Rewriting dataflow diagrams

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Disclaimer

I don’t know anything much about term rewriting.
Where we’re heading

I would like to use rewriting tools to simplify diagrams representing certain kinds of functions on sequences:
Causal functions on sequences

$A^\omega$ is the set of $A$-valued infinite sequences. The entries of $\sigma \in A^\omega$ are $\sigma_k \in A$ for $k \in \mathbb{N}$, so $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_k, \ldots)$.
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$$( \cdot )_{i:j} : \sigma \mapsto (\sigma_i, \sigma_{i+1}, \ldots, \sigma_j)$$
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**Definition**

A **causal** function $f : A^\omega \rightarrow B^\omega$ satisfies

$$\forall \sigma, \tau \in A^\omega, \forall k \in \mathbb{N}, \sigma_{0:k} = \tau_{0:k} \rightarrow f(\sigma)_{0:k} = f(\tau)_{0:k}$$

Intuitively, the first $k$ outputs of $f$ only depend on the first $k$ inputs.
Examples and non-examples

We specialize from arbitrary sets $A$ to $\mathbb{R}$.

1. $\bar{r} : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ defined by $[\bar{r}(\sigma)]_k = r$ is causal. The $k$th output depends on nothing.

2. If $g : \mathbb{R} \rightarrow \mathbb{R}$, then $\text{map}(g) : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ defined by $[\text{map}(g)(\sigma)]_k = g(\sigma_k)$ is causal. The $k$th output of this function depends on the $k$th input only.

3. $\text{runProd} : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ defined by $[\text{runProd}(\sigma)]_k = \prod_{i=0}^{k} \sigma_i$ is causal. The $k$th output depends on all the first $k$ inputs.

4. $\text{tl} : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ defined by $[\text{tl}(\sigma)]_k = \sigma_{k+1}$ is not causal, since output $k$ is input $k + 1$. 
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Dataflow diagrams

Dataflow diagrams are a common visual language for defining causal functions. For example, this is one of the example causal functions:

\[ \times \]

The first diagram we saw is another example, related to the derivative of the diagram above:

\[ \times + 0 \]
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Dataflow diagrams

Dataflow diagrams are constructed from these basic components:

And closed under these operations:

In the above rules, \( r \in \mathbb{R} \) is arbitrary.
Linearized language

We can use a 1-D syntax set up to parallel the 2-D syntax of diagrams. Types are \([n, m]\) (think of \([n, m] \leftrightarrow \mathbb{R}^n \rightarrow \mathbb{R}^m\)), and symbols are:

\[
\begin{align*}
\text{const}_r : [0, 1] & \mid \text{id}_1 : [1, 1] & \mid \text{swap} : [2, 2] & \mid \text{dup} : [1, 2] \mid \\
\text{del} : [1, 0] & \mid + : [2, 1] & \mid \times : [2, 1] & \mid \text{id}_0 : [0, 0]
\end{align*}
\]

Operations take the following types (superscripts usually omitted):

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\begin{align*}
\circ^{\ell, m, n} : [m, n] \times [\ell, m] & \rightarrow [\ell, n] \\
\otimes^{n, p, \ell, m} : [n, p] \times [\ell, m] & \rightarrow [n + \ell, p + m] \\
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In this language, the running product function is \(tr_1(dup \circ \times)\).

We may also use variables in terms.
(Vague) problem statement

Given:

- a set of equations between terms in this language
- a size function $\| \cdot \|$ sending closed terms to non-negative real numbers

Produce:

- a rewrite system taking terms to least-cost terms obtainable by applying equations

We may assume:

- equations are contextually closed under $\circ$, $\otimes$, and $tr_r$
- equations are closed under substitutions
- if $\tau_1$ is a subterm of $\tau_2$, then $\| \tau_1 \| \leq \| \tau_2 \|$
Equations I: composition

1. $\text{id}_1 \circ x = x$ and $y = y \circ \text{id}_1$ for all $x : [n, 1]$, $y : [1, n]$
2. $\text{id}_0 \circ x = x$ and $y = y \circ \text{id}_0$ for $x : [n, 0]$, $y : [0, n]$
3. $\circ$ and $\otimes$ are associative (not necessarily commutative)
4. $\text{id}_0 \otimes x = x = x \otimes \text{id}_0$ for all $x : [m, n]$
5. $(x \circ y) \otimes (z \circ w) = (x \otimes z) \circ (y \otimes w)$ for all ...
Equations II: resource management

1. \( \text{swap} \circ \text{swap} = \text{id}_1 \otimes \text{id}_1 \)
2. \( \text{swap} \circ (x \otimes y) = (y \otimes x) \circ \text{swap} \) for all \( x, y : [1, 1] \)
3. \( \text{del} \circ x = \text{del} \) for all \( x : [1, 1] \)
4. \( \text{dup} \circ x = (x \otimes x) \circ \text{dup} \) for all \( x : [1, 1] \)
5. \( (\text{del} \otimes \text{id}_1) \circ \text{dup} = \text{id}_1 \)
6. \( \text{swap} \circ \text{dup} = \text{dup} \)
7. \( (\text{id}_1 \otimes \text{dup}) \circ \text{dup} = (\text{dup} \otimes \text{id}_1) \circ \text{dup} \)
Equations III: semiring axioms

1. $+ \circ \text{swap} = +$ (also for $\times$)
2. $+ \circ (\text{id}_1 \otimes +) = + \circ (+ \otimes \text{id}_1)$ (also for $\times$)
3. $+ \circ (\text{id}_1 \otimes \text{const}_0) = \text{id}_1$ and $\times \circ (\text{id}_1 \otimes \text{const}_1) = \text{id}_1$
4. $\times \circ (\text{id}_1 \otimes \text{const}_0) = \text{const}_0 \circ \text{del}$
5. $+ \circ (\text{const}_r \otimes \text{const}_s) = \text{const}_{r+s}$ for all $r, s \in \mathbb{R}$
6. $\times \circ (\text{const}_r \otimes \text{const}_s) = \text{const}_{rs}$ for all $r, s \in \mathbb{R}$
7. $\times \circ (\text{id}_1 \otimes +) =$
   $+ \circ (\times \otimes \times) \circ (\text{id}_1 \otimes \text{swap} \otimes \text{id}_1) \circ (\text{dup} \otimes \text{id}_1 \otimes \text{id}_1)$
Equations IV: trace axioms

1. $tr_r(x) \circ y = tr_r(x \circ (id_1 \otimes y))$ for all ...
2. $y \circ tr_r(x) = tr_r((id_1 \otimes y) \circ x)$ for all ...
3. $tr_r(x) \otimes y = tr_r(x \otimes y)$ for all ...
4. $tr_r(dup) = const_r$
Symbol sizes

Nullary symbols represent machine instructions or simple components of circuitry. We are given some basic information about the resources they consume: $∥ \cdot ∥ : \Sigma_0 \rightarrow [0, \infty)$. Think of this as measuring time or power.
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For example,

$\| id_0 \| = 0 \quad \| id_1 \| = 1 \quad \| del \| = 5 \quad \| const_r \| = 7$

$\| + \| = 10 \quad \| dup \| = 15 \quad \| swap \| = 30 \quad \| \times \| = 40$
Term sizes

There are various ways to extend these to terms:

\[ \| \tau \|_{\text{work}} \triangleq \begin{cases} 
\| \tau \| & \text{if } \tau \in \Sigma_0 \\
\| \tau_1 \|_{\text{work}} + \| \tau_2 \|_{\text{work}} & \text{if } \tau = \tau_2 \circ \tau_1 \\
\| \tau_1 \|_{\text{work}} + \| \tau_2 \|_{\text{work}} & \text{if } \tau = \tau_1 \otimes \tau_2 \\
\| \tau_1 \|_{\text{work}} + \epsilon & \text{if } \tau = \text{tr}_r(\tau_1)
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\[ \| \tau \|_{\text{span}} \triangleq \begin{cases} \| \tau \| & \text{if } \tau \in \Sigma_0 \\ \| \tau_1 \|_{\text{span}} + \| \tau_2 \|_{\text{span}} & \text{if } \tau = \tau_2 \circ \tau_1 \\ \max(\| \tau_1 \|_{\text{span}}, \| \tau_2 \|_{\text{span}}) & \text{if } \tau = \tau_1 \otimes \tau_2 \\ \| \tau_1 \|_{\text{span}} + \epsilon' & \text{if } \tau = tr_r(\tau_1) \end{cases} \]
Orienting equations by size

Two important orientable equations include:
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\begin{align*}
\begin{array}{c}
\text{x} \\
\text{→}
\end{array}
& \quad \rightarrow \\
\begin{array}{c}
\text{→}
\end{array}
\end{align*}
\]

Many other equations don’t have an obvious orientation, like:

\[
\begin{align*}
\begin{array}{c}
\text{x} \\
\text{x}
\end{array}
& \quad \rightarrow \\
\begin{array}{c}
\text{x} \\
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\end{align*}
\]
Rewriting to produce smaller diagrams

We would like a rewriting system that helps obtain the following reasoning.
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\[
\begin{array}{c}
\times 1 \\
\rightarrow \ast
\end{array}
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Rewriting to produce smaller diagrams

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Problem extensions

1. Allow infinitely many symbols and equations, often defined by recursion. For example, $\text{id}_n := \text{id}_1 \otimes \text{id}_{n-1}$ and $f = f \circ \text{id}_n$ for all $f : [n,m]$

2. If we add finitely many symbols and equations, can we obtain a rewrite system for the new language from the rewrite system for the old language? (e.g. adding $\sin, \cos : [1,1]$ and related equations)

3. Add further typing constraints to the wires in addition to co/arity, e.g. $\text{if} : \mathbb{B} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$

4. Constrained optimizations, e.g. minimize $\|\tau\|_{\text{work}}$ subject to $\|\tau\|_{\text{span}} \leq T$

5. Multi-objective optimizations, e.g. find the Pareto front for $\|\tau\|_{\text{work}}$ and $\|\tau\|_{\text{span}}$
Thanks!