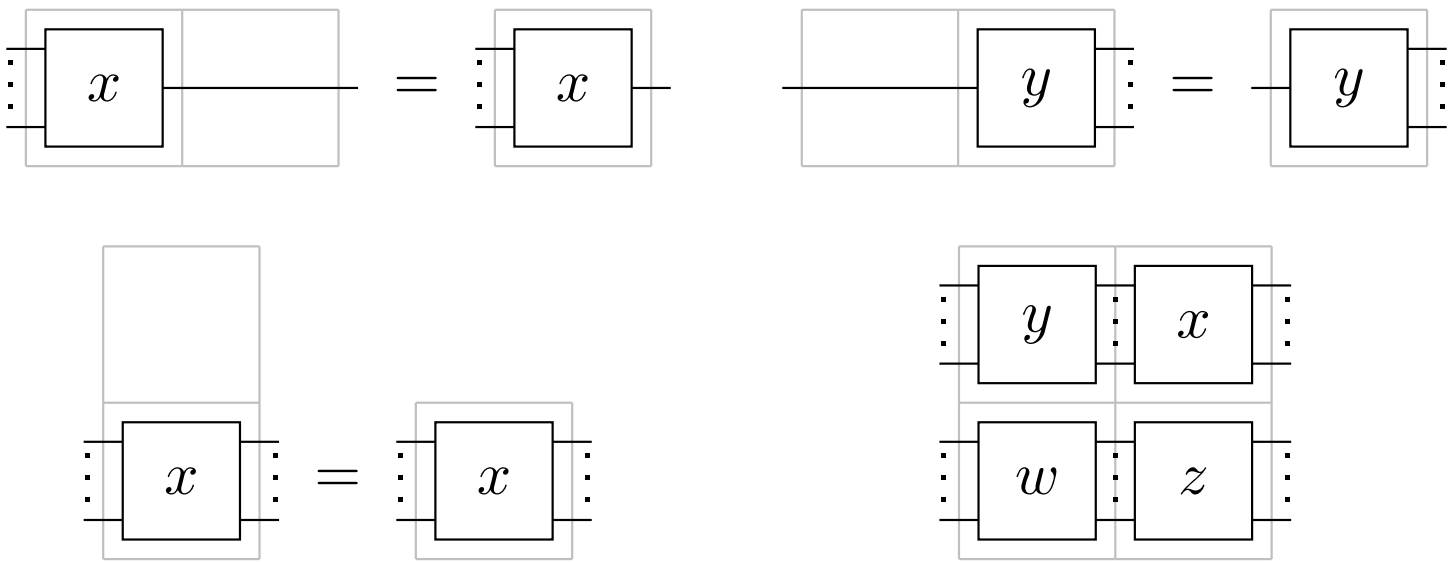
- a rewrite system taking terms to least-cost terms obtainable by applying equations

We may assume:

- equations are contextually closed under $\circ$, $\otimes$, and tr_r
- equations are closed under substitutions
- if τ_1 is a subterm of τ_2, then $\|\tau_1\| \leq \|\tau_2\|$

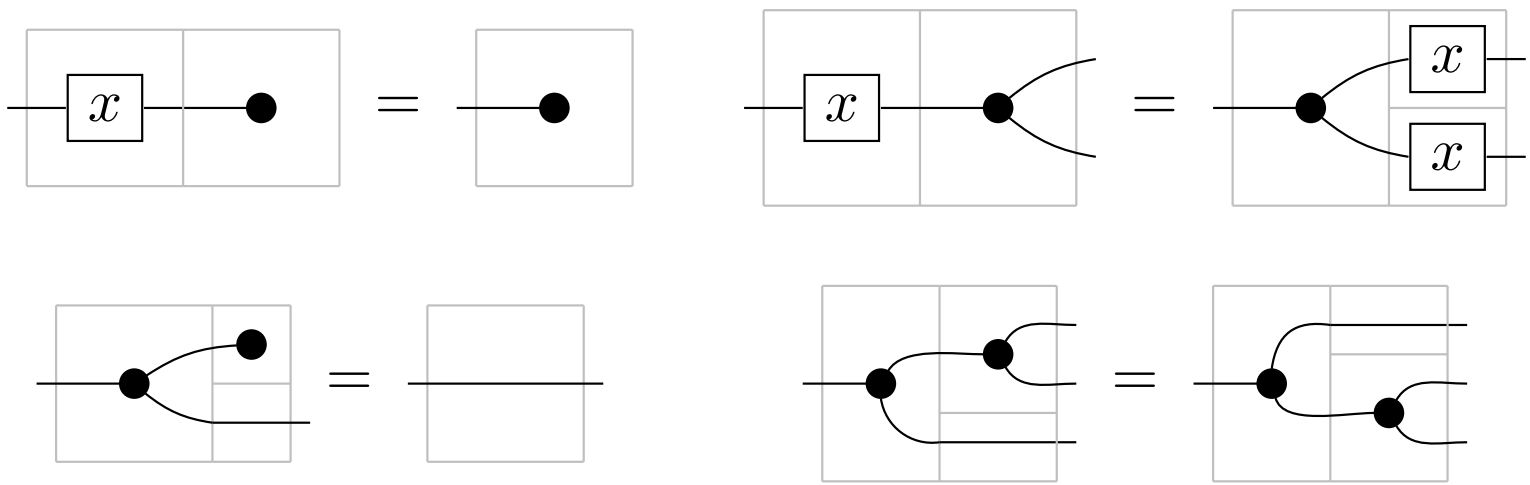
Equations I: composition

1. $\mathtt{id}_1 \circ x = x$ and $y = y \circ \mathtt{id}_1$ for all $x : [n, 1]$, $y : [1, n]$
2. $\mathtt{id}_0 \circ x = x$ and $y = y \circ \mathtt{id}_0$ for $x : [n, 0]$, $y : [0, n]$
3. $\circ$ and $\otimes$ are associative (not necessarily commutative)
4. $\mathtt{id}_0 \otimes x = x = x \otimes \mathtt{id}_0$ for all $x : [m, n]$
5. $(x \circ y) \otimes (z \circ w) = (x \otimes z) \circ (y \otimes w)$ for all ...



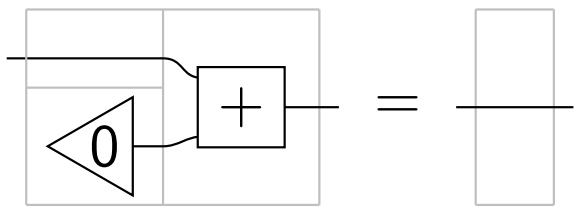
Equations II: resource management

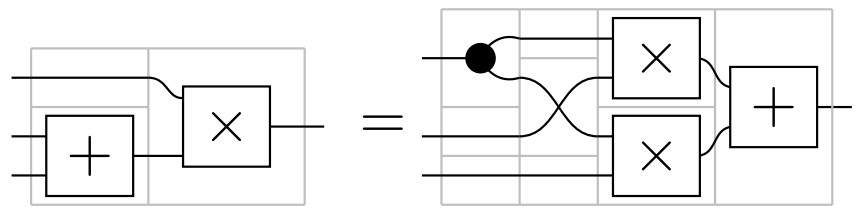
1. $\mathtt{swap} \circ \mathtt{swap} = \mathtt{id}_1 \otimes \mathtt{id}_1$
2. $\mathtt{swap} \circ (x \otimes y) = (y \otimes x) \circ \mathtt{swap}$ for all $x, y : [1, 1]$
3. $\mathtt{del} \circ x = \mathtt{del}$ for all $x : [1, 1]$
4. $\mathtt{dup} \circ x = (x \otimes x) \circ \mathtt{dup}$ for all $x : [1, 1]$
5. $(\mathtt{del} \otimes \mathtt{id}_1) \circ \mathtt{dup} = \mathtt{id}_1$
6. $\mathtt{swap} \circ \mathtt{dup} = \mathtt{dup}$
7. $(\mathtt{id}_1 \otimes \mathtt{dup}) \circ \mathtt{dup} = (\mathtt{dup} \otimes \mathtt{id}_1) \circ \mathtt{dup}$

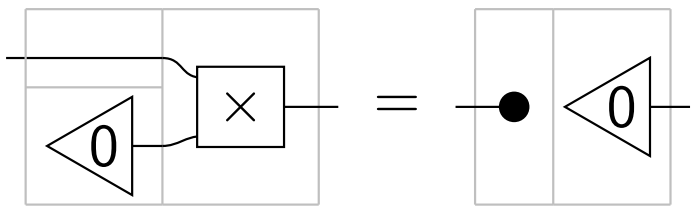


Equations III: semiring axioms

1. $+ \circ \mathtt{swap} = +$ (also for $\times$)
2. $+ \circ (\mathtt{id}_1 \otimes +) = + \circ (+ \otimes \mathtt{id}_1)$ (also for $\times$)
3. $+ \circ (\mathtt{id}_1 \otimes \mathtt{const}_0) = \mathtt{id}_1$ and $\times \circ (\mathtt{id}_1 \otimes \mathtt{const}_1) = \mathtt{id}_1$
4. $\times \circ (\mathtt{id}_1 \otimes \mathtt{const}_0) = \mathtt{const}_0 \circ \mathtt{del}$
5. $+ \circ (\mathtt{const}_r \otimes \mathtt{const}_s) = \mathtt{const}_{r+s}$ for all $r, s \in \mathbb{R}$
6. $\times \circ (\mathtt{const}_r \otimes \mathtt{const}_s) = \mathtt{const}_{rs}$ for all $r, s \in \mathbb{R}$
7. $\times \circ (\mathtt{id}_1 \otimes +) =$
   $+ \circ (\times \otimes \times) \circ (\mathtt{id}_1 \otimes \mathtt{swap} \otimes \mathtt{id}_1) \circ (\mathtt{dup} \otimes \mathtt{id}_1 \otimes \mathtt{id}_1)$

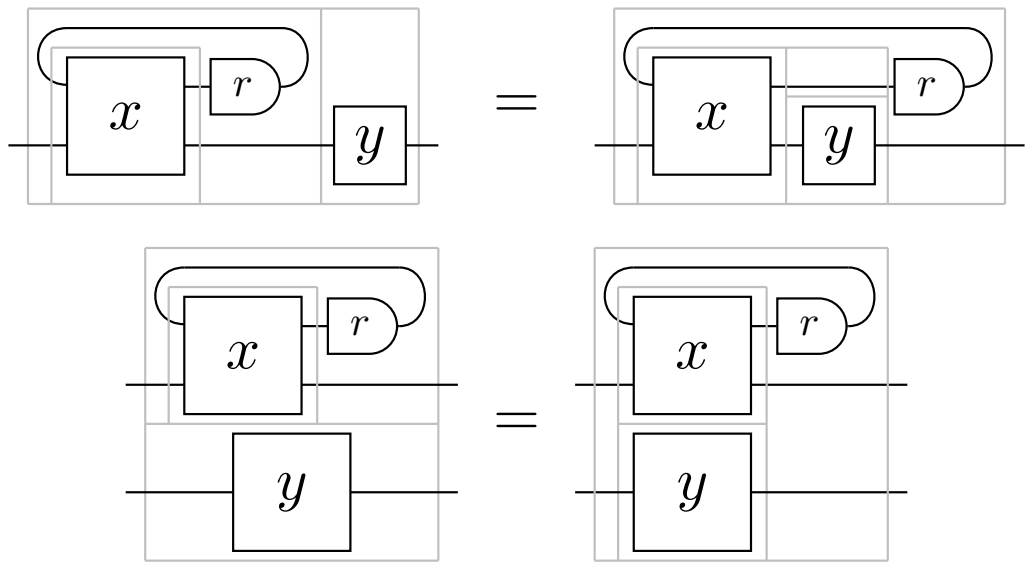






Equations IV: trace axioms

1. $tr_r(x) \circ y = tr_r(x \circ (\mathtt{id}_1 \otimes y))$ for all ...
2. $y \circ tr_r(x) = tr_r((\mathtt{id}_1 \otimes y) \circ x)$ for all ...
3. $tr_r(x) \otimes y = tr_r(x \otimes y)$ for all ...
4. $tr_r(\mathtt{dup}) = \mathtt{const}_r$



Symbol sizes

Nullary symbols represent machine instructions or simple components of circuitry. We are given some basic information about the resources they consume: $\| \cdot \| : \Sigma_0 \to [0, \infty)$. Think of this as measuring time or power.

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For example,

$$\|\mathtt{id}_0\| = 0 \quad \|\mathtt{id}_1\| = 1 \quad \|\mathtt{del}\| = 5 \quad \|\mathtt{const}_r\| = 7$$
$$\|+\| = 10 \quad \|\mathtt{dup}\| = 15 \quad \|\mathtt{swap}\| = 30 \quad \|\times\| = 40$$

Term sizes

There are various ways to extend these to terms:

$$\|\tau\|_{\mathsf{work}} \triangleq \begin{cases} \|\tau\| & \text{if } \tau \in \Sigma_0 \\ \|\tau_1\|_{\mathsf{work}} + \|\tau_2\|_{\mathsf{work}} & \text{if } \tau = \tau_2 \circ \tau_1 \\ \|\tau_1\|_{\mathsf{work}} + \|\tau_2\|_{\mathsf{work}} & \text{if } \tau = \tau_1 \otimes \tau_2 \\ \|\tau_1\|_{\mathsf{work}} + \epsilon & \text{if } \tau = tr_r(\tau_1) \end{cases}$$

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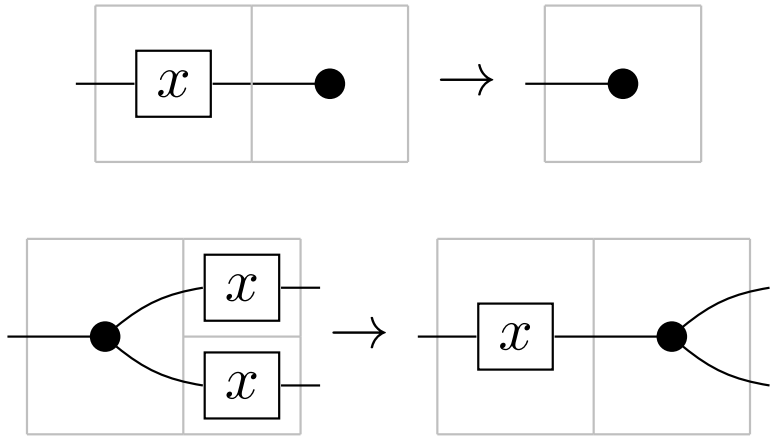
$$\|\tau\|_{\mathsf{span}} \triangleq \begin{cases} \|\tau\| & \text{if } \tau \in \Sigma_0 \\ \|\tau_1\|_{\mathsf{span}} + \|\tau_2\|_{\mathsf{span}} & \text{if } \tau = \tau_2 \circ \tau_1 \\ \max(\|\tau_1\|_{\mathsf{span}}, \|\tau_2\|_{\mathsf{span}}) & \text{if } \tau = \tau_1 \otimes \tau_2 \\ \|\tau_1\|_{\mathsf{span}} + \epsilon' & \text{if } \tau = tr_r(\tau_1) \end{cases}$$

Orienting equations by size

Two important orientable equations include:



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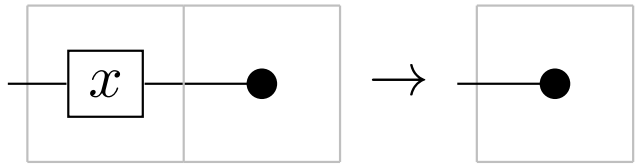
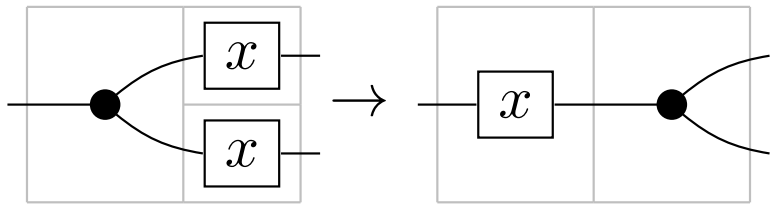
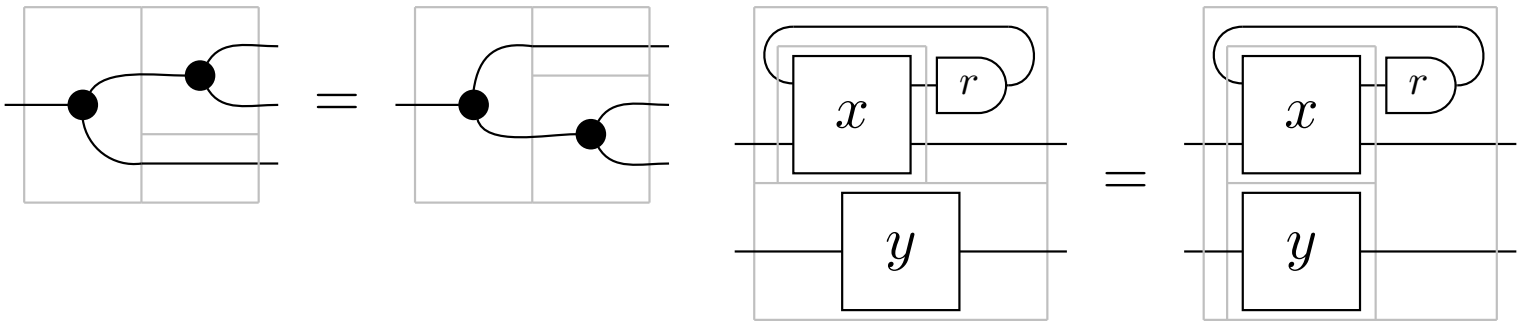




Many other equations don't have an obvious orientation, like:



Rewriting to produce smaller diagrams

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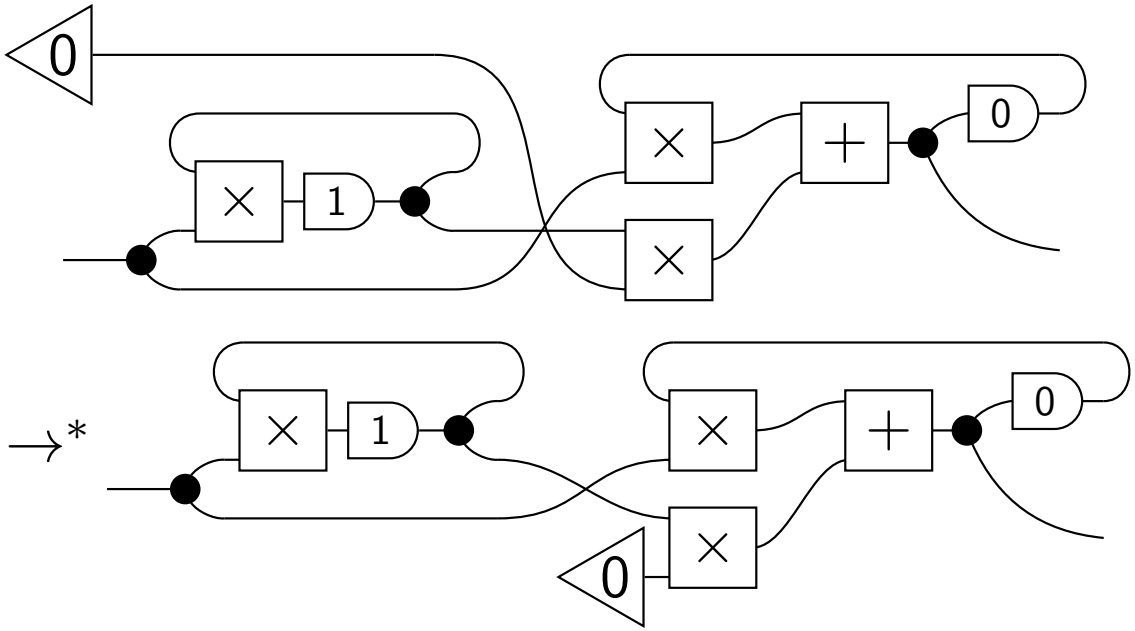

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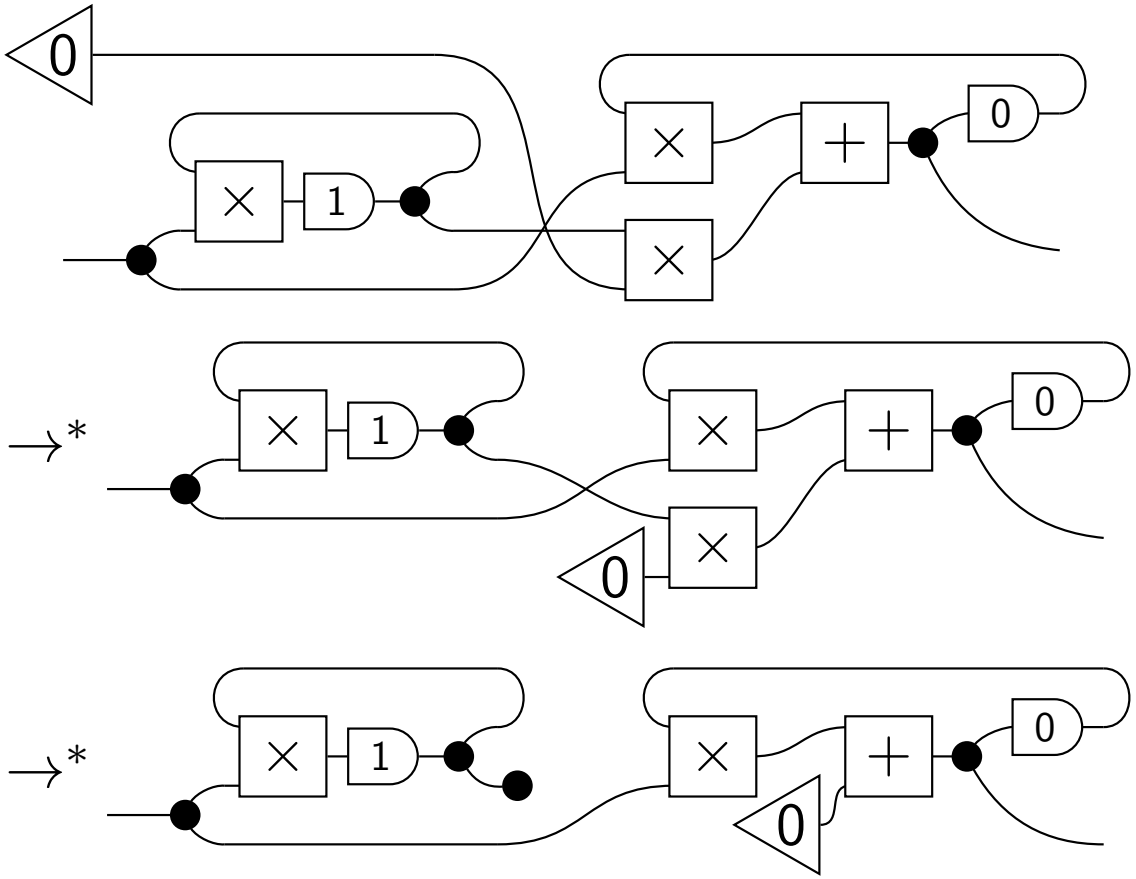

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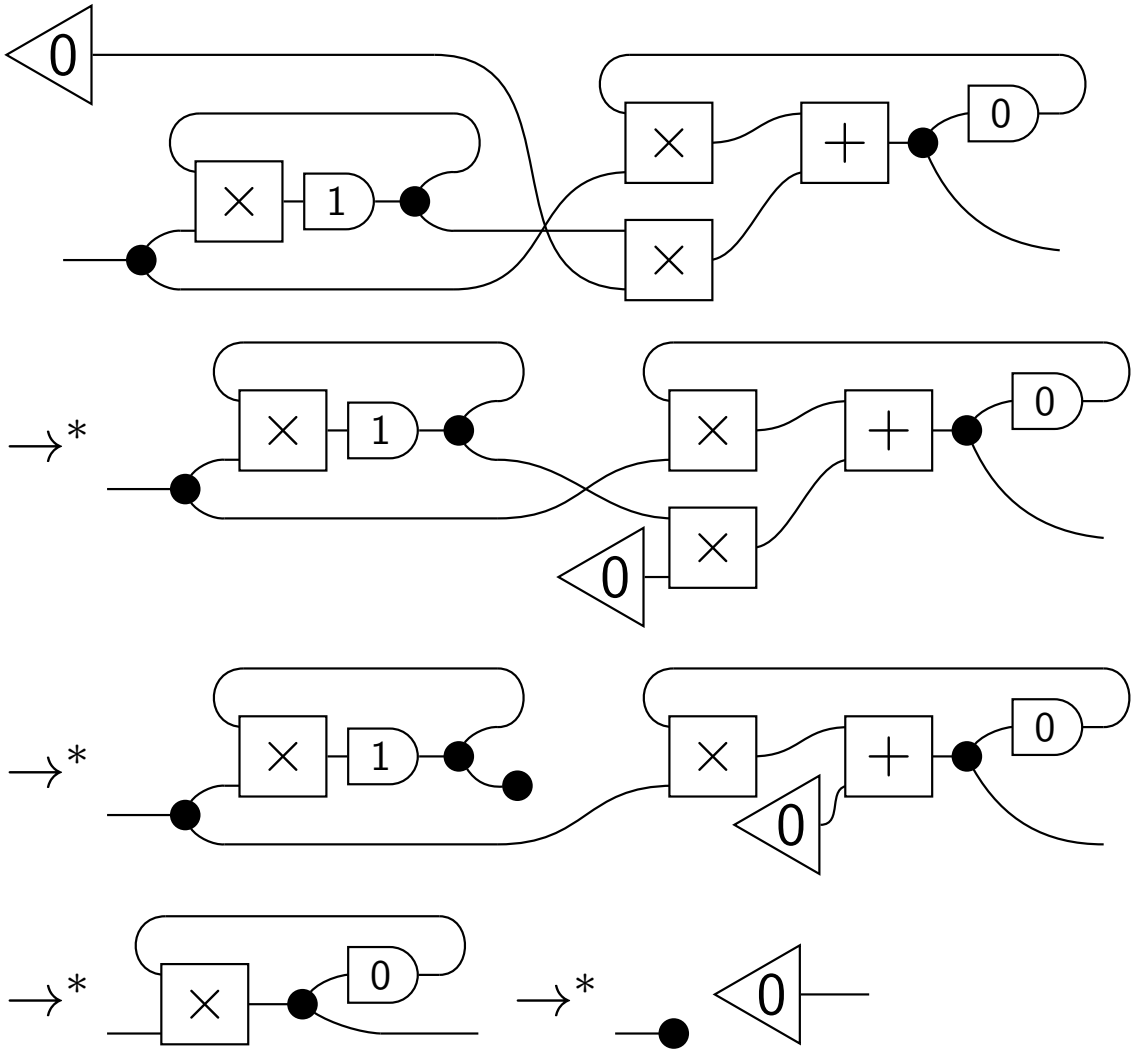

Problem extensions

1. Allow infinitely many symbols and equations, often defined by recursion. For example, $\mathtt{id}_n := \mathtt{id}_1 \otimes \mathtt{id}_{n-1}$ and $f = f \circ \mathtt{id}_n$ for all $f : [n, m]$
2. If we add finitely many symbols and equations, can we obtain a rewrite system for the new language from the rewrite system for the old language? (e.g. adding $\sin, \cos : [1, 1]$ and related equations)
3. Add further typing constraints to the wires in addition to co/arity, e.g. $\mathtt{if} : \mathbb{B} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$
4. Constrained optimizations, e.g. minimize $\|\tau\|_{\text{work}}$ subject to $\|\tau\|_{\text{span}} \leq T$
5. Multi-objective optimizations, e.g. find the Pareto front for $\|\tau\|_{\text{work}}$ and $\|\tau\|_{\text{span}}$

Thanks!