

Eigenvalues and Transduction of Morphic Sequences: Extended Version*

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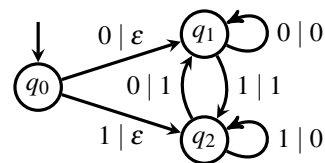
Abstract

We study finite state transduction of automatic and morphic sequences. Dekking [4] proved that morphic sequences are closed under transduction and in particular morphic images. We present a simple proof of this fact, and use the construction in the proof to show that non-erasing transductions preserve a condition called α -substitutivity. Roughly, a sequence is α -substitutive if the sequence can be obtained as the limit of iterating a substitution with dominant eigenvalue α . Our results culminate in the following fact: for multiplicatively independent real numbers α and β , if v is a α -substitutive sequence and w is a β -substitutive sequence, then v and w have no common non-erasing transducts except for the ultimately periodic sequences. We rely on Cobham's theorem for substitutions, a recent result of Durand [5].

1 Introduction

Infinite sequences of symbols are of paramount importance in a wide range of fields, ranging from formal languages to pure mathematics and physics. A landmark was the discovery in 1912 by Axel Thue, founding father of formal language theory, of the famous sequence 0110 1001 1001 0110 1001 0110 \dots . Thue was interested in infinite words which avoid certain patterns, like squares ww or cubes www , when w is a non-empty word. Indeed, the sequence shown above, called the Thue–Morse sequence, is cube-free. It is perhaps the most natural cube-free infinite word.

A common way to transform infinite sequences is by using *finite state transducers*. These transducers are deterministic finite automata with input letters and output words for each transition; an example is shown in Figure 1. Usually we omit the words “finite state” and refer to *transducers*. A transducer maps infinite sequences to infinite sequences by reading the input sequence letter by letter. Each of these transitions produces an output word, and the sequence formed by concatenating each of these output words in the order they were produced is the output sequence. In particular, since this transducer runs for infinite time to read its entire input, this model of transduction



■ **Figure 1** A transducer computing the difference (exclusive or) of consecutive bits.

* This is an extended version of our paper [9] presented at *Developments in Language Theory 2014*. This extended version contains examples and additional remarks.

does not have final states. A transducer is called *k-uniform* if each step produces *k*-letter words. For example, *Mealy machines* are 1-uniform transducers. A transducer is *non-erasing* if each step produces a non-empty word; this condition is prominent in this paper.

Although transducers are a natural machine model, hardly anything is known about their capabilities of transforming infinite sequences. To state the issues more clearly, let us write $x \preceq y$ if there is a transducer taking y to x . This transducibility gives rise to a partial order of *stream degrees* [6] that is analogous to, but more fine-grained than, recursion-theoretic orderings such as *Turing reducibility* \leq_T and *many-one reducibility* \leq_m . We find it surprising that so little is known about \preceq . As of now, the structure of this order is vastly unexplored territory with many open questions. To answer these questions, we need a better understanding of transducers.

The main things that are known at this point concern two particularly well-known sets of streams, namely the *morphic* and *automatic* sequences. Morphic sequences are obtained as the limit of iterating a morphism on a starting word (and perhaps applying a coding to the limit word). Automatic sequences have a number of independent characterizations (see [1]); we shall not repeat these here. There are two seminal closure results concerning the transduction of morphic and automatic sequences:

- (1) The class of morphic sequences is closed under transduction (Dekking [4]).
- (2) For all k , the class of k -automatic sequences is closed under uniform transduction (Cobham [3]).

The restriction in (2) to uniform transducers is shown by the following example.

► **Example 1.** Let $w \in \{0,1\}^\omega$ be defined by $w(n) = 1$ if n is a power of 2 and $w(n) = 0$ otherwise. This sequence is 2-automatic. Let h be the morphism $0 \mapsto 0$ and $1 \mapsto 01$. Taking the image of w under h , that is $h(w)$, yields a sequence that is no longer automatic (but still morphic). Here is a sketch that $h(w)$ is not 2-automatic. Note that the i^{th} digit in $h(w)$ is 1 iff $i = 2^n + n$ for some n . Suppose that M is a finite-state machine with the property that reading in each number i in binary yields the i^{th} digit of $h(w)$. Let N be large enough so that the binary representation of $2^N + N$ has a run of zeroes longer than the number of states in N . Then by pumping, N must accept a number which is not of the form $2^n + n$.

In this paper, we do not attack the central problems concerning the stream degrees. Instead, we are interested in a closure result for non-erasing transductions. Our interest comes from the following easy observation:

- (3) For every morphic sequence $w \in \Sigma^\omega$ there is a 2-automatic sequence $w' \in (\Sigma \cup \{a\})^\omega$ such that w is obtained from w' by erasing all occurrences of a . (See Allouche and Shallit [1, Theorem 7.7.1])

This motivates the question: how powerful is non-erasing transduction?

Our contribution

The main result of this paper is stated in terms of the notion of *α -substitutivity*. This condition is defined in Definition 8 below, and the definition uses the eigenvalues of matrices naturally associated with morphisms on finite alphabets. Indeed, the core of our work is a collection of results on eigenvalues of these matrices.

We prove that the set of α -substitutive words is closed under non-erasing finite state transduction. We follow Allouche and Shallit [1] in obtaining transducts of a given morphic

sequence w by *annotating* an iteration morphism, and then taking a morphic image of the annotated limit sequence. For the first part of this transformation, we show that a morphism and its annotation have the same eigenvalues with non-negative eigenvectors. For the second part, we revisit the proof given in Allouche and Shallit [1] of Dekking's theorem that morphic images of morphic sequences are morphic. We simplify the construction in the proof to make it amenable for an analysis of the eigenvalues of the resulting morphism.

Related work

Durand [5] proved that if w is an α -substitutive sequence and h is a non-erasing morphism, then $h(w)$ is α^k -substitutive for some $k \in \mathbb{N}$. We strengthen this result in two directions. First, we show that k may be taken to be 1; hence $h(w)$ is α^k -substitutive for every $k \in \mathbb{N}$. Second, we show that Durand's result also holds for non-erasing transductions.

2 Preliminaries

We recall some of the main concepts that we use in the paper. For a thorough introduction to morphic sequences, automatic sequences and finite state transducers, we refer to [1, 8].

We are concerned with infinite sequences Σ^ω over a finite alphabet Σ . We write Σ^* for the set of finite words, Σ^+ for the finite, non-empty words, Σ^ω for the infinite words, and $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ for all finite or infinite words over Σ .

2.1 Morphic sequences and automatic sequences

► **Definition 2.** A *morphism* is a map $h : \Sigma \rightarrow \Gamma^*$. This map extends by concatenation to $h : \Sigma^* \rightarrow \Gamma^*$, and we do not distinguish the two notationally. Notice also that $h(vu) = h(v)h(u)$ for all $u, v \in \Sigma^*$. If $h_1, h_2 : \Sigma \rightarrow \Sigma^*$, we have a composition $h_2 \circ h_1 : \Sigma \rightarrow \Sigma^*$.

An *erased letter* (with respect to h) is some $a \in \Sigma$ such that $h(a) = \varepsilon$. A morphism $h : \Sigma^* \rightarrow \Gamma^*$ is called *erasing* if has an erased letter. A morphism is *k-uniform* (for $k \in \mathbb{N}$) if $|h(a)| = k$ for all $a \in \Sigma$. A *coding* is a 1-uniform morphism $c : \Sigma \rightarrow \Gamma$.

A morphic sequence is obtained by iterating a morphism, and applying a coding to the limit word.

► **Definition 3.** Let $s \in \Sigma^+$ be a word, $h : \Sigma \rightarrow \Sigma^*$ a morphism, and $c : \Sigma \rightarrow \Gamma$ a coding. If the limit $h^\omega(s) = \lim_{n \rightarrow \infty} h^n(s)$ exists and is infinite, then $h^\omega(s)$ is a *pure morphic sequence*, and $c(h^\omega(s))$ a *morphic sequence*.

If $h(x_1) = x_1z$ for some $z \in \Sigma^+$, then we say that h is *prolongable* on x_1 . In this case, $h^\omega(x_1)$ is a pure morphic sequence.

If additionally, the morphism h is k -uniform, then $c(h^\omega(s))$ is a *k-automatic sequence*. A sequence $w \in \Sigma^\omega$ is called *automatic* if w is k -automatic for some $k \in \mathbb{N}$.

► **Example 4.** A well-known example of a purely morphic word is the Thue–Morse sequence. This sequence can be obtained as the limit of iterating the morphism $0 \mapsto 01$, $1 \mapsto 10$ on the starting word 0. The first iterations are

$$0 \mapsto 01 \mapsto 0110 \mapsto 01101001 \mapsto 0110100110010110 \mapsto \dots,$$

and they converge, in the limit, to the Thue–Morse sequence. As the morphism h is 2-uniform, the sequence is also 2-automatic.

► **Example 5.** An example of a purely morphic word which is not automatic is provided by the *Fibonacci substitution* $a \mapsto ab, b \mapsto a$. Starting with a , the fixed point is

$$abaababa \dots$$

2.2 Cobham's Theorem for morphic words

► **Definition 6.** For $a \in \Sigma$ and $w \in \Sigma^*$ we write $|w|_a$ for the *number of occurrences of a in w* . Let h be a morphism over Σ . The *incidence matrix* of h is the matrix $M_h = (m_{i,j})_{i \in \Sigma, j \in \Sigma}$ where $m_{i,j} = |h(j)|_i$ is the number of occurrences of the letter i in the word $h(j)$.

► **Theorem 7 (Perron-Frobenius).** *Every non-negative square matrix M has a real eigenvalue $\alpha \geq 0$ that is greater than or equal to the absolute value of any other eigenvalue of M and the corresponding eigenvector is non-negative. We refer to α as the dominating eigenvalue of M .*

► **Definition 8.** The *dominating eigenvalue* of a morphism h is the dominating eigenvalue of M_h . An infinite sequence $w \in \Sigma^\omega$ over a finite alphabet Σ is said to be α -*substitutive* ($\alpha \in \mathbb{R}$) if there exist a morphism $h : \Sigma \rightarrow \Sigma^*$ with dominating eigenvalue α , a coding $c : \Sigma \rightarrow \Sigma$ and a letter $a \in \Sigma$ such that (i) $w = c(h^\omega(a))$, and (ii) every letter of Σ occurs in $h^\omega(a)$.

► **Remark.** Let us remark on the importance of the condition (ii) in Definition 8. Without this condition every α -substitutive sequence $w \in \Sigma^\omega$ would also be β -substitutive for every $\beta > \alpha$ that is the dominating eigenvalue of a non-negative integer matrix.

This can be seen as follows. Let $h : \Sigma \rightarrow \Sigma^*$ be a morphism with dominating eigenvalue α . Let $a \in \Sigma$ such that $w = h^\omega(a)$ exists, is infinite and contains all letters from Σ . Then w is α -substitutive. Now let $\beta > \alpha$ be the dominating eigenvalue of a non-negative integer matrix. Then there exists an alphabet Γ (disjoint from Σ , $\Gamma \cup \Sigma = \emptyset$) and a morphism $g : \Gamma \rightarrow \Gamma^*$ with dominating eigenvalue β . Define $z : (\Sigma \cup \Gamma) \rightarrow (\Sigma \cup \Gamma)^*$ by $z(b) = h(b)$ for all $b \in \Sigma$ and $z(c) = g(c)$ for all $c \in \Gamma$. Then $z^\omega(a) = h^\omega(a) = w$ and the dominating eigenvalue of z is β .

Two complex numbers x, y are called *multiplicatively independent* if for all $k, \ell \in \mathbb{Z}$ it holds that $x^k = y^\ell$ implies $k = \ell = 0$. We shall use the following version of Cobham's theorem due to Durand [5].

► **Theorem 9.** *Let α and β be multiplicatively independent Perron numbers. If a sequence w is both α -substitutive and β -substitutive, then w is eventually periodic.* ◀

2.3 Transducers

► **Definition 10.** A *(sequential finite-state stream) transducer (FST)* $M = (\Sigma, \Delta, Q, q_0, \delta, \lambda)$ consists of

- (i) a finite input alphabet Σ ,
- (ii) a finite output alphabet Δ ,
- (iii) a finite set of states Q ,
- (iv) an initial state $q_0 \in Q$,
- (v) a transition function $\delta : Q \times \Sigma \rightarrow Q$, and
- (vi) an output function $\lambda : Q \times \Sigma \rightarrow \Delta^*$.

► **Example 11.** The transducer $(\Sigma, \Delta, Q, q_0, \delta, \lambda)$ shown in Figure 1 can be defined as follows: $\Sigma = \Delta = \{0, 1\}$, $Q = \{q_0, q_1, q_2\}$ with q_0 the initial state, and the transition function δ and output function λ are given by:

$$\begin{array}{llll} \delta(q_0, 0) = q_1 & \lambda(q_0, 0) = \varepsilon & \delta(q_0, 1) = q_2 & \lambda(q_0, 1) = \varepsilon \\ \delta(q_1, 0) = q_1 & \lambda(q_1, 0) = 0 & \delta(q_1, 1) = q_2 & \lambda(q_1, 1) = 1 \\ \delta(q_2, 0) = q_1 & \lambda(q_2, 0) = 1 & \delta(q_2, 1) = q_2 & \lambda(q_2, 1) = 0 \end{array}$$

We use transducers to transform infinite words. The transducer reads the input word letter by letter, and the transformation result is the concatenation of the output words encountered along the edges.

► **Definition 12.** Let $M = (\Sigma, \Delta, Q, q_0, \delta, \lambda)$ be a transducer. We extend the state transition function δ from letters Σ to finite words Σ^* as follows: $\delta(q, \varepsilon) = q$ and $\delta(q, aw) = \delta(\delta(q, a), w)$ for $q \in Q$, $a \in \Sigma$, $w \in \Sigma^*$.

The output function λ is extended to the set of all words $\Sigma^\infty = \Sigma^\omega \cup \Sigma^*$ by the following definition: $\lambda(q, \varepsilon) = \varepsilon$ and $\lambda(q, aw) = \lambda(q, a) \lambda(\delta(q, a), w)$ for $q \in Q$, $a \in \Sigma$, $w \in \Sigma^\infty$.

We introduce $\delta(w)$ and $\lambda(w)$ as shorthand for $\delta(q_0, w)$ and $\lambda(q_0, w)$, respectively. Moreover, we define $M(w) = \lambda(w)$, the *output of M on $w \in \Sigma^\omega$* . In this way, we think of M as a function from (finite or infinite) words on its input alphabet to infinite words on its output alphabet $M : \Sigma^\infty \rightarrow \Delta^\infty$.

If $x \in \Sigma^\omega$ and $y \in \Delta^\omega$, we write $y \preceq x$ if for some transducer M , we have $M(x) = y$.

Notice that every morphism is computable by a transducer (with one state). In particular, every coding is computable by a transducer.

► **Definition 13.** Let $M = (\Sigma, \Delta, Q, q_0, \delta, \lambda)$ and $N = (\Sigma', \Delta', Q', q'_0, \delta', \lambda')$ be transducers, and assume that $\Sigma' = \Delta$. We define the *composition* $N \circ M$ to be the transducer

$$\begin{aligned} N \circ M = & (\Sigma, \Delta', Q \times Q', (q_0, q'_0), \\ & ((q, q'), a) \mapsto (\delta(q, a), \delta'(q', \lambda(q, a))), \\ & ((q, q'), a) \mapsto \lambda'(q', \lambda(q, a))). \end{aligned}$$

Here δ' and λ' are the extensions of the transition and output functions of N to Σ^* , respectively.

► **Proposition 14.** Concerning the composition relation on transducers and \preceq on finite and infinite words:

- (i) The map $\Sigma^\infty \rightarrow (\Delta')^\infty$ computed by $N \circ M$ is the composition of $M : \Sigma^\infty \rightarrow \Delta^\infty$ followed by $N : \Delta^\infty \rightarrow (\Delta')^\infty$.
- (ii) The relation \preceq is transitive.
- (iii) If $x \in \Sigma^\infty$ and $h : \Sigma \rightarrow \Delta^*$ is a coding, then $h(x) \preceq x$.

3 Closure of Morphic Sequences under Morphic Images

► **Definition 15.** Let $h : \Sigma^* \rightarrow \Sigma^*$ be morphisms, and let $\Gamma \subseteq \Sigma$ be a set of letters. We call a letter $a \in \Sigma$

- (i) *dead* if $h^n(a) \in \Gamma^*$ for all $n \geq 0$,
- (ii) *near dead* if $a \notin \Gamma$, and for all $n > 0$, $h^n(a)$ consists of dead letters,

- (iii) *resilient* if $h^n(a) \notin \Gamma^*$ for all $n \geq 0$,
- (iv) *resurrecting* if $a \in \Gamma$ and $h^n(a) \notin \Gamma^*$ for all $n > 0$.

with respect to h and Γ . We say that the morphism h *respects* Γ if every letter $a \in \Sigma$ is either dead, near dead, resilient, or resurrecting. (Note that all of these definitions are with respect to some fixed h and Γ .)

► **Lemma 16.** *Let $g : \Sigma^* \rightarrow \Sigma^*$ be a morphism, and let $\Gamma \subseteq \Sigma$. Then g^r respects Γ for some natural number $r > 0$.*

Proof. See Lemma 7.7.3 in Allouche and Shallit [1]. ◀

► **Definition 17.** For a set of letters $\Gamma \subseteq \Sigma$ and a word $w \in \Sigma^\omega$, we write $\gamma_\Gamma(w)$ for the word obtained from w by erasing all occurrences of letters in Γ .

► **Definition 18.** Let $g : \Sigma^* \rightarrow \Sigma^*$ be a morphism, and $\Gamma \subseteq \Sigma$ a set of letters. We construct an alphabet Δ , a morphism $\xi : \Delta^* \rightarrow \Delta^*$ and a coding $\rho : \Delta \rightarrow \Sigma$ as follows. We refer to Δ, ξ, ρ as the *morphic system associated with the erasure of Γ from g^ω* .

Let $r \in \mathbb{N}_{>0}$ be minimal such that g^r respects Γ (r exists by Lemma 16). Let \mathcal{D} be the set of dead letters with respect to g^r and Γ . For $x \in \Sigma^*$ we use brackets $[x]$ to denote a new letter. For words $w \in \{g^r(a) \mid a \in \Sigma\}$, whenever $\gamma_{\mathcal{D}}(w) = w_0 a_1 w_1 a_2 w_2 \cdots a_{k-1} w_{k-1} a_k w_k$ with $a_1, \dots, a_k \notin \Gamma$ and $w_0, \dots, w_k \in \Gamma^*$, we define

$$\text{blocks}(w) = [w_0 a_1 w_1] [a_2 w_2] \cdots [a_{k-1} w_{k-1}] [a_k w_k]$$

Here it is to be understood that $\text{blocks}(w) = \varepsilon$ if $\gamma_{\mathcal{D}}(w) = \varepsilon$, and $\text{blocks}(w)$ is *undefined* if $\gamma_{\mathcal{D}}(w) \in \Gamma^+$.

Let the alphabet Δ consist of all letters $[a]$ and all bracketed letters $[w]$ occurring in words $\text{blocks}(g^r(a))$ for $a \in \Sigma$. We define the morphism $\xi : \Delta^* \rightarrow \Delta^*$ and the coding $\rho : \Delta \rightarrow \Sigma$ by

$$\xi([a_1 \cdots a_k]) = \text{blocks}(g^r(a_1)) \cdots \text{blocks}(g^r(a_k)) \quad \rho([w a u]) = a$$

for $[a_1 \cdots a_k] \in \Delta$ and $a \notin \Gamma, w, u \in \Gamma^*$. For $a \in \Gamma$ we can define $\rho([a])$ arbitrarily, for example, $\rho(a) = a$.

► **Remark.** The requirement that g^r respects Γ in Definition 18 guarantees for every $a \in \Sigma$ that either $g^r(a)$ consists of dead letters only or $g^r(a)$ contains at least one near dead or resilient letter. In both cases, $\text{blocks}(g^r(a))$ is well-defined. As a consequence $\xi([w])$ is well-defined for every $[w] \in \Delta$.

► **Example 19.** We let $\Sigma = \{a, b, c\}$ and define a morphism $g : \Sigma \rightarrow \Sigma^*$ by $a \mapsto ab, b \mapsto ac$ and $c \mapsto a$. The word $g^\omega(a) = abacabaabacababacabaabacabacabaabacababa \cdots$ is known as the *tribonacci word*.

Let $\Gamma = \{a\}$, that is, we delete the letter a . The morphism g does not respect Γ since $g(c) = a \in \Gamma^*$ but $g^2(c) = ab \notin \Gamma^*$. However, g^2 respects Γ : $g^2(a) = abac, g^2(b) = aba$ and $g^2(c) = ab$. The letter a is resurrecting and b, c are resilient with respect to g^2 and Γ . Definition 18 yields $\Delta = \{[a], [b], [c], [ab], [aba]\}$ and

$$\begin{aligned} \xi([a]) &= [aba][c] & \xi([b]) &= [aba] & \xi([c]) &= [ab] \\ \xi([ab]) &= [aba][c][aba] & \xi([aba]) &= [aba][c][aba][aba][c] \end{aligned}$$

while $\rho([b]) = \rho([ab]) = \rho([aba]) = b$, $\rho([c]) = c$, and $\rho([a]) = a$. The starting letter for iterating ξ is $[a]$ (since the tribonacci word starts with a). The first iterations of ξ are:

$$\begin{aligned} [a] &\mapsto [aba][c] \mapsto [aba][c][aba][aba][c][ab] \\ &\mapsto [aba][c][aba][aba][c][ab][aba][c][aba][c][aba][c][aba][c][ab][aba][c][aba] \\ &\mapsto \dots \end{aligned}$$

Then an application of the coding ρ yields $\rho(\xi^\omega([a])) = bcbcbcbcbcbcbcbcbcbcb \dots = \gamma_a(g^\omega(a))$.

► **Example 20.** We let $\Sigma = \{a, b, c, d, e\}$ and define $g : \Sigma \rightarrow \Sigma^*$ by $a \mapsto abcde$, $b \mapsto cc$, $c \mapsto b$, $d \mapsto c$ and $e \mapsto ea$. We let $\Gamma = \{b, e\}$. Then g^2 respects Γ : $a \mapsto abcdec bcea$, $b \mapsto bb$, $c \mapsto cc$, $d \mapsto b$ and $e \mapsto eaabcde$. Here b is dead, d near dead, a and c are resilient and e is resurrecting. Definition 18 yields

$$\begin{aligned} \xi([a]) &= [a][c][de][c][c][ce][a] & \xi([b]) &= \varepsilon & \xi([c]) &= [c][c] & \xi([d]) &= \varepsilon \\ \xi([e]) &= [ea][a][c][de] & \xi([ce]) &= [c][c][ea][a][c][de] \\ \xi([de]) &= [ea][a][c][de] & \xi([ea]) &= [ea][a][c][de][a][c][de][c][c][ce][a] \end{aligned}$$

where $\Delta = \{[a], [b], [c], [d], [e], [ce], [de], [ea]\}$. Moreover, we have $\rho([a]) = \rho([ea]) = a$, $\rho([c]) = \rho([ce]) = c$, $\rho([d]) = \rho([de]) = d$, $\rho([b]) = b$, and $\rho([e]) = e$.

► **Proposition 21.** Let $g : \Sigma^* \rightarrow \Sigma^*$ be a morphism, $a \in \Sigma$ such that $g^\omega(a) \in \Sigma^\omega$, and $\Gamma \subseteq \Sigma$ a set of letters. Let Δ , ξ and ρ be the morphic system associated to the erasure of Γ from g^ω in Definition 18. Then

$$\rho(\xi^\omega([a])) = \gamma_\Gamma(g^\omega(a))$$

Proof. For $\ell \in \mathbb{N}$ and $[w_1], \dots, [w_\ell] \in \Delta$ we define $\text{cat}([w_1] \dots [w_\ell]) = w_1 \dots w_\ell$. We prove by induction on n that for all words $w \in \Delta^*$, and for all $n \in \mathbb{N}$, $\text{cat}(\xi^n(w)) = g^{nr}(\text{cat}(w))$. The base case is immediate. For the induction step, assume that we have $n \in \mathbb{N}$ such that for all words $w \in \Delta^*$, $\text{cat}(\xi^n(w)) = g^{nr}(\text{cat}(w))$. Let $w \in \Delta^*$, $w = [a_{1,1} \dots a_{1,\ell_1}] \dots [a_{k,1} \dots a_{k,\ell_k}]$. Then

$$\begin{aligned} \text{cat}(\xi(w)) &= \text{cat}(\xi([a_{1,1} \dots a_{1,\ell_1}]) \dots \xi([a_{k,1} \dots a_{k,\ell_k}])) \\ &= \text{cat}(\text{blocks}(g^r(a_{1,1})) \dots \text{blocks}(g^r(a_{1,\ell_1})) \dots \text{blocks}(g^r(a_{k,1})) \dots \text{blocks}(g^r(a_{k,\ell_k}))) \\ &= g^r(\text{cat}(w)) \end{aligned}$$

By the induction hypothesis, $\text{cat}(\xi^{n+1}(w)) = g^{nr}(\text{cat}(\xi(w))) = g^{nr}(g^r(\text{cat}(w))) = g^{(n+1)r}(\text{cat}(w))$. To complete the proof, note that by definition $\rho([w a u]) = \gamma_\Gamma(w a u)$ and thus $\rho(w) = \gamma_\Gamma(\text{cat}(w))$ for every $w \in \Delta^*$. Hence, for all $n \geq 1$, $\rho(\xi^n([a])) = \gamma_\Gamma(\text{cat}(\xi^n([a]))) = \gamma_\Gamma(g^{nr}(a))$. Taking limits: $\rho(\xi^\omega([a])) = \gamma_\Gamma(g^\omega(a))$. ◀

► **Definition 22.** Let $g, h : \Sigma^* \rightarrow \Sigma^*$ be morphisms such that h is non-erasing. We construct an alphabet Δ , a morphism $\xi : \Delta^* \rightarrow \Delta^*$ and a coding $\rho : \Delta \rightarrow \Sigma$ as follows. We refer to Δ , ξ , ρ as the *morphic system associated with the morphic image of g^ω under h* .

Let $\Delta = \Sigma \cup \{[a] \mid a \in \Sigma\}$. For nonempty words $w = a_1 a_2 \dots a_k \in \Sigma^*$ we define $\text{head}(w) = a_1$ and $\text{tail}(w) = a_2 \dots a_k$. We also define $\text{img}(w) = [a_1]u_1 [a_2]u_2 \dots [a_{k-1}]u_{k-1} [a_k]u_k$ where $u_i = \text{tail}(h(a_i)) \in \Sigma^*$. We define the morphism $\xi : \Delta^* \rightarrow \Delta^*$ and the coding $\rho : \Delta \rightarrow \Sigma$ by

$$\xi([a]) = \text{img}(g(a)) \quad \xi(a) = \varepsilon \quad \rho([a]) = \text{head}(h(a)) \quad \rho(a) = a$$

for $a \in \Sigma$.

Notice here the $\rho([a])$ and u_i , defined using $\text{head}()$ and $\text{tail}()$, are well-defined since h is non-erasing and hence $h(a_i)$ will be nonempty.

► **Example 23.** Here is an example illustrating Definition 22. Let g be the substitution from the Fibonacci word, $g(a) = ab$ and $g(b) = a$. Further, let h be defined so that $h(a) = bb$ and $h(b) = a$. As in Definition 22, let ξ and ρ be defined by

$$\xi([a]) = [a]b[b] \quad \xi([b]) = [a]b \quad \xi(a) = \varepsilon = \xi(b) \quad \rho([a]) = b \quad \rho([b]) = a$$

Then $[a] \mapsto [a]b[b] \mapsto [a]b[b][a]b \mapsto [a]b[b][a]b[a]b[b] \mapsto [a]b[b][a]b[a]b[b][a]b[b][a]b \mapsto \dots$ are the first iterations of ξ on $[a]$. The point here is that applying ρ to the limit word $\xi^\omega([a])$ is the same as $h(g^\omega(a))$:

$$h(g^\omega(a)) = h(abaababaabaabaabab \dots) = bbabbbabbabbbabbbabb \dots$$

► **Proposition 24.** Let $g, h : \Sigma^* \rightarrow \Sigma^*$ be morphisms such that h is non-erasing, and $a \in \Sigma$ such that $g^\omega(a) \in \Sigma^\omega$. Let Δ , ξ and ρ be as in Definition 18. Then

$$\rho(\xi^\omega([a])) = h(g^\omega(a))$$

Proof. We define $z : \Delta \rightarrow \Sigma^*$ by $z(a) = \varepsilon$ and $z([a]) = a$ for all $a \in \Sigma$. By induction on $n > 0$ we show

$$\rho(\xi^n(w)) = h(g^n(z(w))) \quad \text{and} \quad z(\xi^n(w)) = g^n(z(w)) \quad \text{for all } w \in \Delta^* \quad (1)$$

We start with the base case. Note that $\rho(\xi([a])) = h(g(a)) = h(g(z([a])))$ and $\rho(\xi(a)) = \varepsilon = h(g(z(a)))$ for all $a \in \Sigma$, and thus $\rho(\xi(w)) = h(g(z(w)))$ for all $w \in \Delta^*$. Moreover, we have $z(\xi([a])) = g(a) = g(z([a]))$ and $z(\xi(a)) = \varepsilon = g(z(a))$ for all $a \in \Sigma$, and hence $z(\xi(w)) = g(z(w))$ for all $w \in \Delta^*$.

Let us consider the induction step. By the base case and induction hypothesis

$$\begin{aligned} \rho(\xi^{n+1}(w)) &= \rho(\xi(\xi^n(w))) = h(g(z(\xi^n(w)))) = h(g(g^n(z(w)))) = h(g^{n+1}(z(w))) \\ z(\xi^{n+1}(w)) &= z(\xi(\xi^n(w))) = g(z(\xi^n(w))) = g(g^n(z(w))) = g^{n+1}(z(w)) \end{aligned}$$

Thus $\rho(\xi^n([a])) = h(g^n(a))$ for all $n \in \mathbb{N}$, and taking limits yields $\rho(\xi^\omega([a])) = h(g^\omega(a))$. ◀

Every morphic image of a word can be obtained by erasing letters, followed by the application of a non-erasing morphism. As a consequence we obtain:

► **Corollary 25.** *The morphic image of a pure morphic word is morphic or finite.*

Proof. Let $w \in \Sigma^\omega$ be a word and $h : \Sigma \rightarrow \Sigma^*$ a morphism. Let $\Gamma = \{ a \mid h(a) = \varepsilon \}$ be the set of letters erased by h , and $\Delta = \Sigma \setminus \Gamma$. Then $h(w) = g(\gamma_\Gamma(w))$ where g is the non-erasing morphism obtained by restricting h to Δ . Hence for purely morphic w , the result follows from Propositions 21 and 24. ◀

► **Theorem 26** (Cobham [2], Pansiot [7]). *The morphic image of a morphic word is morphic.*

Proof. Follows from Corollary 25 since the coding can be absorbed into the morphic image. ◀

Eigenvalue analysis

The following lemma states that if a square matrix N is an extension of a square matrix M , and all added columns contain only zeros, then M and N have the same non-zero eigenvalues.

$$\left(\begin{array}{c|ccc} M & 0 & \cdots & 0 \\ \hline & 0 & \cdots & 0 \\ & 0 & \cdots & 0 \end{array} \right)$$

► **Lemma 27.** *Let Σ, Δ be disjoint, finite alphabets. Let $M = (m_{i,j})_{i,j \in \Sigma}$ and $N = (n_{i,j})_{i,j \in \Sigma \cup \Delta}$ be matrices such that (i) $n_{i,j} = m_{i,j}$ for all $i, j \in \Sigma$ and (ii) $n_{i,j} = 0$ for all $i \in \Sigma \cup \Delta, j \in \Delta$. Then M and N have the same non-zero eigenvalues.*

Proof. N is a block lower triangular matrix with M and 0 as the matrices on the diagonal. Hence the eigenvalues of N are the combined eigenvalues of M and 0 . Therefore M and N have the same non-zero eigenvalues. ◀

We now show that morphic images with respect to non-erasing morphisms preserve α -substitutivity. This strengthens a result obtained in [5] where it has been shown that the non-erasing morphic image of an α -substitutive sequence is α^k -substitutive for some $k \in \mathbb{N}$. We show that one can always take $k = 1$. Note that every α -substitutive sequence is also α^k -substitutive for all $k \in \mathbb{N}, k > 0$.

► **Theorem 28.** *Let Σ be a finite alphabet, $w \in \Sigma^\omega$ be an α -substitutive sequence and $h : \Sigma \rightarrow \Sigma^*$ a non-erasing morphism. Then the morphic image of w under h , that is $h(w)$, is α -substitutive.*

Proof. Let $\Sigma = \{a_1, \dots, a_k\}$ be a finite alphabet, $w \in \Sigma^\omega$ be an α -substitutive sequence and $h : \Sigma \rightarrow \Sigma^*$ a non-erasing morphism. As the sequence w is α -substitutive, there exist a morphism $g : \Sigma \rightarrow \Sigma^*$ with dominant eigenvalue α , a coding $c : \Sigma \rightarrow \Sigma$ and a letter $a \in \Sigma$ such that $w = c(g^\omega(a))$ and all letters from Σ occur in $g^\omega(a)$. Then $h(w) = h(c(g^\omega(a))) = (h \circ c)(g^\omega(a))$, and $h \circ c$ is a non-erasing morphism. Without loss of generality, by absorbing c into h , we may assume that c is the identity.

From h and g , we obtain an alphabet Δ , a morphism ξ , and a coding ρ as in Definition 22. Then by Proposition 24, we have $\rho(\xi^\omega([a])) = h(g^\omega(a))$. As a consequence, it suffices to show that $\rho(\xi^\omega([a]))$ is α -substitutive. Let $M = (M_{i,j})_{i,j \in \Sigma}$ and $N = (N_{i,j})_{i,j \in \Delta}$ be the incidence matrices of g and ξ , respectively. By Definition 22 we have for all $a, b \in \Sigma$: $|\xi([a])|_{[b]} = |g(a)|_b$ and $|\xi(a)|_b = |\xi(a)|_{[b]} = 0$. Hence we obtain $N_{[b],[a]} = M_{b,a}, N_{b,a} = 0$ and $N_{[b],a} = 0$ for all $a, b \in \Sigma$. After changing the names (swapping a with $[a]$) in N , we obtain from Lemma 27 that N and M have the same non-zero eigenvalues, and thus the same dominant eigenvalue. ◀

► **Example 29.** Let F be the Fibonacci word (generated by the morphism $a \mapsto ab$ and $b \mapsto a$) and let T be the Thue–Morse sequence. We show that there exist no non-erasing morphisms g, h such that $g(F) = h(T)$ and this image is not ultimately periodic. Let g and h be non-erasing morphisms. The Fibonacci word is φ -substitutive where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio, and the Thue–Morse sequence is 2-substitutive. By Theorem 28, $g(F)$ is φ -substitutive and $h(T)$ is 2-substitutive. Note that φ and 2 are multiplicatively independent: using induction on $k \in \mathbb{N}_{>0}$ it follows that every φ^k is of the form $a + b\sqrt{5}$ for rational numbers $a, b > 0$. It follows by Theorem 9 that $g(F) = h(T)$ implies that this word is ultimately periodic.

► **Remark.** The restriction to non-erasing morphisms in Theorem 28 is important since every morphic sequence can be obtained by erasure of letters from a 2-substitutive sequence.

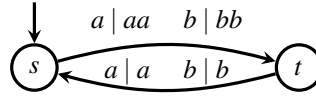
Nevertheless, we can use the above theorem to reason about morphic images with respect to erasing morphisms as follows. Let $w \in \Sigma^\omega$, and $g : \Sigma \rightarrow \Sigma^*$ a morphism. Let Γ be the letters erased by g , and let h be the restriction of g to $\Sigma \setminus \Gamma$. Then h is non-erasing and $g(w) = h(\gamma_\Gamma(w))$. Hence, if $\gamma_\Gamma(w)$ is α -substitutive, then so is $g(w)$ by Theorem 28. As a consequence, it suffices to determine α -substitutivity of all sequences $\gamma_\Gamma(w)$ with $\Gamma \subseteq \Sigma$ (using Definition 18 and Proposition 21).

4 Closure of Morphic Sequences under Transduction

In this section, we give a proof of the following theorem due to Dekking [4].

► **Theorem 30** (Transducts of morphic sequences are morphic). *If $M = (\Sigma, \Delta, Q, q_0, \delta, \lambda)$ is a transducer with input alphabet Σ and $x \in \Sigma^\omega$ is a morphic sequence, then $M(x)$ is morphic or finite.*

This proof will proceed by *annotating* entries in the original sequence x with information about what state the transducer is in upon reaching that entry. This allows us to construct a new morphism which produces the transduced sequence $M(x)$ as output. After proving this theorem, we will show that this process of annotation preserves α -substitutivity.



■ **Figure 2** A transducer that doubles every other letter.

► **Example 31.** To illustrate several points in this section, we will consider the Fibonacci morphism ($h(a) = ab$, $h(b) = a$) and the transducer which doubles every other letter, shown in Figure 2.

4.1 Transducts of morphic sequences are morphic

We show in Lemma 40 that transducts of morphic sequences are morphic. In order to prove this, we also need several lemmas about transducers which are of independent interest. The approach here is adapted from a result in Allouche and Shallit [1]; it is attributed in that book to Dekking. We repeat it here partly for the convenience of the reader, but mostly because there are some details of the proof which are used in the analysis of the substitutivity property.

► **Definition 32** (τ_w , $\Xi(w)$). Given a transducer $M = (\Sigma, \Delta, Q, q_0, \delta, \lambda)$ and a word $w \in \Sigma^*$, we define $\tau_w \in Q^Q$ to be $\tau_w(q) = \delta(q, w)$. Note that $\tau_{wv} = \tau_v \circ \tau_w$. Further, we define $\Xi : \Sigma^* \rightarrow (Q^Q)^\omega$ by $\Xi(w) = (\tau_w, \tau_{h(w)}, \tau_{h^2(w)}, \dots, \tau_{h^{|w|}(w)}, \dots)$.

► **Example 33.** Recall the transducer M from Figure 2. Let $id : Q \rightarrow Q$ be the identity, and let $\nu : Q \rightarrow Q$ be the transposition $\nu(s) = t$ and $\nu(t) = s$. For this transducer, $\tau_w = id$ if $|w|$ is even and $\tau_w = \nu$ if $|w|$ is odd. We have $\Xi(a) = (\tau_a, \tau_{ab}, \tau_{aba}, \tau_{abaab}, \tau_{abaababa}, \dots)$. In this notation,

$$\Xi(a) = (\nu, id, \nu, \nu, id, \nu, \nu, id, \nu, \nu, \dots) \quad \Xi(b) = (\nu, \nu, id, \nu, \nu, id, \dots) \quad \Xi(\epsilon) = (id, id, id, id, \dots)$$

Next, we show that $\{\Xi(w) : w \in \Sigma^*\}$ is finite.

► **Lemma 34.** For any transducer M and any morphism $h : \Sigma \rightarrow \Sigma^*$, there are natural numbers $p \geq 1$ and $n \geq 0$ so that for all $w \in \Sigma^*$, $\tau_{h^i(w)} = \tau_{h^{i+p}(w)}$ for all $i \geq n$.

Proof. Let $\Sigma = \{1, 2, \dots, s\}$. Define $H : (Q^Q)^s \rightarrow (Q^Q)^s$ by $H(f_1, f_2, \dots, f_s) = (f_{h(1)}, f_{h(2)}, \dots, f_{h(s)})$. When we write $f_{h(i)}$ on the right, here is what we mean. Suppose that $h(i) = v_0 \cdots v_j$. Then $f_{h(i)}$ is short for the composition $f_{v_j} \circ f_{v_{j-1}} \circ \cdots \circ f_{v_1} \circ f_{v_0}$. Recall the notation τ_w from Definition 32; we thus have τ_i for the individual letters $i \in \Sigma$. Consider $T_0 = (\tau_1, \tau_2, \dots, \tau_s)$. We define its *orbit* as the infinite sequence $(T_i)_{i \in \omega}$ of elements of $(Q^Q)^s$ given by $T_i = H^i(T_0) = H^i(\tau_1, \dots, \tau_s) = (\tau_{h^i(1)}, \dots, \tau_{h^i(s)})$. Since each of the T_i belongs to the finite set $(Q^Q)^s$, the orbit of T_0 is eventually periodic. Let n be the preperiod length and p be the period length. The periodicity implies that (*) $\tau_{h^i(j)} = \tau_{h^{i+p}(j)}$ for each $j \in \Sigma$ and for all $i \geq n$.

Let $w \in \Sigma^*$ and $i \geq n$. Since $w \in \Sigma^*$, we can write it as $w = \sigma_1 \sigma_2 \cdots \sigma_m$. We prove that $\tau_{h^i(w)} = \tau_{h^{i+p}(w)}$. Note that $\tau_{h^i(w)} = \tau_{h^i(\sigma_1 \cdots \sigma_m)} = \tau_{h^i(\sigma_1)h^i(\sigma_2) \cdots h^i(\sigma_m)} = \tau_{h^i(\sigma_m)} \circ \cdots \circ \tau_{h^i(\sigma_1)}$. We got this by breaking w into individual letters, then using the fact that h is a morphism, and finally using the fact that $\tau_{uv} = \tau_u \circ \tau_v$. Finally we know by (*) that for individual letters, $\tau_{h^i(\sigma_j)} = \tau_{h^{i+p}(\sigma_j)}$. So $\tau_{h^i(w)} = \tau_{h^{i+p}(w)}$, as desired. ◀

► **Definition 35** ($\Theta(w)$). Given a transducer M and a morphism h , we find p and n as in Lemma 34 just above and define $\Theta(w) = (\tau_w, \tau_{h(w)}, \dots, \tau_{h^{n+p-1}(w)})$.

► **Example 36.** We continue with Example 31. As the proof in Lemma 34 demonstrates, to find the p and n for our transducer and the Fibonacci morphism, we only need to find the common period of $\Xi(a)$ and $\Xi(b)$. Using what we saw in Example 33 above, we can take $n = 0$ and $p = 3$. Therefore, $\Theta(a) = (v, id, v)$ and $\Theta(b) = (v, v, id)$. We also note that $\Theta(\epsilon) = (id, id, id)$ and $\Theta(ab) = (id, v, v)$, as we will need these later.

- **Lemma 37.** (i) Given M and h , the set $A = \{\Theta(w) : w \in \Sigma^*\}$ is finite.
(ii) If $\Theta(w) = \Theta(y)$, then $\Theta(h(w)) = \Theta(h(y))$.
(iii) If $\Theta(w) = \Theta(y)$, then for all $u \in \Sigma^*$, $\Theta(wu) = \Theta(yu)$.

Proof. Part (i) comes from the fact that each of the $n + p$ coordinates of $\Theta(w)$ comes from the finite set Q^Q . For (ii), we calculate:

$$\begin{aligned} \Theta(h(w)) &= (\tau_{h(w)}, \tau_{h^2(w)}, \tau_{h^3(w)}, \dots, \tau_{h^{n+p}(w)}) \\ &= (\tau_{h(w)}, \tau_{h^2(w)}, \tau_{h^3(w)}, \dots, \tau_{h^{n+p-1}(w)}, \tau_{h^n(w)}) && \text{by Lemma 34} \\ &= (\tau_{h(y)}, \tau_{h^2(y)}, \tau_{h^3(y)}, \dots, \tau_{h^{n+p-1}(y)}, \tau_{h^n(y)}) = \Theta(h(y)) && \text{since } \Theta(w) = \Theta(y) \end{aligned}$$

Part (iii) uses $\Theta(w) = \Theta(y)$ as follows:

$$\begin{aligned} \Theta(wu) &= (\tau_{wu}, \tau_{h(wu)}, \tau_{h^2(wu)}, \dots, \tau_{h^{n+p-1}(wu)}) \\ &= (\tau_u \circ \tau_w, \tau_{h(u)} \circ \tau_{h(w)}, \tau_{h^2(u)} \circ \tau_{h^2(w)}, \dots, \tau_{h^{n+p-1}(u)} \circ \tau_{h^{n+p-1}(w)}) \\ &= (\tau_u \circ \tau_y, \tau_{h(u)} \circ \tau_{h(y)}, \tau_{h^2(u)} \circ \tau_{h^2(y)}, \dots, \tau_{h^{n+p-1}(u)} \circ \tau_{h^{n+p-1}(y)}) = \Theta(yu) \end{aligned}$$

► **Definition 38** (\bar{h}). Given a transducer M and a morphism h , let A be as in Lemma 37(i). Define the morphism $\bar{h} : \Sigma \times A \rightarrow (\Sigma \times A)^*$ as follows. For for all $\sigma \in \Sigma$, whenever $h(\sigma) = s_1 s_2 s_3 \cdots s_\ell$, let

$$\bar{h}((\sigma, \Theta(w))) = (s_1, \Theta(hw)) (s_2, \Theta((hw)s_1)) (s_3, \Theta((hw)s_1 s_2)) \cdots (s_\ell, \Theta((hw)s_1 s_2 \cdots s_{\ell-1}))$$

By repeated use of Lemma 37, \bar{h} is well-defined. Notice that $|\bar{h}(\sigma, a)| = |h(\sigma)|$ for all σ .

► **Lemma 39.** For all $\sigma \in \Sigma$, all $w \in \Sigma^*$ and all natural numbers n , if $h^n(\sigma) = s_1 s_2 \cdots s_\ell$, then

$$\bar{h}^n((\sigma, \Theta(w))) = (s_1, \Theta(h^n w)) (s_2, \Theta((h^n w) s_1)) (s_3, \Theta((h^n w) s_1 s_2)) \cdots (s_\ell, \Theta((h^n w) s_1 s_2 \cdots s_{\ell-1})).$$

In particular, for $1 \leq i \leq \ell$, the first component of the i^{th} term in $h^n(\sigma, \Theta(w))$ is s_i .

Proof. By induction on n . For $n = 0$, the claim is trivial. Assume that it holds for n . Let $h^n(\sigma) = s_1 s_2 \cdots s_\ell$, and for $1 \leq i \leq \ell$, let $h(s_i) = t_1^i t_2^i \cdots t_{k_i}^i$. Thus $h^{n+1}(\sigma) = h(s_1 s_2 \cdots s_\ell) = t_1^1 t_2^1 \cdots t_{k_1}^1 t_1^2 t_2^2 \cdots t_{k_2}^2 t_1^\ell t_2^\ell \cdots t_{k_\ell}^\ell$. Then:

$$\begin{aligned} \bar{h}(\bar{h}^n(\sigma, \Theta(w))) &= \bar{h}(s_1, \Theta((h^n w))) \bar{h}(s_2, \Theta((h^n w) s_1)) \bar{h}(s_3, \Theta((h^n w) s_1 s_2)) \\ &\quad \cdots \bar{h}(s_\ell, \Theta((h^n w) s_1 s_2 \cdots s_{\ell-1})) \end{aligned}$$

For $1 \leq i \leq \ell$, we have

$$\begin{aligned} &\bar{h}(s_i, \Theta((h^n w) s_1 \cdots s_{i-1})) \\ &= (t_1^i, \Theta((h^n w) h(s_1 \cdots s_{i-1}))) (t_2^i, \Theta((h^n w) h(s_1 \cdots s_{i-1}) t_1^i)) \\ &\quad \cdots (t_{k_i}^i, \Theta((h^n w) h(s_1 \cdots s_{i-1}) t_1^i t_2^i \cdots t_{k_i}^i)) \\ &= (t_1^i, \Theta((h^{n+1} w) t_1^1 t_2^1 \cdots t_{k_1}^1 \cdots t_1^{i-1} t_2^{i-1} \cdots t_{k_{i-1}}^{i-1})) (t_2^i, \Theta((h^{n+1} w) t_1^1 t_2^1 \cdots t_{k_1}^1 \cdots t_1^{i-1} t_2^{i-1} \cdots t_{k_{i-1}}^{i-1} t_1^i)) \\ &\quad \cdots (t_{k_i}^i, \Theta((h^{n+1} w) t_1^1 t_2^1 \cdots t_{k_1}^1 \cdots t_1^{i-1} t_2^{i-1} \cdots t_{k_{i-1}}^{i-1} t_1^i \cdots t_{k_i}^i)) \end{aligned}$$

Concatenating the sequences $\bar{h}(s_i, \Theta((h^n w) s_1 \cdots s_{i-1}))$ for $i = 1, \dots, \ell$ completes our induction step. ◀

► **Lemma 40.** Let $M = (\Sigma, \Delta, Q, q_0, \delta, \lambda)$ be a transducer, let h be a morphism prolongable on the letter x_1 , and write $h^\omega(x_1)$ as $x = x_1 x_2 x_3 \cdots x_n \cdots$. Let Θ be from Definition 35. Using this, let A be from Lemma 37(i), and \bar{h} from Definition 38. Then

- (i) \bar{h} is prolongable on $(x_1, \Theta(\epsilon))$.
- (ii) Let $c : \Sigma \times A \rightarrow \Sigma \times Q$ be the coding $c(\sigma, \Theta(w)) = (\sigma, \tau_w(q_0))$. Then c is well-defined.
- (iii) The image under c of $\bar{h}^\omega((x_1, \Theta(\epsilon)))$ is

$$z = (x_1, \delta(q_0, \epsilon)) (x_2, \delta(q_0, x_1)) (x_3, \delta(q_0, x_1 x_2)) \cdots (x_n, \delta(q_0, x_1 x_2 \cdots x_{n-1})) \cdots \quad (2)$$

This sequence z is morphic in the alphabet $\Sigma \times Q$.

Proof. For (i), write $h(x_1)$ as $x_1 x_2 \cdots x_\ell$. Using the fact that $h^i(\epsilon) = \epsilon$ for all i , we see that

$$\bar{h}((x_1, \Theta(\epsilon))) = (x_1, \Theta(\epsilon)) (x_2, \Theta(x_1)) \cdots (x_\ell, \Theta(x_1, \dots, x_{\ell-1})).$$

This verifies the prolongability.

For (ii): if $\Theta(w) = \Theta(u)$, then τ_w and τ_u are the first component of $\Theta(w)$ and are thus equal.

We turn to (iii). Taking $w = \epsilon$ in Lemma 39 shows that $\bar{h}^\omega((x_1, \Theta(\epsilon)))$ is

$$(x_1, \Theta(\epsilon)) (x_2, \Theta(x_1)) (x_3, \Theta(x_1 x_2)) \cdots (x_m, \Theta(x_1 x_2 \cdots x_{m-1})) \cdots .$$

The image of this sequence under the coding c is

$$(x_1, \tau_\epsilon(q_0)) (x_2, \tau_{x_1}(q_0)) (x_3, \tau_{x_1 x_2}(q_0)) \cdots (x_m, \tau_{x_1 x_2 \cdots x_{m-1}}(q_0)) \cdots .$$

In view of the τ functions' definition (Def. 32), we obtain z in (2). By definition, z is morphic. ◀

This is most of the work required to prove Theorem 30, the main result of this section.

Theorem 30. Since x is morphic there is a morphism $h : \Sigma' \rightarrow (\Sigma')^*$, a coding $c : \Sigma' \rightarrow \Sigma$, and an initial letter $x_1 \in \Sigma'$ so that $x = c(h^\omega(x_1))$. We are to show that $M(c(h^\omega(x_1)))$ is morphic. Since c is computable by a transducer, we have $x = (M \circ c)(h^\omega(x_1))$, where \circ is the composition of transducers from Definition 13. It is thus sufficient to show that given a transducer M , the sequence $M(h^\omega(x_1))$ is morphic.

By Lemma 40, the sequence

$$z = (x_1, \delta(q_0, \epsilon)) \quad (x_2, \delta(q_0, x_1)) \quad (x_3, \delta(q_0, x_1x_2)) \quad \cdots \quad (x_n, \delta(q_0, x_1x_2 \cdots x_{n-1})) \quad \cdots$$

is morphic. The output function of M is a morphism $\lambda : \Sigma \times Q \rightarrow \Delta^*$. By Corollary 25, $\lambda(z)$ is morphic or finite. But $\lambda(z)$ is exactly $M(x)$; indeed, the definition of $M(x)$ is basically the same as the definition of $\lambda(z)$. This proves the theorem. \blacktriangleleft

4.2 Substitutivity of transducts

We are also interested in analyzing the α -substitutivity of transducts. We claim that if a sequence x is α -substitutive, then $M(x)$ is also α -substitutive for all M .

As a first step, we show that annotating a morphism does not change α -substitutivity.

► **Definition 41.** Let Σ be an alphabet and A any set. Let $w = (b_1, a_1) (b_2, a_2) \dots (b_k, a_k) \in (\Sigma \times A)^*$ be a word. We call A the *set of annotations*. We write $[w]$ for the word $b_1b_2 \dots b_k$, that is, the word obtained by *dropping the annotations*.

A morphism $\bar{h} : (\Sigma \times A) \rightarrow (\Sigma \times A)^*$ is an *annotation* of $h : \Sigma \rightarrow \Sigma^*$ if $h(b) = [\bar{h}(b, a)]$ for all $b \in \Sigma, a \in A$.

Note that the morphism \bar{h} from Definition 38 is an annotation of h in this sense. Then from the following proposition it follows that if x is α -substitutive, then the sequence z in Lemma 40 is also α -substitutive.

► **Proposition 42.** If $x = h^\omega(\sigma)$ is an α -substitutive morphic sequence with morphism $h : \Sigma \rightarrow \Sigma^*$ and A is any set of annotations, then any annotated morphism $\bar{h} : \Sigma \times A \rightarrow (\Sigma \times A)^*$ also has an infinite fixpoint $\bar{h}^\omega((\sigma, a))$ which is also α -substitutive.

The proof of this proposition is in two lemmas: first that the eigenvalues of the morphism are preserved by the annotation process, and second that if α is the dominant eigenvalue for h , then no greater eigenvalues are introduced for \bar{h} .

► **Lemma 43.** All eigenvalues for h are also eigenvalues for any annotated version \bar{h} of h .

Proof. Let $M = (m_{i,j})_{i,j \in \Sigma}$ be the incidence matrix of h . Order the elements of the annotated alphabet $\Sigma \times A$ lexicographically. Then the incidence matrix of \bar{h} , call it $N = (n_{i,j})_{i,j \in \Sigma \times A}$, can be thought of as a block matrix where the blocks have size $|A| \times |A|$ and there are $|\Sigma| \times |\Sigma|$ such blocks in N . Note that by the definition of annotation, the row sum in each row of the (a, b) block of N is $m_{a,b}$. To simplify the notation, for the rest of this proof we write J for $|\Sigma|$ and K for $|A|$. Suppose $v = (v_1, v_2, \dots, v_j)$ is a column eigenvector for M with eigenvalue α . Consider $\bar{v} = (v_1, \dots, v_1, v_2, \dots, v_2, \dots, v_n, \dots, v_n)$. This is a "block vector": the first K entries are v_1 , the second K entries are v_2 , and so on, for a total of $K \cdot J$ entries. We claim that \bar{v} is a column eigenvector for N with eigenvalue α .

Consider the product of row k of N with \bar{v} . This is $\sum_{j=1}^{K \cdot J} n_{k,j} \bar{v}_j = \sum_{b=1}^J v_b \cdot (\sum_{j=1}^K n_{k,Kb+j})$. Now $k = Ka + r$. So $\sum_{j=1}^K n_{k,Kb+j}$ is the row sum of the (a, b) block of N and hence is $m_{a,b}$.

Therefore, row k of N times \bar{v} is $\sum_{b=1}^J v_b m_{a,b} = \alpha v_a$, since v is an eigenvector of M . Finally we note that the k th entry of \bar{v} is v_a by its definition. Hence multiplying \bar{v} by N multiplies the k th entry of \bar{v} by α for all k .

We have shown that \bar{v} is a column eigenvector of N with eigenvalue α , so the (column) eigenvalues of M are all present in N . However, since a matrix and its transpose have the same eigenvalues, the (column) qualification on the eigenvalues is unnecessary. ◀ ◀

If \bar{h} is an annotation of h , then we have

$$|h(b)|_{b'} = \sum_{a' \in A} |\bar{h}((b, a))|_{(b', a')} \quad \text{for all } b, b' \in \Sigma \text{ and } a \in A \quad (3)$$

► **Lemma 44.** *Let h, \bar{h} be morphisms such that $\bar{h} : (\Sigma \times A) \rightarrow (\Sigma \times A)^*$ is an annotation of $h : \Sigma \rightarrow \Sigma^*$. Then every eigenvalue of \bar{h} with a non-negative eigenvector is also an eigenvalue for h .*

Proof. Let $M = (m_{i,j})_{i,j \in \Sigma}$ be the incidence matrix of h and $N = (n_{i,j})_{i,j \in \Sigma \times A}$ be the incidence matrix of \bar{h} . Let r be an eigenvalue of N with corresponding eigenvector $v = (v_{(b,a)})_{(b,a) \in \Sigma \times A}$, that is, $Nv = rv$ and $v \neq 0$. We define a vector $w = (w_b)_{b \in \Sigma}$ as follows: $w_b = \sum_{a \in A} v_{(b,a)}$. We show that $Mw = rw$. Let $b' \in \Sigma$, then:

$$\begin{aligned} (Mw)_{b'} &= \sum_{b \in \Sigma} M_{b',b} w_b = \sum_{b \in \Sigma} \left(M_{b',b} \sum_{a \in A} v_{(b,a)} \right) \\ &= \sum_{b \in \Sigma} \sum_{a \in A} M_{b',b} v_{(b,a)} \stackrel{\text{by (3)}}{=} \sum_{b \in \Sigma} \sum_{a \in A} \left(\sum_{a' \in A} N_{(b',a'),(b,a)} \right) v_{(b,a)} \\ &= \sum_{a' \in A} \sum_{b \in \Sigma} \sum_{a \in A} N_{(b',a'),(b,a)} v_{(b,a)} \stackrel{Nv=rv}{=} \sum_{a' \in A} r v_{(b',a')} = r \sum_{a' \in A} v_{(b',a')} = r w_{b'} \end{aligned}$$

Hence $Mw = rw$. If $w \neq 0$ it follows that r is an eigenvalue of M . Note that if v is non-negative, then $w \neq 0$. This proves the claim. ◀

► **Corollary 45.** *Let h, \bar{h} be morphisms such that $\bar{h} : (\Sigma \times A) \rightarrow (\Sigma \times A)^*$ is an annotation of $h : \Sigma \rightarrow \Sigma^*$. Then the dominant eigenvalue for h coincides with the dominant eigenvalue for \bar{h} .*

Proof. By Lemma 43 every eigenvalue of h is an eigenvalue of \bar{h} . Thus the dominant eigenvalue of \bar{h} is greater or equal to that of h . By Theorem 7, the dominant eigenvalue of a non-negative matrix is a real number $\alpha > 1$ and its corresponding eigenvector is non-negative. By Lemma 43, every eigenvalue of \bar{h} with a non-negative eigenvector is also an eigenvalue of h . Thus the dominant eigenvalue of h is also greater or equal to that of \bar{h} . Hence the dominant eigenvalues of h and \bar{h} must be equal. ◀

► **Theorem 46.** *Let α and β be multiplicatively independent real numbers. If v is a α -substitutive sequence and w is a β -substitutive sequence, then v and w have no common non-erasing transducts except for the ultimately periodic sequences.*

Proof. Let h_v and h_w be morphisms whose fixed points are v and w , respectively. By the proof of Theorem 30, x is a morphic image of an annotation \bar{h}_v of h_v , and also of an annotation \bar{h}_w of h_w . The morphisms must be non-erasing, by the assumption in this theorem. By Corollary 45 and Theorem 28, x is both α - and β -substitutive. By Durand's Theorem 9, x is eventually periodic. ◀

4.3 Example

We conclude the section with an example of Theorem 30 and the lemmas in this section.

► **Example 47.** We saw the Fibonacci sequence in Example 5:

$$x = abaababaabaababaabaababaabaababaaba \dots$$

We conclude our series of examples pertaining to this sequence and the transducer M which doubles every other letter (see Example 31 and Figure 2). We want to exhibit \bar{h} , following the recipe of Lemma 40. First, some examples of how \bar{h} works:

$$\begin{aligned} (b, \Theta(a)) &\mapsto (a, \Theta(ab)) & (a, \Theta(\epsilon)) &\mapsto (a, \Theta(\epsilon))(b, \Theta(a)) \\ (b, \Theta(ab)) &\mapsto (a, \Theta(aba)) = (a, \Theta(b)) & (a, \Theta(a)) &\mapsto (a, \Theta(ab))(b, \Theta(aba)) = (a, \Theta(ab))(b, \Theta(b)) \end{aligned}$$

It turns out that only a few elements from this A end up appearing in the expressions for $\bar{h}(\sigma, \Theta(w))$: It is convenient to abbreviate some of the elements of $\Sigma \times A$: Let us use x as an element of $\{a, b\}$, and also write $(x, \Theta(\epsilon))$ as x_0 , $(x, \Theta(a))$ as x_1 , $(x, \Theta(b))$ as x_2 and $(x, \Theta(ab))$ as x_3 . It turns out that we do not need to exhibit \bar{h} in full because only eight points are reachable from a_0 . We may take \bar{h} to be

$$\begin{array}{cccc} a_0 \mapsto a_0 b_1 & a_1 \mapsto a_2 b_3 & a_2 \mapsto a_3 b_2 & a_3 \mapsto a_1 b_0 \\ b_0 \mapsto a_0 & b_1 \mapsto a_2 & b_2 \mapsto a_3 & b_3 \mapsto a_1 \end{array}$$

The fixpoint of this morphism starting with a_0 starts as

$$y = \bar{h}^{a_0}(a_0) = a_0 b_1 a_2 a_3 b_2 a_1 b_0 a_3 a_2 b_3 a_0 a_1 b_0 a_3 b_2 a_1 a_0 b_1 a_2 b_3 a_0 a_1 b_0 a_3 a_2 b_3 a_0 b_1 \dots$$

Turning to the coding c , recall that the set Q of states of M is $\{s, t\}$. Let us abbreviate the elements of $\Sigma \times Q$ the same way we did with $\Sigma \times A$. It is not hard to check that $c(\sigma_0) = \sigma_s$, $c(\sigma_1) = \sigma_t$, $c(\sigma_2) = \sigma_s$, and $c(\sigma_3) = \sigma_t$. Then the state-annotated sequence z from Lemma 40 is

$$z = c(y) = a_s b_t a_s a_t b_s a_t b_s a_t a_s b_t a_s a_t b_s a_t b_s a_t a_s b_t a_s a_t b_s a_t a_s b_t a_s b_t \dots$$

Recall that $\lambda : \Sigma \times Q \rightarrow \Delta^* = \Sigma^*$ in our transducer doubles whatever letter it sees while in state s and copies whatever letter it sees while in state t . That is, $\lambda(x_s) = xx$, and $\lambda(x_t) = x$. Thus when we apply the morphism λ to the sequence z , we get

$$\lambda(z) = aa b aa a bb a bb a aa b aa a bb a bb a aa b aa b aa a bb a aa b aa b \dots$$

As we saw in the proof of Theorem 30, this sequence

$$aabaabbabbaabaabaabbabbaabaabaabbabbaabaab \dots$$

is exactly $M(x)$.

5 Conclusion

We have re-proven some of the central results in the area of morphic sequences, the closure of the morphic sequences under morphic images and transduction. However, the main results in this paper come from the eigenvalue analyses which followed our proofs in Sections 3 and 4. These are some of the only results known to us which enable one to prove

negative results on the transducibility relation \preceq . One such result is in Theorem 46; this is perhaps the culmination of this paper.

The next step in this line of work is to weaken the hypothesis in some of results that the transducers be *non-erasing*. Although our results can be used to reason about erasing morphisms, see Remark 3, this does not help us with erasing transducers since annotating a morphism can yield an unbounded large alphabet. As a consequence, to reason about erasing transducers, we need to understand better what form of annotated morphisms arise from transducers, and how these interact with the erasure of letters (Proposition 21).

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